Relation of the Plasma Kinetic Equation for the Darwin Hamiltonian to the Relativistic Landau Equation*

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The connection between the two-particle ring term for the kinetic equation with Darwin interaction and the $(v/c)^2$ approximation of the relativistic Landau equation is demonstrated. It is suggested that for practical calculations of $(v/c)^2$ corrections, an equation (with cutoffs) given might be used instead of the more complete kinetic equation (with ring-sum term and without cutoffs) derived recently.

I. INTRODUCTION

IN a recent paper[1] we derived a kinetic equation for the Darwin Hamiltonian, including both long- and short-range contributions. It was further observed that this kinetic equation was convergent (as was also shown earlier to be the case for the static interaction[2]) without the use of the Debye cutoff and the short-range cutoff.

Recent interesting work by Mangeney,[3] by De Gottal,[4] and by De Gottal and Prigogine,[5] has investigated the question of relativistic interactions from the standpoint of the Prigogine formalism, which includes fields as independent quantities. One of the results of Mangeney[3] has been the derivation of the Beliaev-Budker equation,[6] which is the relativistic generalization of the Landau equation.[7] The derivation of this equation for the velocity distribution assumes that the particles effectively move at constant velocity under the action of the Liénard-Wiechert potential. It is known that the question of radiation damping will come in if one proceeds beyond the lowest order in e^2, which coincides with the $(v/c)^2$ approximation, in a pure particle theory. In the approach cited above,[3,8] the derivation of the kinetic equation effectively goes beyond this approximation in $(v/c)^2$ (although retaining the lowest order in e^2), and yet physically the situation is acknowledged to be equivalent to particles in a "physical vacuum" ($T=0$ for the field variables). Indeed, the energy of the vacuum remains the same. The question then arises as to whether there is consistency in including these higher orders while maintaining the lowest order in e^2 if indeed there is no discernible effect in the physical situation from the inclusion of independent field variables.

Without pursuing further the above interesting question, we remain within the confines of a $(v/c)^2$ approximation for which a particle Hamiltonian is well defined. In the absence of an adequate relativistic theory of interacting particles[9] (either classical or quantal, with fields or without), we sacrifice the desideratum of exact Lorentz covariance [although the kinetic equation given previously is Lorentz covariant to the order $(v/c)^2$]. Thus while the Beliaev-Budker equation is exactly Lorentz covariant, its physical content beyond a $(v/c)^2$ approximation is an open question.

II. RELATION TO RELATIVISTIC LANDAU EQUATION

It is our intent here to show that the Darwin-Hamiltonian kinetic equation[1] contains a term which is the Beliaev-Budker equation when the latter is taken to order $(v/c)^2$.

The Beliaev-Budker equation for the velocity distribution as given by Mangeney[3] is written as (in the notation of Ref. 1),

$$\frac{\partial\varphi_1(\mathbf{P}_1)}{\partial t}=16\pi^3e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{l}\cdot\mathbf{D}_1\left\{\frac{\mathbf{v}_1\cdot\mathbf{v}_2-c^2}{(\mathbf{l}\cdot\mathbf{v}_1)^2-l^2c^2}\right\}^2 \times\delta(\mathbf{l}\cdot\mathbf{g}_{12})\mathbf{l}\cdot\mathbf{D}_{12}\varphi_1(\mathbf{P}_1)\varphi_1(\mathbf{P}_2)\,. \quad (2.1)$$

The bracketed term in Eq. (2.1) is simply the Fourier transform of the interaction energy of one unit charge in the field produced by uniform motion of another, in Lorentz gauge. In terms of the Coulomb gauge, this transform is

$$\frac{1}{l^2}\left[\frac{l^2(\mathbf{v}_1\cdot\mathbf{v}_2)-(\mathbf{v}_1\cdot\mathbf{l})(\mathbf{v}_2\cdot\mathbf{l})+(\mathbf{l}\cdot\mathbf{v}_1)^2-l^2c^2}{(\mathbf{l}\cdot\mathbf{v}_1)^2-l^2c^2}\right]. \quad (2.2)$$

Expanding the denominator in terms of (v/c) and retaining only terms to the second order, Eq. (2.2)

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[8] I. Prigogine, in *Statistical Mechanics of Equilibrium and Non-Equilibrium*, edited by J. Meixner (North-Holland Publishing Company, Amsterdam, 1965), p. 20.

[9] P. Havas, in *Statistical Mechanics of Equilibrium and Non-equilibrium*, edited by J. Meixner (North-Holland Publishing Company, Amsterdam, 1965), p. 1.

becomes

$$\frac{1}{l^2c^2}\left[\mathbf{v}_1\cdot\mathbf{v}_2-c^2-\frac{(\mathbf{v}\cdot\mathbf{l})(\mathbf{v}_2\cdot\mathbf{l})}{l^2}\right]$$

$$\approx\frac{1}{l^2(mc)^2}\left[\mathbf{P}_1\cdot\mathbf{P}_2-(mc)^2-\frac{(\mathbf{P}_1\cdot\mathbf{l})(\mathbf{P}_2\cdot\mathbf{l})}{l^2}\right]. \quad (2.3)$$

Under the assumption made in Ref. 1 involving spherical symmetry of the distribution function in the homogeneous plasma (this assumption was necessary in order to obtain the long-range ring term, owing to the particular momentum dependence of the Darwin Hamiltonian), we find that closer examination of (2.1) using (2.3) reveals that the only terms involving the momenta remaining after squaring (2.3) and integrating are $P_{1x}^2P_{2x}^2$ and $P_{1y}^2P_{2y}^2$. But then the right-hand side is just the two-particle ring term of Eq. (3.8) in Ref. 1 (when symmetry is again taken into account), so that (2.1) becomes

$$\frac{\partial\varphi_1(\mathbf{P}_1)}{\partial t}=16\pi^3e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{R}_1(\mathbf{l})$$

$$=16\pi^3e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{l}\cdot\mathbf{D}_1\frac{\delta(\mathbf{l}\cdot\mathbf{g}_{12})}{l^4}\mathbf{l}\cdot\mathbf{D}_{12}$$

$$\times[1+2h(1)h(2)]\varphi_1(\mathbf{P}_1)\varphi_1(\mathbf{P}_2), \quad (2.4)$$

where

$$h(\alpha)\equiv[2(mc)^2]^{-1}[\mathbf{P}_\alpha\cdot\mathbf{P}_\alpha-(\mathbf{l}\cdot\mathbf{P}_\alpha)^2/l^2].$$

Performing the integration over $\mathbf{l}$, (2.4) may be written in the more usable form,

$$\frac{\partial\varphi_1(\mathbf{P}_1)}{\partial t}=2\pi e^4C\ln(l_m/k)\int d\mathbf{P}_2$$

$$\times D_{1r}[g_{12}^{-3}(g_{12}^2\delta_{rs}-g_{12r}g_{12s})]$$

$$\times\left\{1+\frac{P_1^2P_2^2}{2(mc)^4}[1-(3/4)(\eta_1^2+\eta_2^2)+(5/8)\eta_1^2\eta_2^2]\right\}$$

$$\times D_{1r,2s}\varphi_1(\mathbf{P}_1)\varphi_1(\mathbf{P}_2), \quad (2.5)$$

where

$$\eta_i\equiv(\mathbf{g}\times\mathbf{P}_i)/gP_i, \quad i=1,2.$$

Note that (2.4) only contains the two-particle ring term and that this $(v/c)^2$ generalization of the Landau equation by itself requires large and small l cutoffs (as do the ordinary Landau and the Beliaev-Budker equations). On the other hand, considered as part of the total $(v/c)^2$ kinetic equation given in Ref. 1 (including the total ring summation and Boltzmann terms), namely,

$$\frac{\partial\varphi_1(\mathbf{P}_1)}{\partial t}=16\pi^3e^4C\int d\mathbf{P}_2\int d\mathbf{l}[\mathbf{R}(\mathbf{l})+\mathbf{B}_1(\mathbf{l})-\mathbf{R}_1(\mathbf{l})], \quad (2.6)$$

it is seen that no such cutoffs are needed.[1]

The nonrelativistic Landau equation has been considered as being a special case of the Boltzmann equation. Balescu[10] has examined it as a limiting form of the nonrelativistic ring term. If we use the more general kinetic equation (2.6) in either the nonrelativistic or $(v/c)^2$ approximation, we see that in the first instance if we allow the Boltzmann term $\mathbf{B}_1(\mathbf{l})$ to go over to the Landau form $\mathbf{R}_1(\mathbf{l})$, then only the ring sum term $\mathbf{R}(\mathbf{l})$ remains.[11] If we allow the ring sum term to reduce to the two-particle ring term, then we are left with the Boltzmann equation. In either event, we lose the desirable convergence properties of (2.6), and the ability to treat simultaneously both long- and short-range contributions without cutoffs.[12]

In conclusion, we remark that at the present time it appears that one cannot as yet discriminate between effects of order $(v/c)^2$ and higher orders for hot laboratory plasmas, and so for all practical purposes (apart from the question of the consistency of using the Beliaev-Budker equation) the $(v/c)^2$ equation (2.6) should be sufficient. However, since the ring term alone introduces complicated nonlinearity into the problem, Eq. (2.5) with cutoffs may be valuable for calculation of $(v/c)^2$ corrections.

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[10] R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience Publishers, Inc., New York, 1963), Chap. 11.

[11] For the explicit forms for $\mathbf{B}_1(\mathbf{l})$ and $\mathbf{R}(\mathbf{l})$ in the $(v/c)^2$ approximation, see Ref. 1.

[12] We also remark that, with the interaction term as given in Eq. (2.1), it does not appear, in contrast with the $(v/c)^2$ approximation, that the ring sum can be made.