the process of integrating over Ω_k exactly. If there is no information available about the direction of $\mathbf{k}$ in the t matrices appearing in Eq. (22), then the phase of T_{DS} is completely unknown.

It is stressed that the double-scattering effect will be smaller at values of θ_γ away from 180°.

IV. CONCLUSION

We have suggested that integrals over two energy denominators be carried out by combining the denominators as in Eq. (5) and (6), using the trick common in quantum electrodynamics. This technique has the advantage of simplifying the calculation considerably and allowing greater insight into the approximate evaluation of transition matrices which must be replaced by average values. If the integration variable is an intermediate state momentum, one may be able to carry out angular integrations which are of greatest importance in determining the sign, or phase, of double-scattering contributions.

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Baryon Mass Spectra in a Dynamical S-Wave $SU(6)$ Model*

JUERGEN G. KOERNER†
Northwestern University, Evanston, Illinois
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Some results on the mass spectra of the $\mathbf{56}^{(+)}$ and the proposed $\mathbf{70}^{(-)}$ and $\mathbf{20}^{(+)}$ $SU(6)$ baryon multiplets are presented as calculated from a simple S-wave $SU(6)$-invariant dynamical model involving first-order mass deviations. We take the meson masses from experiment and obtain reasonable agreement with the experimentally known baryon masses. The masses of the missing members of the multiplets are estimated. The mass spectrum of the **70** is discussed in detail. A two-parameter calculation using the experimental **56** and **35** mass spectra yields a mass spectrum in qualitative agreement with a recent nondynamical calculation of Gyuk and Tuan, in which 6 parameters have been used. The three mixing angles occurring in the **70** are estimated from the model and found to be small. On this basis some new approximate mass rules are given. We derive explicit representations for the reduced matrix elements of the strong interaction mass operator for the $SU(6)$ **35**, **56**, **70**, and **20** representations.

I. INTRODUCTION

FOR several reasons it is advantageous to combine particle symmetries with dynamical calculations. For example, the breaking of the symmetry can be studied systematically with the help of simple models. The symmetry group $SU(6)$ makes such a program difficult because of the well-known troubles resulting from the inclusion of dynamical variables into the symmetry. These intrinsic difficulties are avoided if $SU(6)$ invariance is conjectured only for S-wave interactions in the physical region. In this paper such an assumption is used to derive the mass spectra of some $SU(6)$ baryon representations which may correspond to physical multiplets. The dynamical model assumes the dominance of direct channels for the pole positions of the baryon states under consideration. More specifically, the probability matrix approach is used for quantitative computations.[1] In the probability matrix model the sums over the weighted masses of the two (direct channel) constituent multiplets determine the masses corresponding to the resultant particle poles. The weights are determined from Clebsch-Gordan coefficients and the assumed relative importance of the two constituent multiplets. A large part of the results is derived from the approximation of direct channel dominance alone. Such an approximation has the specific feature that a common mass operator structure can be used for the different representations considered, a feature well known from $SU(3)$ theory, but not true in general for $SU(6)$-type theories.

Specifically we consider the two S-wave processes $\mathbf{56}\otimes\mathbf{35}$ and $\mathbf{70}\otimes\mathbf{35}$, and the static model process $\mathbf{56}\otimes\mathbf{35}sw$. The notation $\mathbf{35}sw$ refers to a reidentification of the mesons in the **35** representation. The $L=1$ angular momentum in this interaction can be coupled to the spin quantum numbers of the meson **35** multiplet in such a way that it is possible to construct an $SU(6)$ invariant interaction for the static approximation. This is the model of Capps, Belinfante, and Cutkosky and

* Supported by the National Science Foundation.
† Present address: Columbia University, New York, New York.
[1] R. H. Capps, Phys. Rev. 134, B460 (1964); 134, B1396 (1964).

TABLE I. Reduced matrix elements of the mass and mass-squared operators for the baryon **8** and **10** representations in $SU(3)$.

	(Mass)² [(BeV)²]	Mass [BeV]
B_1^8	1.344	1.151
B_{8s}^8	0.065	0.033
B_{8a}^8	0.302	0.134
B_{27}^8	0.008	−0.003
B_1^{10}	1.933	1.382
B_8^{10}	0.418	0.147
B_{27}^{10}	0.020	0.000
B_{64}^{10}	0.001	0.000

will be defined more precisely in Sec. III.[2,3] Capps has shown that attractive forces can be expected in the representations **70** for **56**⊗**35**, **20** for **70**⊗**35** and **56** for the **56**⊗**35***sw* interactions by considering the effects of crossed channels.[4] There is no definite evidence for the existence of a $\mathbf{20}^{(+)}$ multiplet. However, there exists some evidence for the $\mathbf{70}^{(-)}$ multiplet; such a multiplet has been postulated by many authors.[5–7] Although the three spin $\frac{3}{2}$ particles conjectured to be members of the **70** decay into **56** and **35** mostly by D wave, this does not exclude the possibility that S-wave might be important in the production of the **70**. D-wave decay might be favored by the larger phase space available to these decay modes.[8] Nevertheless, there might be some doubt in attempting to calculate the **70** mass spectrum from a pure S-wave model. No certain prescription for the inclusion of various channels in dispersion calculations exists yet. However, we note that in the case of S-wave $SU(6)$ symmetry a certain class of channels has to be included in total, whereas for D waves, no such rules exist. The effect of added D-wave channels is quite difficult to estimate, so we leave them out. This is not in contradiction with any current dispersion approach; we effectively have to rely on an *a posteriori* justification by comparing the calculation with experiment. From this discussion it is apparent that the experimental existence of a complete **70** plays a parallel but not as important a role for $SU(6)$ as the existence of a completed **10** had played for $SU(3)$. A partially filled **70** would indicate only that the S-wave $SU(6)$-invariant approach is not complete and, in fact, is broken by some other mechanism.

In Sec. II, we list the expansion of $SU(6)$ and $SU(3)$ reduced matrix elements of the mass operators into particle masses of the multiplets under consideration. The method of using reduced matrix elements to classify mass spectra has come to be favored recently, since it offers the advantage that with a common normalization of the reduced matrix elements, comparison of different reduced matrix elements is possible in a well-defined way.[9,10] Although we will need mass-squared values for both mesons and baryons in the dynamical calculations, we also list the baryon reduced matrix elements for linear masses. In Sec. III, the probability matrix method is explained and applied to the $SU(6)$ baryon bootstrap model. The same method is used in Secs. IV and V to derive the mass spectra of the **70** and **20** representations. The results of the calculation involving the **70** multiplet are compared with those obtainable from three-quark models.

II. THE REDUCED MATRIX ELEMENTS

We use α to denote an irreducible representation of $SU(3)$ or $SU(6)$. We wish to classify the strong mass breaking operators, which are those members of the irreducible tensors appearing in $\alpha\otimes\bar{\alpha}$ with eigenvalues of $I=Y=J=0$. The Wigner-Eckart theorem then gives the masses (masses squared) of members of α in terms of reduced matrix elements of these particular tensor operators. Inversely, the reduced matrix elements can be expressed as a linear sum over the particle masses (masses squared). Likewise, this sum can be viewed as an orthogonal transformation of a direct product basis to a basis transforming irreducibly under $\alpha\otimes\bar{\alpha}$. Then the members of the direct product basis have to be normalized to $(r)^{-1}$, where r is the dimension of the degenerate subspace (in the strong interaction limit) which they span.

For $SU(3)$, one can write

$$m_i=N\sum_\beta \begin{pmatrix} \alpha & \beta & \Big| & \alpha \\ i & I=Y=0 & \Big| & i \end{pmatrix} D_\beta{}^\alpha, \qquad (1)$$

where m_i is the mass (mass squared) of a member of α, $D_\beta{}^\alpha$ is the reduced matrix element of the irreducible tensor operator β $(I=Y=0)$ occurring in $\alpha\otimes\bar{\alpha}$, and N is a common normalization factor. Using the inversion properties of the Clebsch-Gordan coefficients we can write[9,10]

$$D_\beta{}^\alpha=\sum_i \xi(-1)^{I_i+Y_i/2}\frac{(2I_i+1)^{1/2}}{n_{3\alpha}}\begin{pmatrix} \alpha & \bar{\alpha} & \Big| & \beta \\ i & \bar{i} & \Big| & I=Y=0 \end{pmatrix} m_i, \qquad (2)$$

where the normalization now has been chosen such that the average multiplet mass corresponds to the reduced matrix element transforming like a singlet. This complies with the usual convention. The $SU(3)$ dimension of the multiplet α is written as $n_{3\alpha}$. We prefer to choose a phase ξ that will not coincide with convention, but will be adjusted such that positive reduced matrix elements result for masses squared. Equation (2) can

[2] R. H. Capps, Phys. Rev. Letters **14**, 31 (1964).
[3] J. G. Belinfante and R. E. Cutkosky, Phys. Rev. Letters **14**, 33 (1965).
[4] R. H. Capps, Phys. Rev. Letters **14**, 842 (1965).
[5] A. Pais, Phys. Rev. Letters **13**, 175 (1964).
[6] M. A. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964).
[7] T. K. Kuo and T. Yao, Phys. Rev. Letters **13**, 415 (1964).
[8] The experimental data are taken from A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965). We have used average isomultiplet masses.
[9] For $SU(3)$ see, for example, J. J. deSwart, Rev. Mod. Phys. **35**, 916 (1963).
[10] For $SU(6)$ see H. Harari and M. A. Rashid, Phys. Rev. **143**, 1354 (1966).

TABLE II. Reduced matrix elements of the mass-squared operator for the mesons in $SU(3)$.

	(Mass)^2 [BeV]^2
$P_1{}^8$	0.168
$P_8{}^8$	0.166
$P_8{}^{27}$	0.006
$V_1{}^8$	0.727
$V_8{}^8$	0.117
$V_8{}^{27}$	0.009
$V_1{}^1$	0.755
$V_8{}^{1\leftrightarrow 8}$	0.284

easily be generalized to cases involving off-diagonal mass tensor elements.

Equations (3a) and (3b) give the expressions for the reduced matrix elements of the baryon multiplets **8** and **10**. The particle labels stand for the quantum numbers. The general representation symbol D will always be replaced by a symbol appropriate to the case discussed. Here we use B for the low-lying baryons. The values for the known baryon **8** and **10** multiplets are given in Table I. We have

$$\begin{bmatrix} 8B_1{}^8 \\ 2(10)^{1/2}B_{8_s}{}^8 \\ 2(2)^{1/2}B_{8_a}{}^8 \\ 8(5/3)^{1/2}B_{27}{}^8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 2 \\ -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & 1 \\ 2 & -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} N \\ \Sigma \\ \Lambda \\ \Xi \end{bmatrix}, \quad (3a)$$

$$\begin{bmatrix} 10B_1{}^{10} \\ 5B_8{}^{10} \\ 10(7/3)^{1/2}B_{27}{}^{10} \\ 5(7/2)^{1/2}B_{64}{}^{10} \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & -5 & -2 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} N^* \\ Y^* \\ \Xi^* \\ \Omega \end{bmatrix}, \quad (3b)$$

$$\begin{pmatrix} 2(10)^{1/2}B_8{}^{8\leftrightarrow 10} \\ 2(5/3)^{1/2}B_{27}{}^{8\leftrightarrow 10} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Sigma \leftrightarrow Y^* \\ \Xi \leftrightarrow \Xi^* \end{pmatrix}. \quad (3c)$$

The two terms $8s$ and $8a$ are the symmetric and antisymmetric octet representations occurring in the reduction of $8\otimes 8$. In Eq. (3c) it is not implied that there exists a mixing between the known physical octet and decuplet states. The expression for $\mathbf{8}\leftrightarrow\mathbf{10}$ mixing will be used later on in the discussion of the **70** representation, where such mixing occurs. Off-diagonal elements are always combined to $(a_{ij}+a_{ji})(2)^{-1/2}$; such terms are denoted by $i\leftrightarrow j$. Equations (4a) and (4b) contain the expressions for the pseudoscalar meson **8** and the vector meson **8** ($P8$ comprising π, K, and η, and $V8+V1$ comprising ρ, K^*, φ, and ω). For the vector meson singlet-octet mixing we use the results of a mixing calculated from an $SU(6)$-type quark model.[11] This point will be discussed later in this section. The numerical results are given in Table II. We have

$$\begin{bmatrix} 8P_1{}^8 \\ 2(10)^{1/2}P_{8_s}{}^8 \\ 2(2)^{1/2}P_{8_a}{}^8 \\ 8(5/3)^{1/2}P_{27}{}^8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 1 & -3 & 1 \\ -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} K \\ \bar{K} \\ \pi \\ \eta \end{bmatrix}, \quad (4a)$$

$$\begin{bmatrix} 8V_1{}^8 \\ 2(10)^{1/2}V_{8_s}{}^8 \\ 2(2)^{1/2}V_{8_a}{}^8 \\ 8(5/3)^{1/2}V_{27}{}^8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 1 & -3 & 1 \\ -1 & 1 & 0 & 0 \\ -2 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} K^* \\ \bar{K}^* \\ \rho \\ \varphi \end{bmatrix}, \quad (4b)$$

$$V_8{}^{1\leftrightarrow 8} = -(\omega\varphi+\varphi\omega)(2)^{-1/2}. \quad (4c)$$

The particle states have the norm $(2I+1)^{-1}$, since we are considering isotopic subspaces of $SU(3)$. The reduced matrix elements are normalized to $1/n_3$, where n_3 is the dimension of the $SU(3)$ representation considered. In the case of $\mathbf{8}\leftrightarrow\mathbf{10}$ mixing, $n_3=8$; for $\mathbf{1}\leftrightarrow\mathbf{8}$ mixing, $n_3=1$.

For $SU(6)$, the reduced matrix elements are defined in a similar way. For the **56** and **35** one could use the Clebsch-Gordan coefficients of the direct products $\mathbf{56}\otimes\overline{\mathbf{56}}$ and $\mathbf{35}\otimes\mathbf{35}$ compiled by Cook and Murtaza.[12] The tables are not sufficient, since some additional information about phases has to be obtained. Hence we have used a method which employs the fact that the probability matrix (defined in Sec. III) acts as a (rectangular) singlet operator in the sense that it produces from an irreducible tensor in the space $\beta\otimes\bar{\beta}$ or $\gamma\otimes\bar{\gamma}$ an irreducible tensor of the same transformation property in the space $\alpha\otimes\bar{\alpha}$, where α, β, and γ are irreducible representations and α is contained in the reduction of $\beta\otimes\gamma$.[13] The $SU(6)$ Clebsch-Gordan coefficients of $(\mathbf{56}\,\mathbf{35}|\mathbf{56})$, $(\mathbf{56}\,\mathbf{35}|\mathbf{70})$, and $(\mathbf{70},\mathbf{35}|\mathbf{20})$ are thus needed to construct the irreducible mass tensors of the representations **35**, **56**, **70**, and **20**.[12,14] These representations are identified by M, B, R, and Z, respectively, and the reduced matrix elements belonging to these representations are normalized to $(35)^{-1}$, $(56)^{-1}$, $(70)^{-1}$, and $(20)^{-1}$, respectively. We note that the particle states obtain an additional normalization factor of $(2J+1)^{-1}$ due to the enlargement of the degenerate subspaces, so that a particle state now is normalized to $[(2I+1)\times(2J+1)]^{-1}$. Again the over-all phases for the **56** and **35** have been chosen such that positive values result for the mass squared reduced matrix elements, except for two cases in which further applications make such a choice inconvenient. The phases of the **70** and **20** were chosen arbitrarily. The notation of our $SU(6)$ reduced matrix elements is $D_{i;j}$, where D is the $SU(6)$ representation

[11] See, for example, G. Zweig in *Symmetries in Elementary Particle Physics*, edited by A. Zichichi (Academic Press Inc., New York, 1965).

[12] C. L. Cook and G. Murtaza, Nuovo Cimento 34, 531 (1965).

[13] This method was proposed by R. H. Capps, Phys. Rev. 134, B649 (1964).

[14] J. C. Carter, J. J. Coyne, and S. Meshkov (to be published). I am grateful to the authors for sending me their (**35 70**|**20**) Clebsch-Gordan table prior to publication.

under consideration (i.e., B, R, Z, or M), and i denotes the $SU(6)$ transformation property and j the $SU(3)$ transformation property of the decomposed space $D\otimes\bar{D}$. Where no confusion can arise we omit the symbol D and define the same reduced matrix element by ^{ij}.

The reduced matrix elements of the **56** and **35** have been discussed extensively by Harari and Rashid.[10] They express their mass tensors along components of $SU(2)\otimes SU(2)\otimes U(1)$ eigenstates, whereas we prefer to expand our mass tensors along $SU(2)\otimes SU(3)$ eigenstates. These two procedures do not differ, since our mass tensors are block-diagonalized expressions of mass tensors expanded along the particle states. Thus, we have for the baryon **56** representation

$$\begin{pmatrix} 56B_{1;1} \\ 7(10)^{-1/2}B_{405;1} \end{pmatrix} = \begin{pmatrix} 40 & 16 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} B_1^{10} \\ B_1^{8} \end{pmatrix}, \tag{5a}$$

$$\begin{bmatrix} (42)^{1/2}(5)^{-1}B_{35;8} \\ (14)^{1/2}B_{405;8} \\ (21)^{1/2}B_{2695;8} \end{bmatrix} = \begin{bmatrix} 1 & 0 & (2)^{1/2}(5)^{-1} \\ -1 & 2(2/5)^{1/2} & (2)^{1/2} \\ 1 & 3(2/5)^{1/2} & -(2)^{1/2} \end{bmatrix} \begin{bmatrix} B_8^{10} \\ B_{8s}^{8} \\ B_{8a}^{8} \end{bmatrix}, \tag{5b}$$

$$\begin{pmatrix} (3/2)^{1/2}B_{405;27} \\ (21)^{1/2}B_{2695;27} \end{pmatrix} = \begin{pmatrix} 1 & (1/35)^{1/2} \\ 1 & -14(1/35)^{1/2} \end{pmatrix} \begin{pmatrix} B_{27}^{10} \\ B_{27}^{8} \end{pmatrix}, \tag{5c}$$

$$B_{2695;64} = (5/7)^{1/2}B_{64}^{10}. \tag{5d}$$

For the meson **35** representation the results are

$$\begin{bmatrix} 35M_{1;1} \\ 5(7/2)^{1/2}M_{189;1} \\ 7(5/2)^{1/2}M_{405;1} \end{bmatrix} = \begin{bmatrix} 24 & 8 & 3 \\ 1 & -3 & 2 \\ 5 & -3 & -2 \end{bmatrix} \begin{bmatrix} V_1^{8} \\ P_1^{8} \\ V_1^{1} \end{bmatrix}, \tag{6a}$$

$$\begin{bmatrix} 2(35)^{1/2}M_{35;8} \\ (70)^{1/2}M_{189;8} \\ 2(35)^{1/2}M_{405;8} \end{bmatrix} = \begin{bmatrix} 3(5)^{1/2} & (5)^{1/2} & (9/2)^{1/2} \\ 1 & 3 & -(5/2)^{1/2} \\ -7 & 3 & (5/2)^{1/2} \end{bmatrix} \begin{bmatrix} V_8^{8} \\ P_8^{8} \\ V_8^{1\leftrightarrow 8} \end{bmatrix}, \tag{6b}$$

$$\begin{pmatrix} (35/2)^{1/2}M_{189;27} \\ (35/6)^{1/2}M_{405;27} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} V_{27}^{8} \\ P_{27}^{8} \end{pmatrix}. \tag{6c}$$

In Tables III and IV we exhibit the numerical values for the **56** and **35** reduced matrix elements. For further applications we list in Table IV, second column, the numerical values of the meson "*sw*" assignment, as explained in Sec. I and applied in Sec. III. Both mass and mass squared values have been calculated in the baryon **56** case. It is noteworthy that the relative magnitude of the different mass-breaking terms does not change drastically if masses squared are used instead of linear masses. A similar observation has been made for $SU(3)$, where mass squared values for the baryons still give a good fit to the Gell-Mann–Okubo rules.

In the case of the **35** we have used the same identification of ω and φ as Harari and Rashid, so that φ is a pure $SU(4)_I$ singlet.[15] This identification leads to a negative off-diagonal mass element $\omega\varphi$, a result similarly derivable from a quark model.[11] This is to be expected, since the basic quark splitting can only manifest itself in the $M_{35;8}$, whereas a positive $\omega\varphi$ gives all three $M_{35;8}$, $M_{189;8}$, and $M_{405;8}$ appreciable strength. The argument in Ref. 10 to determine the sign of $\omega\varphi$ by the absence of $\varphi\rightarrow\rho\pi$ decay is not complete, since the particles are coupled by p-wave and simple $SU(6)$ arguments cannot be applied. On the other hand, the multiplicative mass rule derived by Bég and Singh from the absence of $M_{189;8}$ holds well experimentally.[6] We consider this sufficient evidence that the sign of $\omega\varphi$ has to be chosen negative.

The $SU(6)$ representation **70** reduces under $SU(3)\otimes SU(2)$ into $(10;2)+(8;4)+(8;2)+(1;2)$. The irreducible

[15] $SU(4)_I$ is referring to the $SU(4)$ group appearing in $[SU(4)\times SU(2)]_Y$. This group is a subgroup of $SU(6)$ and was first discussed by F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964). We follow the notation of Ref. 26.

parts of the mass tensor belonging to the **70** representation will be denoted as follows:

$$\mathbf{1}^1,\ \mathbf{189}^1,\ \mathbf{405}^1,\ \mathbf{3675}^1,\ \mathbf{35H}^8,\ \mathbf{35NH}^8,\ \mathbf{189}^8,\ (\mathbf{280}+\overline{\mathbf{280}})^8,\ (\mathbf{280}-\overline{\mathbf{280}})^8,$$
$$\mathbf{405}^8,\ \mathbf{3675}^{8b},\ \mathbf{3675}^{8c},\ \mathbf{3675}^{8d},\ \mathbf{189}^{27},\ \mathbf{405}^{27},\ \mathbf{3675}^{27B},\ \mathbf{3675}^{27C},\ \mathbf{3675}^{27D},\ \text{and}\ \mathbf{3675}^{64}.$$

These 19 irreducible components can be written as an orthogonal transformation on the 13 particle and 2×3 mixing states contained in the multiplet. The mass matrix is symmetric and therefore pairs of off-diagonal elements are equal. We incorporate this information by splitting off three components along differences of equal off-diagonal elements. These three must be chosen from $(\mathbf{280}-\overline{\mathbf{280}})^8$, $\mathbf{3675}^8$ and $\mathbf{3675}^{27}$. A 16×16 matrix results which can be block-diagonalized to 4×4 for the singlet terms, 7×7 for the octet terms, 4×4 for the **27** and 1×1 for the **64** terms. Several ambiguities occur. The tensor **35** appears twice in $\mathbf{70}\otimes\overline{\mathbf{70}}$. We have fixed one direction to represent hypercharge splitting (i.e., $R_{35H;8}$) and chosen the other one (i.e., $R_{35NH;8}$) orthogonal to it. This is in analogy to the D and F separation of the two **8** occurring in $\mathbf{8}\otimes\mathbf{8}$. The $SU(6)$ representation **3675** contains three $SU(3)$ octet and three $SU(3)$ **27** terms. Since no meaningful diagonalizing operator could be found, we chose *different* convenient orthogonal sets in each case, denoted by b, c, and d and by B, C, and D, respectively. We denote the spin-$\frac{3}{2}$ octet in the **70** by $\mathbf{8}_\gamma$; the other $SU(3)$ multiplets are designated by their $SU(3)$ representation symbol. The complete set of reduced matrix elements is

$$\begin{bmatrix} 70R_{1;1} \\ 10(14)^{1/2}R_{189;1} \\ 14(5)^{1/2}R_{405;1} \\ 2(14)^{1/2}R_{3675;1} \end{bmatrix} = \begin{bmatrix} 20 & 32 & 16 & 2 \\ 15 & -16 & 2 & -1 \\ 5 & 8 & -10 & -3 \\ 1 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} R_1{}^{10} \\ R_1{}^{8\gamma} \\ R_1{}^{8} \\ R_1{}^{1} \end{bmatrix}, \tag{7a}$$

$$\begin{bmatrix} (385)^{1/2}R_{35H;8} \\ 8(770)^{1/2}R_{35NH;8} \\ 40(14)^{1/2}R_{189;8} \\ 2(210)^{1/2}R_{(280+\overline{280});8} \\ 40(14)^{1/2}R_{405;8} \\ 20(14)^{1/2}R_{3675;8b} \\ 40(42)^{1/2}R_{3675;8c} \end{bmatrix} = \mathbf{M} \begin{bmatrix} R_8{}^{10} \\ R_{8s}{}^{8\gamma} \\ R_{8a}{}^{8\gamma} \\ R_{8s}{}^{8} \\ R_{8a}{}^{8} \\ R_8{}^{8\leftrightarrow10} \\ R_8{}^{1\leftrightarrow8} \end{bmatrix}, \tag{7b}$$

where

$$\mathbf{M} = \begin{bmatrix} 5(2)^{1/2} & 0 & 8 & 0 & 4 & 0 & 0 \\ 10(2)^{1/2} & 44(5)^{1/2} & -28 & 0 & 8 & -22(10)^{1/2} & 11 \\ -30(2)^{1/2} & 20(5)^{1/2} & 60 & 8(5)^{1/2} & 0 & 6(10)^{1/2} & 5 \\ 0 & 4(5)^{1/2} & -4 & -4(5)^{1/2} & 4 & 2(10)^{1/2} & -1 \\ -10(2)^{1/2} & 4(5)^{1/2} & -20 & 16(5)^{1/2} & 40 & -2(10)^{1/2} & -15 \\ 10(2)^{1/2} & 4(5)^{1/2} & -20 & 8(5)^{1/2} & 0 & 6(10)^{1/2} & 5 \\ 30(2)^{1/2} & 28(5)^{1/2} & 20 & 8(5)^{1/2} & -80 & 2(10)^{1/2} & -25 \end{bmatrix},$$

$$\begin{bmatrix} 2(70)^{1/2}R_{189;27} \\ 2(105)^{1/2}R_{405;27} \\ 14(5)^{1/2}R_{3675;27B} \\ 2(210)^{1/2}R_{3675;27C} \end{bmatrix} = \begin{bmatrix} (35)^{1/2} & 0 & 2 & 4(2)^{1/2} \\ (35)^{1/2} & 8 & 2 & -4(2)^{1/2} \\ (35)^{1/2} & 0 & -14 & 0 \\ (35)^{1/2} & -16 & +2 & -4(2)^{1/2} \end{bmatrix} \begin{bmatrix} R_{27}{}^{10} \\ R_{27}{}^{8\gamma} \\ R_{27}{}^{8} \\ R_{27}{}^{8\leftrightarrow10} \end{bmatrix}, \tag{7c}$$

$$R_{3675;64} = (2/7)^{1/2}R_{64}{}^{10}. \tag{7d}$$

The situation for the **20** is simpler. The reduction under $SU(3)\otimes SU(2)$ is $\mathbf{20}=(8;2)+(1;4)$. The mass tensors must be contained in $\mathbf{20}\otimes\overline{\mathbf{20}}=\mathbf{1}\oplus\mathbf{35}\oplus\mathbf{175}\oplus\mathbf{189}$ and the components are $\mathbf{1}^1$, $\mathbf{189}^1$, $\mathbf{35}^8$, $\mathbf{189}^8$, and $\mathbf{189}^{27}$. Even without the help of Clebsch-Gordan tables, one can immediately write down the mass tensors by using

TABLE III. Reduced matrix elements of the mass and mass-squared operators for the baryon **56** representation in $SU(6)$.

	(Mass)^2 [BeV]^2	Mass [BeV]
$B_{1;1}$	1.765	1.316
$B_{405;1}$	0.266	0.104
$B_{35;8}$	0.388	0.142
$B_{405;8}$	0.024	0.022
$B_{2695;8}$	0.025	0.004
$B_{405;27}$	0.017	0.000
$B_{2695;27}$	0.000	0.001
$B_{2695;64}$	0.001	0.000

the fact that $Z_{1;1}$ must be the average multiplet mass and $Z_{35;8}$ must only contain the hypercharge splitting $Z_{8a}{}^8$, since **35** is contained only once in $\mathbf{20}\otimes\overline{\mathbf{20}}$. The remaining components can be constructed by orthogonalization:

$$\begin{pmatrix} 20Z_{1;1} \\ 5/2Z_{189;1} \end{pmatrix} = \begin{pmatrix} 16 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Z_1{}^8 \\ Z_1{}^1 \end{pmatrix}, \tag{8a}$$

$$\begin{pmatrix} (5)^{1/2}(2)^{-1}Z_{35;8} \\ (5)^{1/2}(2)^{-1}Z_{189;8} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z_{8s}{}^8 \\ Z_{8a}{}^8 \end{pmatrix}, \tag{8b}$$

$$Z_{189;27} = 2(5)^{-1/2} Z_{27}{}^8. \tag{8c}$$

III. THE 56 REPRESENTATION

The possibility of an $SU(6)$ invariant static bootstrap model of the **56** representation has been demonstrated by Capps and by Belinfante and Cutkosky.[2,3] The $L=1$ angular momentum in the interaction $\mathbf{56}^{(+)}\otimes\mathbf{35}^{(-)}\to\mathbf{56}^{(+)}$ is absorbed in the meson **35** and effectively behaves like an internal quantity of the mesons. Through this formal identification $SU(6)$ invariance can be applied. We follow the approach of Ref. 16 and assume that in this interaction K, π, and η effectively behave like the (8;3) multiplet, K^*, ρ, and φ_6 [$\varphi_6=(2/3)^{1/2}\varphi+(1/3)^{1/2}\omega$] like the (8;1) multiplet and X_0 (959 MeV) like the (1;3). This model holds strictly for the static limit only. However, it has the advantage that the calculation includes many simultaneously coupled channels. We emphasize this point because mass estimates based on dispersion relations have often been unreliable due to the fact that only few channels could be included because of calculational difficulties.

A simple feature of this bootstrap scheme is the prediction that the **2695** mass-breaking term is absent in the mass spectrum of the **56** representation. We note that, fortunately for our model, the one $SU(3)$ singlet term that lifts the degeneracy of the mean octet and decuplet masses does not appear in the **2695**, but in the **405**. The absence of the $\mathbf{2695}^8$ term leads to the equality of Gell-Mann–Okubo type splitting of the two $SU(3)$ octet and decuplet multiplets. This can be seen by writing out the Gell-Mann–Okubo mass operator in $SU(3)$:

$$m=a_\alpha+b_\alpha Y+c_\alpha[I(I+1)-\tfrac{1}{4}Y^2], \tag{9}$$

where a, b, and c only depend on the $SU(3)$ representation α chosen. Inserting $B_{2695;8}=0$ we obtain the equation $b_{10}+\frac{3}{2}c_{10}=b_8+\frac{3}{2}c_8$. Furthermore we notice that only the quantity $b_{10}+\frac{3}{2}c_{10}$ is physically defined, since the mass spectrum for the **10** is overparametrized. This allows us to choose $b_{10}=b_8$, and $c_{10}=c_8$ follows. This result is not new, since the general mass operator of Bég and Singh simplifies in the **56** representation to[6]

$$M^{56}=a+bJ(J+1)+cY+d[I(I+1)-\tfrac{1}{4}Y^2], \tag{10}$$

which has the same consequence. The two methods used have to be consistent of course, since the same number of terms in the mass-breaking operator are excluded. The method of reduced matrix elements has the advantage that small deviations from zero in supposedly absent terms can be traced more easily. The absence of the $\mathbf{2695}^{27}$ mass-splitting term gives a relation between the $B_{27}{}^8$ and $B_{27}{}^{10}$. With a more restrictive assumption in the bootstrap model, the $B_{405;27}$ term can be related to the small $M_{405;27}$ term; this results in smallness of both $B_{27}{}^{10}$ and $B_{27}{}^8$. Finally the absence of the $\mathbf{2695}^{64}$ term directly leads to the absence of the **64** component in the decuplet. As can be seen from Table III these results hold well for both mass squared and linear masses, except for the case of the $\mathbf{2695}^8$ mass squared. Qualitatively speaking, a model which predicts the absence of terms other than **1** and **8** in the $SU(3)$ mass operator can be termed successful.

In order to arrive at quantitative results for the remaining mass-breaking terms, we use the probability matrix approach of Capps, explained in Ref. 16. For the sake of completeness we reproduce the basic formula in our notation:

$$R_{ij}/(\bar{R})=\alpha\sum_{k,l,m,n}C_{ikm}C_{jln}\delta_{mn}B_{kl}/(\bar{B}) + (1-\alpha)\sum_{k,l,m,n}C_{ikm}C_{jln}\delta_{kl}M_{mn}/(\bar{M}), \tag{11}$$

where the C_{ikm} are the Clebsch-Gordan coefficients connecting a representation R with representations B and M. The expressions R_{ij}, B_{kl}, and M_{mn} are the matrix elements of the mass squared operator in the spaces of the R, B, and M representations. The $(\bar{R})$, $(\bar{B})$, and $(\bar{M})$ denote average mass squared $SU(6)$

TABLE IV. Reduced matrix elements of the mass-squared operator for the meson **35** representation in $SU(6)$.

	(Mass)^2 [BeV]^2	(Mass)^2 [BeV]^2 "sw"-Assignment
$M_{1;1}$	0.602	0.360
$M_{189;1}$	0.185	⋯
$M_{405;1}$	0.146	−0.288
$M_{35;8}$	0.139	0.088
$M_{189;8}$	0.002	⋯
$M_{405;8}$	−0.001	−0.039
$M_{189;27}$	0.008	⋯
$M_{405;27}$	−0.001	0.001

[16] R. H. Capps, Phys. Rev. Letters 14, 456 (1964). Section III of our paper is essentially an elaboration of Ref. 16.

multiplet masses. The parameter α may have values between 0 and 1 and fixes the relative contributions of the masses in B and M to the masses in R. For our bootstrap model, $R=B$. It is advantageous to work with mass tensors having definite $SU(6)$ transformation properties in the spaces $B\otimes\bar{B}$ and $M\otimes\bar{M}$. The equation then becomes

$$B_\nu=\frac{(1-\alpha)C_\nu(M)}{[1-C_\nu(B)\alpha]}\frac{(\bar{B})}{(\bar{M})}M_\nu. \tag{12}$$

Since Eq. (12) is independent of the $SU(3)$ subclassification, these indices are omitted. $C_\nu(B)$ and $C_\nu(M)$ are the remnants of the multiplied Clebsch-Gordan coefficients after the diagonalization has been carried out. They too are independent of the $SU(3)$ indices and are given by

$$\begin{aligned} C_1(B)&=1, & C_1(M)&=1,\\ C_{35}(B)&=11/15, & C_{35}(M)&=2(1/15)^{1/2},\\ C_{405}(B)&=17/45, & C_{405}(M)&=-2/9,\\ C_{2695}(B)&=-1/15, & C_{2695}(M)&=0. \end{aligned} \tag{13}$$

Now the convenience of operations in the linear space of the reduced matrix elements becomes apparent. After the reduced matrix elements are calculated from Eq. (12), the individual baryon masses can be computed by simple inversion. For $\nu=2695$ we obtain the aforementioned result of $B_{2695}=0$, since $C_{2695}(M)=0$ and the denominator is larger than zero. The term $\nu=1$ is redundant, since it is used to normalize the relative meson and baryon contributions. The remaining two terms $\nu=35$ and 405 yield

$$B_{35}=\frac{15(1-\alpha)}{(15-11\alpha)}2.53M_{35}, \tag{14}$$

$$B_{405}=\frac{45(1-\alpha)}{(45-17\alpha)}(-1.09)M_{405}. \tag{15}$$

The ratio of the two factors $15(1-\alpha)/(15-11\alpha)$ and $45(1-\alpha)/(45-17\alpha)$ remains fairly constant over the range of α, so that the ratio of the various baryon terms does not depend on the choice of α in a crucial manner. We find that for two of the three $\nu=405$ terms the experimental baryon mass spectrum is quite well reproduced when α is adjusted to approximately 0.4, whereas the **35** term comes out consistently too small. For $\alpha=0.4$ we obtain

$$\begin{aligned} B_{35;8}&=0.189\ [\text{BeV}]^2,\\ B_{405;1}&=0.222\ [\text{BeV}]^2,\\ B_{405;8}&=0.030\ [\text{BeV}]^2. \end{aligned} \tag{16}$$

The smallness of $B_{35;8}$ manifests itself in an insufficient predicted spread in hypercharge. For instance we can calculate the $(\Sigma-\Lambda)$ and $(\Xi-N)$ mass differences by inversion:

$$\Sigma-\Lambda=\tfrac{4}{5}(14)^{1/2}B_{405;8},$$
$$\Xi-N=2(7/6)^{1/2}B_{35;8}+(14)^{1/2}B_{405;8},$$

leaving out $\mathbf{405}^{27}$ and the **2695** terms. We see that both $(\Sigma-\Lambda)$ and $(\Xi-N)$ are predicted with the right sign, but are too small, whereas their ratio is close to the experimental value over most of the range of α.[17] Although this may be accidental, the model still is able to generate a substantial $(\Sigma-\Lambda)$ mass difference of the right sign. The term $B_{405;27}$ is calculated to be small. Together with the condition $B_{2695;27}=0$, this leads to a small **27** type mass splitting in the $SU(3)$ baryon multiplets.

We conclude that this bootstrap model is able to account for all the qualitative features of the baryon **56** mass spectrum. The lack of quantitative accuracy is not disturbing, since crude approximations have been made throughout the calculations. From Eqs. (12) and (13) it can be seen that an $SU(6)$ bootstrap model which accounts for the $L=1$ angular momentum by some other mechanism and does not use the prescribed relabeling of quantum-number assignments for the mesons will immediately run into trouble, since the $B_{405;1}$ reduced matrix element will be predicted to be negative. This of course is not feasible since experimentally the mean **10** mass is higher than the mean **8** mass. In this context, it seems even more striking that the relabeling argument does lead to a qualitatively correct bootstrapped mass spectrum of the **56**.

IV. THE 70 REPRESENTATION

The $SU(6)$ **70** representation reduces into the four multiplets (10;2), (8;4), (8;2), and (1;2) under $SU(3)\otimes SU(2)$. We use the notation of Bég and Singh for the particle states, i.e., $\tilde{N}^*$, $\tilde{Y}^*$, $\tilde{\Xi}^*$, and $\tilde{\Omega}$ are used for (10;2), N_γ, Σ_γ, Λ_γ, and Ξ_γ for (8;4), $\tilde{N}$, $\tilde{\Sigma}$, $\tilde{\Lambda}$, and $\tilde{\Xi}$ for (8;2) and Λ' for (1;2).[18] Where mixing occurs the suffix R denotes the diagonalized particle states.

The assumptions leading to the consideration of an invariant S-wave production of the **70** have been discussed in detail in Sec. I. In this section we consider the consequences of these assumptions. Mass-breaking terms transforming like $\mathbf{3675}^1$, $\mathbf{3675}^{8b}$, $\mathbf{3675}^{8c}$, $\mathbf{3675}^{27B}$, $\mathbf{3675}^{27C}$, $\mathbf{3675}^{64}$, and $(\mathbf{280}+\overline{\mathbf{280}})^8$ should be absent, since none of these appear in $\mathbf{56}\otimes\overline{\mathbf{56}}$ or in $\mathbf{35}\otimes\mathbf{35}$. Unfortunately we have mixing among three pairs of particles. In order to obtain relations between the diagonalized particle states we have to eliminate the mixing terms by substitution. This leads to 4 cumbersome multiplicative mass rules. Alternatively one can assume that in addition all the **27** and **64** type mass-breaking terms are absent. The results of this assumption have to be treated cautiously, especially for small mass differences.

[17] This is not in contradiction with the statement that $B_{405;8}$ is quite well reproduced, since experimentally $B_{2695;8}$ is large enough to be effective in the $\Sigma-\Lambda$ mass determination.

[18] M. A. B. Bég and V. Singh, Phys. Rev. Letters 13, 509 (1964).

For instance, in the γ octet the $(\Sigma_\gamma-\Lambda_\gamma)$ mass difference is likely to be of the same order of magnitude as the violations of the Gell-Mann–Okubo mass formula. We list only the resulting linear rules.[19] Using $(\mathbf{280}+\overline{\mathbf{280}})^8=\mathbf{3675}^{8b}=\mathbf{3675}^{8c}=0$, we obtain

$$\tilde{\Xi}_R^*-\tilde{Y}_R^*=\tilde{\Xi}_R-\tilde{\Sigma}_R\,, \tag{17}$$

and the equivalent relation

$$\tfrac{1}{3}(\tilde{\Omega}-\tilde{N}^*)=\tilde{\Xi}_R-\tilde{\Sigma}_R\,. \tag{18}$$

From $\mathbf{3675}^1=0$ we derive

$$3\tilde{\Lambda}_R+3\Lambda_R'-\tilde{Y}_R^*-\tilde{\Sigma}_R-4\tilde{\Xi}_R-4\tilde{\Xi}_R^*-4\tilde{N}+4\tilde{N}^*+4\tilde{\Omega}=0\,. \tag{19}$$

For the γ octet we have the usual Gell-Mann–Okubo formula

$$2(N_\gamma+\Xi_\gamma)=3\Lambda_\gamma+\Sigma_\gamma\,, \tag{20}$$

since no mixing is involved.

There is good evidence that the $\mathbf{189}^8$ term is absent in the meson mass spectrum, as pointed out in Sec. II. So we may drop the $\mathbf{189}^8$ contribution to the **70** mass spectrum. An additional rule results in

$$\tilde{\Xi}_R^*-\tilde{Y}_R^*=\Xi_\gamma-\Lambda_\gamma \tag{21}$$

and the $\mathbf{8}\leftrightarrow\mathbf{10}$ mixing simplifies to

$$\tilde{\Xi}^*\leftrightarrow\tilde{\Xi}=\tilde{Y}^*\leftrightarrow\tilde{\Sigma}=-(2)^{-1/2}(\Sigma_\gamma-\Lambda_\gamma)\,. \tag{22}$$

The $\Lambda'\tilde{\Lambda}$ mixing may not be easily expressed in terms of diagonalized particles. Instead of listing a long uninstructive formula we give the mixing in terms of undiagonalized particles;

$$\Lambda'\leftrightarrow\tilde{\Lambda}=(2)^{-1/2}[\Sigma_\gamma-\Lambda_\gamma-2(\tilde{\Sigma}-\tilde{\Lambda})]\,. \tag{23}$$

To be more specific we use the probability matrix model as described in Sec. III [Eq. (11)]. The appropriate Clebsch-Gordan coefficients are given in Ref. 12. The matrix equation is then diagonalized and we obtain

$$R_\nu/(\bar{R})=\alpha C_\nu(B)B_\nu/(\bar{B})+(1-\alpha)C_\nu(M)M_\nu/(\bar{M})\,, \tag{24}$$

with

$$\begin{aligned}
C_1(B)&=1\,, & C_1(M)&=1\,,\\
C_{35H}(B)&=(27/55)^{1/2}, & C_{35H}(M)&=3(44)^{-1/2},\\
C_{35NH}(B)&=-(6/55)^{1/2}, & C_{35NH}(M)&=7(352)^{-1/2},\\
C_{189}(B)&=0\,, & C_{189}(M)&=\tfrac{1}{4}\,,\\
C_{405}(B)&=(2)^{1/2}(3)^{-1}, & C_{405}(M)&=-(2)^{1/2}(24)^{-1},\\
C_{3675}(B)&=0\,, & C_{3675}(M)&=0\,,\\
C_{(280+\overline{280})}(B)&=0\,, & C_{(280+\overline{280})}(M)&=0\,.
\end{aligned} \tag{25}$$

[19] Equations (17) to (21) have been derived by Gyuk and Tuan. Their derivation is complementary to ours in that they start from the positive assertion of the mass operator. Since in both calculations the same mass-breaking terms are left out, the end result must be the same. Two of their corollary statements are incorrect. First, they argue that all off-diagonal elements are equal. Second, they claim that Eqs. (17) and (18) have to be united, if the $\mathbf{189}^8$ mass breaking is included. These mistakes have not entered their numerical calculations. I. P. Gyuk and S. F. Tuan, Phys. Rev. **140**, B164 (1965).

TABLE V. Probability matrix calculation of the $SU(6)$ **70** multiplet. Choice of parameters: $\alpha=0.631$, $(\bar{R})=2.797$ BeV^2.

	$R_{ij}(B)$	$R_{ij}(M)$	Mass [BeV]	
Λ'	0.762	0.976	1.526	(1.532)
$\tilde{N}$	0.709	0.888	1.471	
$\tilde{\Sigma}$	0.909	1.038	1.633	(1.635)
$\tilde{\Lambda}$	0.888	1.033	1.628	(1.622)
$\tilde{\Xi}$	1.087	1.191	1.770	(1.772)
N_γ	0.867	0.768	1.523	
Σ_γ	0.951	1.026	1.653	
Λ_γ	1.119	0.813	1.676	
Ξ_γ	1.307	0.959	1.815	
$\tilde{N}^*$	0.867	0.953	1.585	
$\tilde{Y}^*$	1.035	1.095	1.720	(1.718)
$\tilde{\Xi}^*$	1.213	1.248	1.853	(1.851)
$\tilde{\Omega}$	1.401	1.409	1.980	
$\Lambda'\leftrightarrow\tilde{\Lambda}(2)^{-1/2}$	−0.105	0.106		
$\tilde{\Sigma}\leftrightarrow\tilde{Y}^*(2)^{-1/2}$	0.084	−0.102		
$\tilde{\Xi}\leftrightarrow\tilde{\Xi}^*(2)^{-1/2}$	0.094	−0.111		

Equation (24) and the eigenvalues (25) are independent of the $SU(3)$ indices. The eigenvalues corresponding to **3675** and $(\mathbf{280}+\overline{\mathbf{280}})$ are zero; these terms will not contribute to the **70** mass spectrum. The $SU(3)$ **27**-type mass-breaking terms are present in general. In Table V, column 1, we have listed the various $\sum_{k,l,m,n}C_{ikm}C_{jln}\times\delta_{mn}B_{kl}/(\bar{B})$ [referred to as $R_{ij}(B)$] and in column 2 the various $\sum_{k,l,m,n}C_{ikm}C_{jln}\delta_{kl}M_{mn}/(\bar{M})$ [referred to as $R_{ij}(M)$]. It is apparent that the calculated spectrum of the **70** is quite sensitive to the specification of α. Particularly the mixing parameters depend strongly on the choice of α. The only significant result obtainable for general α $(0\leq\alpha\leq1)$ relates the mean $SU(3)$ multiplet masses. We find $R_1^{10}>R_1^8>R_1^1$ and $R_1^{10}>R_1^{8\gamma}$. It is evident that α must be fixed before any hierarchy can be found in the spectrum. We take α to be

$$\alpha=\frac{(\bar{B})^{1/2}}{(\bar{B})^{1/2}+(\bar{M})^{1/2}}\,, \tag{26}$$

as suggested in Ref. 20. Since it is desirable to fit the experimentally known members of the γ octet, i.e., $N_\gamma(1512)$, $\Sigma_\gamma(1660)$, and $\Xi_\gamma(1817)$, into the **70** multiplet, we adjust the average calculated (8; 4) mass to the average experimental γ-octet mass.

In Table V, column 3, these values have been used for the two parameters α and $(\bar{R})$ to calculate the masses of the individual members of the **70.** The numbers appearing are the square roots of the numbers calculated in the model in order that a comparison with the experimental data is faciliated. The brackets after the mass values of particles involved in the mixing contain the mass values for the corresponding undiagonalized mass levels. From Table V it is apparent that the experimentally known members of the γ octet fit into our calculation quite well. The $Y_0^*(1405)$ is

[20] R. H. Capps, Phys. Rev. **137**, B125 (1965).

generally taken to be the (1; 2) member of the **70**. Our calculation puts the mass of the (1; 2) member at 1526 MeV. Evidently the probability matrix model fails to produce sufficient spread in the different mean masses of the $SU(3)$ multiplets contained in the **70**. This is not too disturbing, since relativistic spin-dependent effects might play a role in determining the correct $SU(3)$ mean mass levels. This deficiency can be separated in the probability matrix model. Apart from the complete diagonalization of the probability matrix achieved in Eq. (24) by considering irreducible tensors in $SU(6)$, we can block-diagonalize the probability matrix with regard to irreducible tensors of the subgroup $SU(3) \otimes SU(2)$. In this fashion the probability matrix can be separated into 4 parts relating $SU(3)$ tensors transforming like **1**, **8**, **27**, and **64**. For the singlet part we obtain

$$\begin{bmatrix} R_1^{10} \\ R_1^{8\gamma} \\ R_1^{8} \\ R_1^{1} \end{bmatrix} = \frac{\alpha(\bar{R})}{48(\bar{B})} \begin{bmatrix} 40 & 8 \\ 40 & 8 \\ 20 & 28 \\ 0 & 48 \end{bmatrix} \begin{pmatrix} B_1^{10} \\ B_1^{8} \end{pmatrix} + \frac{(1-\alpha)(\bar{R})}{48(\bar{M})} \begin{bmatrix} 34 & 6 & 8 \\ 31 & 15 & 2 \\ 35 & 9 & 4 \\ 36 & 12 & 0 \end{bmatrix} \begin{bmatrix} V_1^{8} \\ P_1^{8} \\ V_1^{1} \end{bmatrix}. \tag{27}$$

For the octet part we have

$$\begin{bmatrix} R_8^{10} \\ R_{8_s}^{8\gamma} \\ R_{8_a}^{8\gamma} \\ R_{8_s}^{8} \\ R_{8_a}^{8} \\ R_8^{8\leftrightarrow10} \\ R_8^{1\leftrightarrow8} \end{bmatrix} = \frac{\alpha(\bar{R})}{96(\bar{B})} \begin{bmatrix} 64 & -8(10)^{1/2}(5)^{-1} & 8(2)^{1/2} \\ -8(10)^{1/2} & 2 & -2(5)^{1/2} \\ 40(2)^{1/2} & -2(5)^{1/2} & 10 \\ -4(10)^{1/2} & 16 & 8(5)^{1/2} \\ 20(2)^{1/2} & 8(5)^{1/2} & 32 \\ 16(5)^{1/2} & -2(2)^{1/2} & 2(10)^{1/2} \\ 0 & -24(5)^{1/2} & -72 \end{bmatrix} \begin{bmatrix} B_8^{10} \\ B_{8_s}^{8} \\ B_{8_a}^{8} \end{bmatrix}$$

$$+\frac{(1-\alpha)(\bar{R})}{96(\bar{M})} \begin{bmatrix} 10(10)^{1/2} & 6(10)^{1/2}(5)^{-1} & 16 \\ 22 & 12 & (10)^{1/2} \\ 8(5)^{1/2} & 6(5)^{1/2} & (2)^{1/2} \\ 5 & 3 & -(10)^{1/2} \\ 13(5)^{1/2} & 3(5)^{1/2} & 5(2)^{1/2} \\ -19(2)^{1/2} & -3(2)^{1/2} & -5(5)^{1/2} \\ 12(5)^{1/2} & 12(5)^{1/2} & 6(2)^{1/2} \end{bmatrix} \begin{bmatrix} V_8^{8} \\ P_8^{8} \\ V_8^{1\leftrightarrow8} \end{bmatrix}. \tag{28}$$

The expressions for the mass tensors transforming like **27** and **64** will not be considered here. If the 4 experimental particles N_γ, Σ_γ, Ξ_γ, and Y_0^* are regarded as evidence for the **70** multiplet, we are compelled to consider Eq. (27) covering the singlet terms as only qualitatively correct, whereas we can hope that Eq. (28) for the octet parts holds well quantitatively. Using α from Eq. (26) we calculate

$$\begin{aligned} R_8^{10}/(\bar{R}) &= 0.167, \\ R_{8_s}^{8\gamma}/(\bar{R}) &= -0.013, \\ R_{8_a}^{8\gamma}/(\bar{R}) &= 0.124, \\ R_{8_s}^{8}/(\bar{R}) &= 0.004, \\ R_{8_a}^{8}/(\bar{R}) &= 0.124, \\ R_8^{8\leftrightarrow10}/(\bar{R}) &= 0.019, \\ R_8^{1\leftrightarrow8}/(\bar{R}) &= -0.038. \end{aligned} \tag{29}$$

Apart from the members of the γ octet, which were shown to fit the probability matrix prediction quite well, no further reliable experimental evidence is available to test the remaining figures. For the aforementioned reasons we consider these numbers to be more reliable than those derivable from Eq. (27).

It is gratifying that the masses calculated in Table V agree qualitatively with the masses calculated in Gyuk and Tuan's paper.[19] They obtain their results by neglecting a number of mass breaking terms in the **70** mass spectrum and using 6 input masses. Neither of these assumptions is made in our calculation. The approximations in our model are dynamical in nature. Our results can be taken as supporting evidence that Gyuk and Tuan have essentially used the right input masses. An extensive discussion of the experimental situation concerning the candidates of the **70** representation has been given in Ref. 19. The (1; 2) and (8; 4) multiplets have been considered above. Because of the predicted closeness of the Σ_γ and Λ_γ levels it is possible that Λ_γ is hidden under the peak of Σ_γ. A direct test of this possibility would be the existence of a $\Sigma^0\pi^0$ decay mode of the 1660 resonance. The experimental verification of this decay mode would be quite difficult. A further possible consequence of the predicted smallness of the $(\Sigma_\gamma-\Lambda_\gamma)$ mass difference is a possible appreciable electromagnetic mixing of the neutral members of the two isomultiplets. The $N\eta$ and the $\Lambda\eta$ enhancements near their respective thresholds may be due to resonances. They are candidates for the (8; 2) multiplet.[21] Their positions are not well known but the different estimates of their positions found in Ref. 19 (i.e., $\tilde{N}\sim1480$ MeV and $\tilde{\Lambda}\sim1660$ MeV) are consistent with our predictions. Cline and Olsson recently reported some evidence for an S-wave enhancement near the $\Sigma\eta$ threshold.[22] This enhancement may be due to the $\tilde{\Sigma}$ resonance. There is no estimate on the position of the $\tilde{\Sigma}$. It will be interesting to see whether the predicted proximity of the $\tilde{\Sigma}$ and $\tilde{\Lambda}$ levels is borne out experimentally. Bareyre *et al.* have found two suitable candidates for the **70** in their phase shift analysis of πN scattering.[23] Both S_{11} and S_{31} partial waves show resonance behavior around 1690 MeV. The S_{31} resonance at 1690 MeV was confirmed by Donnachie *et al.* in their phase-shift analysis.[24] Recently Cence has cast some doubt on these results by obtaining a phase-shift fit to the existing data without the S_{11} and S_{31} resonant behavior.[25] Apparently more data have to be analyzed before any definite conclusions can be reached. The conjectured S_{31} (1690) of Bareyre *et al.* fits into our calculated mass spectrum, whereas the conjectured S_{11} (1690) is too heavy to be the $\tilde{N}$ member of the (8; 2) multiplet calculated in our model. As a candidate for the $\tilde{N}$, the resonance believed to be responsible for the $N\eta$ threshold effect has to be favored from our calculation. We conclude that there is no essential disagreement of our predictions with experimental evidence. However, the existing experimental information does not suffice to test the model effectively.

We return to Eqs. (24) and (25) and calculate the reduced matrix elements of the **70** representation from our model. Again α is taken as in Eq. (26). We obtain

$$\begin{aligned}
R_1/(\bar{R})&=\ \ 1\,,\\
R_{35H;8}/(\bar{R})&=\ \ 0.136\,,\\
R_{35NH;8}/(\bar{R})&=-0.014\,,\\
R_{189;1}/(\bar{R})&=\ \ 0.028\,,\\
R_{189;8}/(\bar{R})&=\ \ 0.000\,,\\
R_{405;1}/(\bar{R})&=\ \ 0.040\,,\\
R_{405;8}/(\bar{R})&=\ \ 0.004\,,\\
R_{(280+\overline{280});8}/(\bar{R})&\equiv\ \ 0\,,\\
R_{3675;1}/(\bar{R})&\equiv\ \ 0\,,\\
R_{3675;8}/(\bar{R})&\equiv\ \ 0\,.
\end{aligned}\tag{30}$$

The reduced matrix elements transforming like $SU(3)$ singlets are likely to be too small for the reasons given above. The leading mass-breaking components of the mass tensor are $R_{35H;8}$, on the one hand, and $R_{189;1}$ and $R_{405;1}$, on the other hand. There is a similar pattern for the **56** and the **35** representations. The leading mass-breaking components of the **56** are $B_{35;8}$ and $B_{405;1}$ (see Table III). For the **35** representation the leading terms are $M_{35;8}$, $M_{189;1}$ and $M_{405;1}$ (Table IV). All the leading terms within one $SU(6)$ multiplet are of the same order. Dyson remarked that $SU(6)$ is broken along $SU(3)\otimes SU(2)$ as much as along $[SU(4)\otimes SU(2)]_Y$.[26] If $\mathbf{35}^8$ measures the breaking along $[SU(4)\otimes SU(2)]_Y$ and the $\mathbf{189}^1$ and $\mathbf{405}^1$ measure the breaking along $SU(3)\otimes SU(2)$ all our $\mathbf{35}^8$, $\mathbf{189}^1$, and $\mathbf{405}^1$ reduced matrix elements comply with this statement. It is interesting to note that only one of the two **35** components appearing in $\mathbf{70}\otimes\overline{\mathbf{70}}$ has a sizeable reduced matrix element, namely, the $\mathbf{35H}^8$ component, which produces the hypercharge splitting. The $SU(6)$ symmetry of our model does not exclude either $R_{35H;8}$ or $R_{35NH;8}$. Both $R_{35H;8}$ and $R_{35NH;8}$ are coupled to the $B_{35;8}$ and $M_{35;8}$ mass terms. The smallness of $R_{35NH;8}$ in our calculated mass spectrum is a dynamical effect.[27] A sizable $R_{35NH;8}$ would generate both appreciable mixing and non-hypercharge splitting. In a simple quark model the only $SU(3)$ mass breaking term in the **70** would be the $R_{35H;8}$, leading to pure hypercharge splitting with no mixing. The existence of a large $R_{35NH;8}$ term then would be in contradiction with the quark model, but not with $SU(6)$. Our model shows how the contributions of $B_{35;8}$ and $M_{35;8}$ to $R_{35NH;8}$ cancel each other to a large degree, so that (as in the **56** baryon representation) hypercharge becomes the leading $SU(3)$ breaking operator. Any further comparisons of quark model versus $SU(6)$ models for $SU(3)$ breaking effects have to be made in the realm of second-order effects. An example

[21] In the limit of exact $SU(6)$ symmetry the coupling of $\tilde{N}$ to the $N+\eta$ mode has to vanish. In Ref. 19 it is shown that symmetry breaking effects may serve to restore this coupling.

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[27] It is interesting that the **8** baryon representation in $SU(3)$ which corresponds to the same Young diagram (i.e., first row: 2 boxes; second row: 1 box) as the **70** in $SU(6)$ has a similar mass structure. The reduced matrix element corresponding to the **8a** mass term (i.e., hypercharge term) is appreciably larger than the reduced matrix element corresponding to the **8s** term.

is the mass difference between the isoscalar and the isovector members of an octet.

We now turn to the calculation of the mixing angles. The γ octet provided us with some evidence that the $SU(3)$ octet-type splitting is reproduced well in our model. Since the main contribution to the off-diagonal elements of the **70** mass matrix are octet type, the calculated off-diagonal mass values are likely to be reliable. We can compute the mixing angles from the calculated mass spectrum:

$$\Lambda'\tilde{\Lambda} \text{ mixing } \alpha\sim -14°, \quad (31a)$$

$$\tilde{\Sigma}\tilde{Y}^* \text{ mixing } \beta\sim 9°, \quad (31b)$$

$$\tilde{\Xi}\tilde{\Xi}^* \text{ mixing } \gamma\sim 10°. \quad (31c)$$

We know from perturbation theory that the mixing angle is inversely dependent on the difference of the diagonal unperturbed levels. Since in our model these differences are apparently calculated too small, the mixing angles in (31a), (31b), and (31c) provide a reasonable upper limit to the actual mixing angles.

The smallness of the mixing angles can be used to derive some more approximative mass formulas. It is not desirable to set the off-diagonal elements to zero for such an approximation. Under this assumption the only surviving $SU(3)$ octet mass-breaking term would be $\mathbf{35H}^8$, if all $\mathbf{189}^8$, $(\mathbf{280}+\overline{\mathbf{280}})^8$, $\mathbf{3675}^{8b}$, and $\mathbf{3675}^{8c}$ are set equal to zero. This corresponds to pure hypercharge breaking. Since we are interested in $\Sigma-\Lambda$-breaking effects, such an approximation is not very useful. Instead we consider the small mixing angles by approximating the diagonalized mass values by the unperturbed diagonal masses. With this approximation the Gell-Mann–Okubo mass rule can be applied to the different $SU(3)$ multiplets occurring in the **70**. We present the intermultiplet mass formulas resulting from this approximation. For the octet part we obtain

$$\tilde{\Xi}^*-\tilde{Y}^*=\tilde{\Xi}-\tilde{\Sigma}, \quad (32)$$

setting $(\mathbf{280}+\overline{\mathbf{280}})^8=\mathbf{3675}^{8b}=\mathbf{3675}^{8c}=0$ and

$$\tilde{\Xi}^*-\tilde{Y}^*=\Xi_\gamma-\Lambda_\gamma, \quad (33)$$

if additionally $R_{189;8}=0$ is used. Equations (32) and (33) are identical with Eqs. (17) and (21). This can be seen by using $R_{27}{}^{8\leftrightarrow 10}=0$, i.e., $\tilde{\Xi}\leftrightarrow\tilde{\Xi}^*=\tilde{\Sigma}\leftrightarrow\tilde{Y}^*$, and the secular equations used in diagonalizing the two 2×2 matrices occuring in the $\mathbf{8}\leftrightarrow\mathbf{10}$ mixing. Unless $SU(3)$ **27**-type contributions to the mass breaking are taken into account no new intermultiplet formulas can be obtained for the octet parts by using this approximation. For the singlet part we obtain the illustrative mass rule

$$2R_1{}^8=R_1{}^{10}+R_1{}^1 \quad (34)$$

from $R_{3675;1}=0$. The $(8;2)$ multiplet is predicted to lie halfway between the $(1;2)$ and the $(10;2)$ multiplets.

In Sec. III we have seen that the $B_{405;8}$ term is responsible for the $(\Sigma-\Lambda)$ mass difference. A similar result can be obtained for the **70**. We substitute $\mathbf{189}^8=(\mathbf{280}+\overline{\mathbf{280}})^8=\mathbf{3675}^{8b}=\mathbf{3675}^{8c}=0$ into the remaining three octet-type $SU(6)$ mass tensors and obtain

$$R_{35H;8}=(385)^{-1/2}[5(2)^{1/2}R_8{}^{10}+8R_{8a}{}^{8\gamma}+4R_{8a}{}^8], \quad (35)$$

$$R_{35NH;8}=8(154)^{-1/2}(5)^{-1}(-R_{8s}{}^8+8R_{8s}{}^{8\gamma}), \quad (36)$$

$$R_{405;8}=8(70)^{-1/2}(R_{8s}{}^8). \quad (37)$$

These three equations are to be understood numerically and do not reflect any tensorial properties. If we neglect mass level shifts due to mixing, $R_{35NH;8}$ gives mainly the mass difference $\Sigma_\gamma-\Lambda_\gamma$, whereas $R_{405;8}$ is proportional to the mass difference $\tilde{\Sigma}-\tilde{\Lambda}$. It is clear that the $\Sigma-\Lambda$ signs in the $(8;2)$ and $(8;4)$ are opposite, if the values from Eq. (30) are used for $R_{35NH;8}$ and $R_{405;8}$. In a similar fashion we may substitute $R_{3675;1}=0$ into the remaining $SU(3)$ singlet type mass tensors [Eq. (7a)]. We obtain

$$R_{189;1}=8(14)^{-1/2}(5)^{-1}(R_1{}^{10}-R_1{}^{8\gamma}),$$
$$R_{405;1}=4(5)^{-1/2}(7)^{-1}(R_1{}^{8\gamma}-R_1{}^1).$$

Equation 30 gives us $R_{189;1}<R_{405;1}$. This is quite reasonable since $R_{189;1}$ gets a contribution only from the **35** mass spectrum, whereas $R_{405;1}$ has contributions from both the **35** and the **56** spectra. Thus we have $R_1{}^{10}-R_1{}^{8\gamma}>R_1{}^{8\gamma}-R_1{}^1$, and since $2R_1{}^8=R_1{}^{10}+R_1{}^1$ [Eq. (34)], we obtain

$$R_1{}^{10}>R_1{}^{8\gamma}>R_1{}^8>R_1{}^1. \quad (38)$$

We have shown that the mixing effects are most likely quite small. Equation (38) then can be assumed to hold for the physical particles as well.

V. THE 20 REPRESENTATION

There have been several speculations on the possible existence of a $SU(6)$ $\mathbf{20}^{(+)}$ baryon multiplet. Capps has shown that there exists attraction in the **20** of the $\mathbf{70}\otimes\mathbf{35}$ S-wave $SU(6)$ invariant interaction.[4] If the so-called (1480) Roper resonance has the quantum numbers $\frac{1}{2}(\frac{1}{2}^+)$ for $I(J^P)$, it could be a member of the $\mathbf{20}^{(+)}$. We denote the $\mathbf{20}^{(+)}$ representation by Z. It reduces into $(8;2)+(1;4)$ under $SU(3)\otimes SU(2)$. The members of the octet Z^8 are denoted by N', Σ', Λ^*, and Ξ' and the Z^1 singlet is denoted by Y_0'. We note that the **20** is not contained in the reduction of $\mathbf{56}\otimes\mathbf{35}$. Since the mass spectrum of the **70** is still quite uncertain, we can only give a very qualitative picture of the mass spectrum of the $\mathbf{20}^{(+)}$ as it arises in such a model.

As before, we use the approximation of the probability model. With the appropriate entries Eq. (11) is diagonalized and we obtain a relation of the **20** reduced matrix elements in terms of the reduced matrix elements of the **70** and **35** representations:

$$\begin{bmatrix} Z_1 \\ Z_{35} \\ Z_{189} \end{bmatrix} = \alpha(\bar{Z})/(\bar{R}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5(77)^{-1/2} & (8/77)^{1/2} & 0 \\ 0 & 0 & 0 & 4/(126)^{1/2} \end{bmatrix} \begin{bmatrix} R_1 \\ R_{35H} \\ R_{35NH} \\ R_{189} \end{bmatrix} + (1-\alpha)(\bar{Z})/(\bar{M}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & (7)^{-1/2} & 0 \\ 0 & 0 & (126)^{-1/2} \end{bmatrix} \begin{bmatrix} M_1 \\ M_{35} \\ M_{189} \end{bmatrix}. \quad (39)$$

It can be seen from Eqs. (24) and (25) in Sec. IV that irrespective of the α chosen in deriving the **70** mass spectrum and irrespective of the α chosen in the above equation we obtain the qualitative results $Z_{189;1}>0$, $Z_{35;8}>0$, and $Z_{189;8}\sim 0$. The first, i.e., $Z_{189;1}>0$, leads to $Z_1{}^8>Z_1{}^1$; the mean octet mass is predicted higher than the singlet mass. The second result, i.e., $Z_{35;8}>0$, gives $\Xi'>N'$. This is the usual hypercharge ordering and should be expected. $Z_{189;8}\sim 0$ together with the Gell-Mann–Okubo rule implies $\Lambda^*\sim\Sigma'$. The closeness of the two levels Λ^* and Σ' suggests a possible appreciable electromagnetic mixing effect between the neutral members of the two isomultiplets.

A possible supermixing between the two positive-parity **20** and **56** representations has to be excluded if one believes that a common $SU(6)$ mass operator can be used for all representations. This follows from the decomposition of $\mathbf{20}\otimes\overline{\mathbf{56}}=\mathbf{840}+\mathbf{280}$. Neither of these mass-breaking terms appears in the **56** and **35** mass spectrum, nor in the **70** as calculated in our model.

VI. SUMMARY AND CONCLUSION

We have used $SU(6)$ invariance in a dynamical model to relate the mass differences in the 35-fold meson multiplet to the mass differences in the **56** baryon multiplet and the conjectured $\mathbf{70}^{(-)}$ and $\mathbf{20}^{(+)}$ baryon resonance multiplets. We consider the calculation on the mass spectrum of the $\mathbf{70}^{(-)}$ multiplet the main result of this paper. The computed mass values should be of use in the identification of experimentally found resonances which have the right quantum numbers to fit into the **70** multiplet. In a phenomenological analysis similar to that of Gyuk and Tuan there can be some ambiguity in such a choice. For example, two $I^{JP}[1/2^{1/2(-)}]$ resonances have been reported recently, suitable to be members of the (8; 2) multiplet. In our calculation the $N\eta$ resonance has to be favored over the S_{11} $N\pi$ resonance found in the $N\pi$ phase-shift analysis.[21] The calculated numbers are in quite good agreement with the experimentally established members of the γ octet. We used the method of reduced matrix elements to compare different mass breaking terms. For the baryon multiplets, hypercharge splitting was found to be the most important $SU(3)$ breaking effect. It was shown that a mass operator containing the $\mathbf{35}^8$, $\mathbf{189}^1$, $\mathbf{405}^1$, and $\mathbf{405}^8$ mass-breaking terms is consistent with the experimental data and with the mass spectra computed from our model. The success of our calculation may be an indication that an approximation including only the lowest angular momenta may be sufficient to determine the position of the low-lying baryon poles (our models for the $\mathbf{70}^{(-)}$ and $\mathbf{20}^{(+)}$ multiplets are nonrelativistic in this sense only). Furthermore, the detailed dynamical structure of the exchange processes may not be as important in this determination as the contributions from the mass differences in the various direct channels.

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