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Separation of Strong and Electromagnetic Effects in Charged-Particle Scattering*

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Data on the scattering of charged hadrons reflect both the strong and electromagnetic interactions. We examine the problem of unraveling the "strong" part of the scattering amplitude from the electromagnetic part. Specific prescriptions, involving only the use of observed scattering data, are derived for obtaining the connection between the idealized "strong" scattering amplitude and the observed full scattering amplitude. The importance of the "off-the-mass-shell" or model-dependent corrections, which cannot be obtained directly from observation, are also discussed briefly. These results are equally applicable in the domains of low-energy nuclear physics and high-energy particle physics.

INTRODUCTION

SCATTERING experiments ordinarily do not yield information about the phase of the scattering amplitude but only yield the magnitude. An exception to this is the Coulomb interference region where the known phase of the Coulomb scattering gives information about both the phase and the amplitude of the "nuclear" scattering. Two problems arise immediately. First, we must know the purely electromagnetic scattering amplitude. Secondly, we must be able to relate the residual "nuclear" scattering amplitude to the pure "nuclear" scattering. By pure "nuclear" scattering we mean the scattering which would result were there no electromagnetic forces between projectile and target.

The problem of finding the phase of the purely "nuclear" scattering amplitude has lately assumed special importance at high energies where we wish to obtain the ratio of the real to the imaginary part of the purely "nuclear" scattering amplitude in the forward direction.

Our purpose here is to outline how the residual "nuclear" amplitude may be obtained directly from elastic-scattering data, and further to show how the residual "nuclear" scattering can be related to the purely "nuclear" scattering. This problem is not completely trivial because the residual scattering amplitude does contain electromagnetic effects which are not necessarily small. Bethe[1] discussed the connection between the residual and pure "nuclear" scattering amplitude in a semiclassical calculation using a specific model of the nuclear interaction. We here discuss a general technique for relating these scattering amplitudes and present an approximation which depends only on observable data and therefore is independent of any detailed assumptions about the nuclear forces.

In Sec. I we study central potential scattering where the model-independent connection between the residual and pure nuclear amplitudes is found by assuming that the phase shifts are additive, i.e., $\delta_{\text{total}}=\sigma_{\text{Coul}}+\delta_{\text{nuc}}$. A generalization of Bethe's phase is derived which involves an integral over the experimentally observable residual amplitude.

The model-dependent effects are estimated for central potential scattering in Sec. II. These effects are found to be much smaller than the model independent effects at high energies.

The modifications of the potential scattering results due to relativity and the connection between the technique of Sec. I and the Feynman graph formalism is discussed in Sec. III. The modifications are found to be straightforward.

In Sec. IV the effect of spin on our formulas is briefly outlined. We observe that for small angle scattering the complications of spin can be ignored.

Finally, in Sec. V, we outline the procedure by which the residual amplitude is obtained from the scattering data and exhibit model-independent formulas connecting the parameters of the residual and nuclear amplitudes for high-energy scattering.

I. RELATION BETWEEN RESIDUAL AND "NUCLEAR" AMPLITUDES

Suppose that a Coulomb potential and a short-range central nuclear potential are acting together to produce an observed scattering cross section.

$$\frac{d\sigma}{d\Omega}=|f(\theta)|^2. \tag{1.1}$$

The scattering amplitude is given by its phase-shift representation, as

$$f(\theta)=\sum_{l=0}^{\infty}(2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right)P_l(\theta). \tag{1.2}$$

The total phase shift δ_l can be decomposed as

$$\delta_l=\sigma_l+\delta_l{}^N, \tag{1.3}$$

where $\sigma_l\equiv(1/2i)\ln[\Gamma(l+1+i\eta)/\Gamma(l+1-i\eta)]$ is the pure Coulomb phase shift[2] and $\delta_l{}^N$ the "nuclear" phase shift as defined by Eq. (1.3). Then Eq. (1.2) becomes

$$f(\theta)=f_c(\theta)+R(\theta), \tag{1.4}$$

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[1] H. A. Bethe, Ann. Phys. (N. Y.) 3, 190 (1958).

[2] For nonrelativistic scattering $\eta=Z_1Z_2m\alpha/k$, where $\hbar=c=1$ and $\alpha\simeq1/137$. The relativistic form for η appears in Eq. (3.4).

where f_c, the pure Coulomb amplitude, is

$$f_c(\theta)\equiv\sum_l(2l+1)\left(\frac{e^{2i\sigma_l}-1}{2ik}\right)P_l(\theta)$$
$$=-\eta\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\frac{e^{-i\eta\ln[\sin^2(\theta/2)]}}{2k\sin^2(\theta/2)},\quad(\theta\neq 0)\quad(1.5)$$

and the *residual amplitude* $R(\theta)$ is

$$R(\theta)\equiv\sum_{l=0}^{\infty}(2l+1)e^{2i\sigma_l}f_l^NP_l(\theta),\quad(1.6)$$

with

$$f_l^N=(e^{2i\delta_l^N}-1)/2ik.\quad(1.7)$$

In a charged particle scattering experiment, we can observe the differential scattering cross section $d\sigma/d\Omega$:

$$d\sigma/d\Omega=|R|^2+2\,\mathrm{Re}(f_c{}^*R)+|f_c|^2.\quad(1.8)$$

At scattering angles small enough so that f_c and R are comparable in magnitude (Coulomb interference region) these data allow us to determine both $\mathrm{Re}R(\theta)$ and $\mathrm{Im}R(\theta)$ since $f_c(\theta)$ is known and $R(\theta)$ is relatively slowly varying.

One problem of great interest in nuclear physics is to determine what the nuclear scattering would be in the absence of electromagnetic interactions between projectile and target. A knowledge of $R(\theta)$ does not suffice to give us that information. However, from $R(\theta)$ we may obtain the scattering amplitude

$$f_N(\theta)\equiv\lim_{\sigma_l(\eta)\to 0}R(\theta)=\sum_{l=0}^{\infty}(2l+1)f_l^NP_l(\theta).\quad(1.9)$$

In the next section we will discuss the circumstances under which $f_N(\theta)$ as given by Eq. (1.9) represents a good approximation to the nuclear amplitude of interest.

Equation (1.6) can be inverted by using the completeness and orthonormality of the spherical-harmonic functions. Since $R_l\equiv f_l^N\exp(2i\sigma_l)$ and

$$f_l^{c*}\equiv[1-\exp(-2i\sigma_l)]/2ik,$$

we can write

$$f_l^N=(1-2ikf_l^{c*})R_l,\quad(1.10)$$

from which it follows directly that

$$\int d\Omega_q[f_N(\Omega_{qp})-R(\Omega_{qp})]Y_l^{m*}(\Omega_{qp})$$
$$=-2ikf_l^{c*}\int d\Omega_qR(\Omega_{qp})Y_l^{m*}(\Omega_{qp}).\quad(1.11)$$

From the completeness of the spherical harmonics we obtain

$$f_N(\Omega_{p'p})-R(\Omega_{p'p})$$
$$=-2ik\int d\Omega_qR(\Omega_{qp})\sum_l f_l^{c*}\sum_m Y_l^m(\Omega_{p'p})Y_l^{m*}(\Omega_{qp})$$
$$=-2ik\int\frac{d\Omega_q}{4\pi}R(\Omega_{qp})\sum_l(2l+1)f_l^{c*}P_l(\theta_{p'q}).\quad(1.12)$$

The last sum on the right-hand side of Eq. (1.12) is easily recognized to be $f_c{}^*(\theta_{p'q})$. In the notation we are using, Ω_{ab} represents the polar angles between vectors a and b and $\theta_{p'q}$ is the angle between $\mathbf{p}'$ and $\mathbf{q}$. As might be expected, Eq. (1.12) is a "unitarity" integral of the form

$$f_N(\Omega_{p'p})=R(\Omega_{p'p})-2ik\int\frac{d\Omega_q}{4\pi}f_c{}^*(\theta_{p'q})R(\Omega_{qp}).\quad(1.13)$$

We note, however, that Eq. (1.13) is not unambiguously defined since the treatment of the singularity of $f_c(\theta)$ at $\theta=0$ has not been described. On the other hand, both R and f_N are perfectly well behaved since their partial-wave series both cut off at some finite angular momentum as a consequence of the finite range of the nuclear force. Furthermore when $R(\theta)=\mathrm{const}$ (S-wave scattering), f_N is simply $f_N(\theta)=\exp(-2i\sigma_0)R$, which indicates that we must take

$$e^{-2i\sigma_0}=1-2ik\int\frac{d\Omega_q}{4\pi}f_c{}^*(\theta_{p'q})$$

as the defining treatment of an integral over f_c. That is, all the problems of the non-uniformity of convergence associated with the Coulomb amplitude are contained in the S wave.[3] The rearrangement of Eq. (1.13) which gives the correct, unambiguous result is

$$f_N(\Omega_{p'p})=R(\Omega_{p'p})e^{-2i\sigma_0}$$
$$-2ik\int\frac{d\Omega_q}{4\pi}f_c{}^*(\theta_{p'q})[R(\Omega_{qp})-R(\Omega_{p'p})].\quad(1.14)$$

At very low energies where only a few l states contribute, we can work directly with Eq. (1.6). Depending on the problem, however, there may be an energy region where η is large and a moderate number of angular-momentum states enter into Eq. (1.6), in which case it would be necessary to use Eq. (1.13) in order to obtain f_N from R.

For very high energies where a very large number of angular-momentum states contributes to Eq. (1.6) we must use Eq. (1.14). However, in this case, certain simplifications enter. First, η is small at high energies ($\eta\to Z_1Z_2\alpha$; Sec. III). Secondly, R and f_N will be very sharply peaked in the forward direction in which case it is quite reasonable to focus our attention on $f_N(\theta=0)$ for which Eq. (1.14) simplifies to

$$f_N(0)=R(0)e^{-2i\sigma_0}+i\eta e^{-2i\sigma_0}$$
$$\times\int_{-1}^{1}\frac{d\cos\theta}{2}\left(\frac{1-\cos\theta}{2}\right)^{-1+i\eta}[R(\theta)-R(0)].\quad(1.15)$$

In order to illustrate the difference between $f_N(0)$ and $R(0)$ predicted by Eq. (1.15), let us suppose that an experiment determines that $R(\theta)$ has the form[4]

[3] J. T. Holdeman and R. M. Thaler, Phys. Rev. **139**, B1186 (1965).

[4] This choice for $R(\theta)$ is typical of high-energy scattering results.

$R(\theta)=R(0)\exp(+\frac{1}{2}bt)$, where b is real $(bk^2\gg 1)$ and the square of the four-momentum transfer is $t\equiv -2k^2\times(1-\cos\theta)$. Integration by parts in Eq. (1.15) gives the result that

$$f_N(0)=R(0)e^{-2i\sigma_0}\left[1-(1-e^{-2bk^2})+2^{-i\eta}bk^2\int_{-1}^{1}du(1-u)^{+i\eta}e^{-bk^2(1-u)}\right]. \quad (1.16)$$

By introducing a new integration variable $x=bk^2(1-u)$ and neglecting all terms of order $\exp(-2bk^2)\ll 1$, we then find that Eq. (1.16) is approximated by

$$f_N(0)\simeq R(0)e^{-2i\sigma_0-i\eta\ln(2bk^2)}\int_0^\infty dx\, x^{+i\eta}e^{-x}$$
$$=R(0)e^{-i\eta\ln(2bk^2)+\ln\Gamma(1-i\eta)}\equiv R(0)e^{i\phi(k)}, \quad (1.17)$$

since $\sigma_0=(1/2i)\ln[\Gamma(1+i\eta)/\Gamma(1-i\eta)]$. The phase ϕ of Eq. (1.17) can be expanded in powers of η, using the logarithmic derivatives of $\Gamma(z)$, to give

$$\phi(k)=-\eta\ln(2bk^2/\gamma)+i\eta^2\pi^2/12+\cdots, \quad (1.18)$$

where $\ln\gamma\equiv 0.5772\cdots$ is the Euler-Mascheroni constant. Since $bk^2\gg 1$ and η is small, it is clear that the lowest order term of Eq. (1.18) dominates. Furthermore, since the dominant term of the phase is purely real, the effect of the long-range Coulomb force is to rotate the real and imaginary parts of the forward nuclear-scattering amplitude to produce the experimentally observed quantities $\mathrm{Re}R(0)$ and $\mathrm{Im}R(0)$.

$$\begin{aligned}\mathrm{Im}f_N(0)&=\mathrm{Im}R(0)\cos\phi(k)+\mathrm{Re}R(0)\sin\phi(k);\\ \mathrm{Re}f_N(0)&=\mathrm{Re}R(0)\cos\phi(k)-\mathrm{Im}R(0)\sin\phi(k),\end{aligned} \quad (1.19)$$

where $\phi(k)\simeq -\eta\ln(2bk^2/\gamma)$ can be appreciable at high energies. If we are interested in the ratio of real to imaginary part of $f_N(0)$ we may write

$$\frac{\mathrm{Re}f_N(0)}{\mathrm{Im}f_N(0)}=\tan\left[\tan^{-1}\left(\frac{\mathrm{Re}R(0)}{\mathrm{Im}R(0)}\right)-\phi(k)\right]. \quad (1.20)$$

At ultrahigh energies, if we expect that $\mathrm{Re}f_N(0)/\mathrm{Im}f_N(0)\to 0$, then we would expect to *observe experimentally* the converse of Eq. (1.20), viz.,

$$\frac{\mathrm{Re}R(0)}{\mathrm{Im}R(0)}\underset{k\to\infty}{\sim}\tan\phi(k)\sim -\tan 2\eta(\ln E+\mathrm{const}). \quad (1.21)$$

Therefore to ignore the Coulomb effect described here, i.e., to assume in Eq. (1.21) that $R=f_N$, would lead us to the spurious conclusion that

$$(\mathrm{Re}f_N/\mathrm{Im}f_N)_0\underset{k\to\infty}{\longrightarrow}\tan\phi(k).$$

The observations of the preceding paragraph indicate that a useful approach to Eq. (1.14) at high energies would be for us to write the nuclear amplitude as

$$f_N(\theta)\equiv R(\theta)e^{i\phi(k,\theta)}, \quad (1.22)$$

where $\phi(k,\theta)$ is given in general by

$$\phi(k,\theta)=-\eta\left[\int_{-1}^{1}d\cos\bar\theta\frac{1-[R(\bar\theta)/R(\theta)]}{|\cos\bar\theta-\cos\theta|}-2\ln\gamma\right]+O(\eta^2). \quad (1.23)$$

Equation (1.23) is readily derived from Eq. (1.14) through an expansion in powers of η and exponentiation as in Eq. (1.22). The $O(\eta^2)$ terms of Eq. (1.23), as we have seen, are nugatory at high energies. Finally, we note that the phase of Eq. (1.23) is in general complex since $R(\theta)$ is complex.

II. ADDITIVITY OF THE PHASE SHIFTS

In the scattering of a charged particle by a complex target, we would like to find out what the scattering amplitude would be if the electromagnetic part of the interaction between the target and projectile were switched off. Even assuming that the only electromagnetic interaction of consequence is the Coulombic, this scattering amplitude is not the amplitude defined by Eq. (1.22). The scattering amplitude, Eq. (1.22), would only be the true nuclear scattering amplitude if the phase shifts were additive. However, it is clear that

$$\delta_l^{(\mathrm{tot})}-\sigma_l\equiv\delta_l^N\neq\delta_l^N(\eta=0), \quad (2.1)$$

if $\delta_l^N(\eta=0)$ is taken to be the phase shift that would obtain the absence of the Coulombic interaction between projectile and target. However, at high energies additivity of the phase shifts becomes an excellent approximation and the scattering amplitude given by Eq. (1.22) becomes very close to the true nuclear scattering amplitude.

Let us consider a charged particle moving in the field of another charged particle. The two particles interact via a short-range central nuclear potential $V_N(r)$ in addition to their Coulombic interaction. At high energies, we may use the J.W.K.B. approximation for the phase shift

$$\delta_l^N\simeq\int_{r_0}^\infty\left(k^2-U(r)-\frac{2k\eta}{r}-\frac{l(l+1)}{r^2}\right)^{1/2}dr-\int_{r_1}^\infty\left(k^2-\frac{2k\eta}{r}-\frac{l(l+1)}{r^2}\right)^{1/2}dr, \quad (2.2)$$

and

$$\delta_l^N(\eta=0)\simeq\int_{r_0'}^\infty\left(k^2-U(r)-\frac{l(l+1)}{r^2}\right)^{1/2}dr-\int_{r_1'}^\infty\left(k^2-\frac{l(l+1)}{r^2}\right)^{1/2}dr, \quad (2.3)$$

where $U(r)=(2m/\hbar^2)V_N(r)$. Clearly the difference between the phase shifts given by Eqs. (2.2) and Eq. (2.3) is of order η. In the high-energy limit the phase shifts δ_l^N and $\delta_l^N(\eta=0)$ both become

$$\delta_l^N \simeq \delta_l^N(\eta=0) \simeq -\frac{1}{2k}\int_{r_0}^{\infty} U(r)dr, \qquad (2.4)$$

where $r_0 \simeq l/k$. In this same high-energy limit the difference between $\delta_l^N(\eta=0)$ and δ_l^N becomes

$$\Delta_l \equiv \delta_l^N - \delta_l^N(\eta=0) \simeq -\frac{\eta}{2k^2}\int_{r_0}^{\infty} U(r)dr/r. \quad (2.5)$$

In the forward direction the difference between the amplitudes $f_N(\eta=0)$ and f_N is

$$\Delta f_N \equiv f_N - f_N(\eta=0) = \sum(2l+1)\frac{e^{2i\delta_l^N} - e^{2i\delta_l(\eta=0)}}{2ik} \simeq \frac{1}{k}\sum(2l+1)e^{2i\delta_l^N}\Delta_l. \quad (2.6)$$

Since at high energies the nuclear phase shift goes as k^{-1}, we may approximate Eq. (2.6) by

$$\Delta f_N \simeq \frac{1}{k}\sum(2l+1)\Delta_l \simeq -\frac{\eta}{2k^3}\int_0^{\infty} 2ldl \int_{l/k}^{\infty} U(r)\frac{dr}{r} \simeq -\frac{\eta}{2k^3}\int_0^{\infty} dr\frac{U(r)}{r}\int_0^{kr} 2ldl \simeq -\frac{\eta}{2k}\int_0^{\infty} rdrU(r). \quad (2.7)$$

In the same approximation the nuclear forward scattering amplitude is

$$f_N \simeq -\frac{1}{2k^2}\int_0^{\infty} drU(r)\int_0^{kr} 2ldl \simeq -\frac{1}{2}\int_0^{\infty} dr\, r^2U(r), \quad (2.8)$$

and

$$\frac{\Delta f_N}{f_N} \simeq \left(\frac{\eta}{k}\right)\frac{\int_0^{\infty} dr\, rV_N(r)}{\int_0^{\infty} dr\, r^2V_N(r)} \simeq \frac{\eta}{k}\frac{\langle r\rangle}{\langle r^2\rangle}, \qquad (2.9)$$

where $\langle r\rangle$ and $\langle r^2\rangle$ are average values weighted by the nuclear interaction potential. We conclude that at high energy

$$\Delta f_N/f_N \sim N\eta/kR, \qquad (2.10)$$

where N is a numerical factor of order unity and R is a distance of the order of the range of the nuclear force. For the scattering of pions by protons at 1 BeV/c incident momentum, Eq. (2.10) yields the rough numerical result $(\Delta f_N/f_N) \sim 10^{-5}$. For proton-proton scattering, the correction, Eq. (2.10), is less than 1% for $E \gtrsim 100$ MeV.

The high-energy limit, Eqs. (2.4) and (2.5), that we have used does not apply for infinitely strong interaction potentials, as for example the hard sphere. For the hard-sphere case

$$|\Delta f_N/f_N| \sim \eta, \qquad (2.11)$$

in the forward direction. In either case, the corrections given by Eq. (2.10) or Eq. (2.11) are negligible at high energies relative to the effect exhibited in Sec. I.

III. RELATIVITY

So far we have discussed the relationship between f_N and R in the context of nonrelativistic potential scattering. The basic result of Sec. I, Eq. (1.14), is also valid relativistically for spinless particles since it derives solely from the partial wave representation for scattering amplitudes and does not depend on the details of the interactions. The only nonrelativistic aspects of our observations in Sec. I were involved with the identification of f_c in Eq. (1.5), i.e., $\eta = Z_1Z_2\alpha m/k$. In order to obtain the spinless, relativistic analog of Eq. (1.14) and the high-energy results, Eqs. (1.22) and (1.23), it is only necessary to use for f_c in Eq. (1.14) the relativistically correct electromagnetic scattering amplitude normalized to satisfy Eq. (1.1). In so doing, it is convenient to work in a specific Lorentz frame, in our case the center-of-momentum system, and then express the resulting formulae in terms of the invariant scattering variables, $s=(k_{1\,\text{initial}}+k_{2\,\text{initial}})^2$ and $t=(k_{1\,\text{final}}-k_{1\,\text{initial}})^2$ to obtain a result that can be evaluated easily for any Lorentz frame.

For purposes of illustration, let us obtain the relativistic analog of Eq. (1.23) for the scattering of two spinless particles of charge and mass Z_1, m_1 and Z_2, m_2. Equation (1.23) was dependent only on the lowest order term of f_c, in which case we use the one-photon exchange Feynman graph which gives[5]

$$f_c(s,t) = (Z_1Z_2\alpha/t)\{[s-(m_1^2+m_2^2)]s^{-1/2}\}. \quad (3.1)$$

In the center of momentum system $t=-2k^2(1-\cos\theta)$ and $4sk^2=[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$, so that it is possible to convert Eq. (1.14) and hence Eq. (1.23) into a relativistic form simply by replacing $2k\eta$ by $2k\eta \to Z_1Z_2\alpha[s-(m_1^2+m_2^2)]/\sqrt{s}$. Instead of Eq. (1.22) and (1.23) we then have

$$f_N(s,t) = R(s,t)e^{i\phi(s,t)} \qquad (3.2)$$

[5] We have omitted the t-independent part of Eq. (3.1) present in boson-boson scattering since it will contribute to an undetermined S-wave scattering constant, a typical problem of boson-boson scattering.

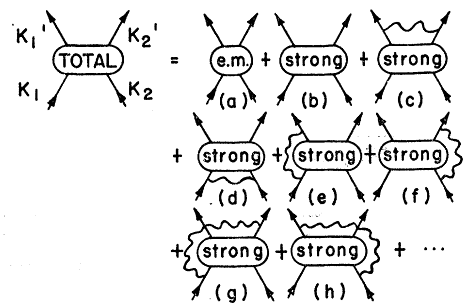

FIG. 1. Graphical decomposition of the scattering amplitude. A virtual photon is represented by a wiggly line. Graph (a) corresponds to $f_c(s,t)$, while (b) corresponds to f_N and (b), (c), etc., to the residual amplitude R.

with

$$\phi(s,t) = -\eta(s)\left[\int_{-t(\pi)}^{0} d\bar{t}\,\frac{1-[R(\bar{t})/R(t)]}{|\bar{t}-t|} - 2\ln\gamma\right] + O(\eta^2) \tag{3.3}$$

and

$$\eta(s) = Z_1Z_2\alpha[s-(m_1^2+m_2^2)]/[s-(m_1+m_2)^2]^{1/2}[s-(m_1-m_2)^2]^{1/2}, \tag{3.4}$$

where

$$t(\pi) = [s-(m_1+m_2)^2][s-(m_1-m_2)^2]/s.$$

Equations (3.2), (3.3), and (3.4) can be used in any Lorentz frame and differ from the nonrelativistic result only by a new form for η. We note[6] that $\eta(s) \to Z_1Z_2\alpha$ as $s \to \infty$. Taking into account the extended charge distribution in Eq. (3.1) causes us no difficulty beyond complicating the integral of Eq. (3.3). On the other hand, form factors will not affect the value of $\phi(s,\, t=0)$ very markedly because they represent a short-range effect, and the dominant behavior of ϕ is determined by the long-range point Coulomb force. For example, if we multiply Eq. (3.1) by the factor

$$F(t) = \left(\frac{M_1^2}{M_1^2-t}\right)\left(\frac{M_2^2}{M_2^2-t}\right) \tag{3.5}$$

so as to account for the form factors and assume that $R(t) \equiv R(0)\exp(\frac{1}{2}bt)$ as in the example of Sec. I, we find that $\phi(s,\, t=0)$ of Eq. (3.3) is now given by

$$\phi_{\text{form factor}}(s,0) = \phi_{\text{point charge}}(s,0) - \eta(s)\left\{\frac{Q_2}{Q_2-Q_1}e^{Q_1}\,\mathrm{Ei}(-Q_1) - \frac{Q_1}{Q_2-Q_1}e^{Q_2}\,\mathrm{Ei}(-Q_2)\right\}, \tag{3.6}$$

where $\phi_{\text{point charge}}$ is given by Eq. (3.3),

$$\mathrm{Ei}(-z) \equiv -\int_z^\infty \frac{dx}{x}e^{-x},$$

and $Q_i \equiv (bM_i^2/2) \sim 1$. Once again, in Eq. (3.6), terms of order $\exp[-\frac{1}{2}bt(\pi)] \ll 1$ have been neglected.

At this point, let us digress momentarily to show the connection between Eqs. (3.2) and (3.3) and the Feynman-graph formalism. For simplicity we will consider spinless, equal-mass scattering. The Feynman graphs of interest are indicated in Fig. 1. The graphs of first order in α not explicitly indicated in Fig. 1 all contain a photon buried in the strong-interaction blobs and therefore represent *model-dependent* corrections, since their calculation requires knowledge of the strong-interaction amplitude off the mass shell. First, let us consider the contribution to the residual amplitude from graphs (b), (c), and (d) of Fig. 1.

$$R(s,t) = f_N(s,t) - 8\pi i Z_1Z_2\alpha \int \frac{d^4k}{(2\pi)^4}\,\frac{4(k_1k_2)f_N(k_1-k,\, k_2+k;\, k_1',\, k_2')}{k^2[(k_1-k)^2-m^2+i\epsilon][(k_2+k)^2-m^2+i\epsilon]}, \tag{3.7}$$

where (d) has been included by counting (c) twice.[7] The order α contribution to R in Eq. (3.7) is infrared divergent. This divergence is treated by using the prescription of Yennie, Frautschi, and Suura[8] in which a common divergent factor is removed from f, f_c, and R in order to obtain the infrared *finite* quantities $\hat{f}$, $\hat{f}_c$, and $\hat{R}$. That is, we write

$$\begin{Bmatrix} f \\ f_c \\ R \end{Bmatrix} \equiv I(s,t) \begin{Bmatrix} \hat{f} \\ \hat{f}_c \\ \hat{R} \end{Bmatrix}, \tag{3.8}$$

where[9]

$$I(s,t) = 1 - 8\pi i\alpha Z_1Z_2 \int \frac{d^4k}{(2\pi)^4}\,\frac{4(k_1k_2)}{k^2[(k_1-k)^2-m^2+i\epsilon][(k_2+k)^2-m^2+i\epsilon]} + i\eta(s)2\ln\gamma + \cdots, \tag{3.9}$$

[6] For an alternative derivation of $\eta(s)$ for relativistic scattering, see W. Rolnick, Phys. Rev. **148**, 1539 (1966).

[7] Once again we have removed from the photon numerator those terms which contribute a t-independent part to Eq. (3.1).

[8] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) **13**, 379 (1961).

[9] The infrared factor $I(s,t)$ is not unique. We have chosen it so as to give the usual Coulomb amplitude from graphs (a) of Fig. 1 in the nonrelativistic limit.

in which case $\hat{R}$ is given by

$$\hat{R}(s,t)=f_N(s,t)[1-2i\eta(s)\ln\gamma]$$
$$-8\pi i Z_1Z_2\alpha\int\frac{d^4k}{(2\pi)^4}\frac{4(k_1k_2)\{f_N(k_1-k,\,k_2+k;\,k_1',\,k_2')-f_N(k_1,k_2;\,k_1',k_2')\}}{k^2[(k_1-k)^2-m^2+i\epsilon][(k_2+k)^2-m^2+i\epsilon]}+\cdots. \quad (3.10)$$

The order α term of $\hat{R}$ in Eq. (3.10) involves knowledge of f_N off the mass shell and would appear to be model-dependent. However, there is a mass-shell contribution to Eq. (3.10) which comes from the imaginary parts of the denominators. Explicitly, we write $(x+i\epsilon)^{-1}=\mathcal{P}/x-i\pi\delta(x)$ and substitute the positive-energy δ-function parts in Eq. (3.10) to obtain the model-*independent* contribution to $\hat{R}$;

$$\hat{R}(s,t)=f_N(s,t)(1-2i\eta(s)\ln\gamma)+\frac{iZ_1Z_2\alpha}{2\pi}\int d^4k\frac{4(k_1k_2)}{k^2}\delta_p((k_1-k)^2-m^2)(k_2+k)^2$$
$$\times[f_N(k_1-k,\,k_2+k;\,k_1',\,k_2')-f_N(k_1,k_2;\,k_1',k_2')]+\text{model-dependent contributions}. \quad (3.11)$$

The integral of Eq. (3.11) can be evaluated easily in the center-of-momentum system in which $\mathbf{k}_1=-\mathbf{k}_2$, by the substitution $k=k_1-p$. The integrations over p_0 and $|\mathbf{p}|$ give

$$\hat{R}(s,t)=f_N(s,t)[1-2i\eta(s)\ln\gamma]-i\frac{Z_1Z_2\alpha 2(k_1k_2)}{2\sqrt{s}|\mathbf{k}_1|}\int_{-1}^{1}d\cos\theta\int_0^{\pi}d\phi/\pi$$
$$\times[f_N(s,\cos\theta)-f_N(s,\cos\theta_s)][1-\cos\theta\cos\theta_s-\sin\theta\sin\theta_s\cos\phi]^{-1}, \quad (3.12)$$

where $\cos\theta_s=1-t/2|\mathbf{k}_1|^2$. The ϕ integration then just reproduces the lowest order form of Eqs. (3.2) and (3.3), since in the center-of-momentum system $\eta(s)=Z_1Z_2\alpha(2k_1k_2)/2s^{1/2}|\mathbf{k}_1|$ and $\hat{R}=f_N$ to zeroth order in α.

We have just shown how the technique of Yennie, Frautschi, and Suura for removing the infrared divergence may be applied to the problem symbolized by the Feynman graphs in Fig. 1. In the work described above we have taken a further step in much the same spirit to calculate the model-independent, or "on-the-mass-shell," electromagnetic corrections to the so-called "strong" scattering amplitude. The result so obtained is identical to the result obtained by the equivalent arguments for potential scattering.

These same techniques may be applied to the calculation of the model independent radiative corrections as shown in graphs (e)–(h) of Fig. 1. These corrections are the inner bremsstrahlung terms. They have no analog in potential scattering, so that we know of no other way of estimating the importance of such corrections.

IV. SPIN

In the scattering of two particles with nonzero spin, the scattering amplitude can be written as

$$M(s,t;\,\sigma_1,\sigma_2)=f(s,t)+g(s,t;\,\sigma_1,\sigma_2), \quad (4.1)$$

where $f(s,t)$ is the spin-independent part of the scattering amplitude, $M(s,t;\,\sigma_1,\sigma_2)$. For spin zero on spin $\frac{1}{2}$, $M(s,t;\,\sigma_1)$ is

$$M(s,t,\sigma_1)=f(s,t)+(\hat{n}\cdot\boldsymbol{\sigma}_1)g(s,t), \quad (4.2)$$

where $\hat{n}=(\mathbf{k}_i\times\mathbf{k}_f)/|\mathbf{k}_i\times\mathbf{k}_f|$. For the scattering of two spin-$\frac{1}{2}$ particles, the scattering amplitude becomes

$$\begin{aligned}M(s,t;\,\sigma_1,\sigma_2)=&f(s,t)+A(s,t)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{n}\\&+B(s,t)(\boldsymbol{\sigma}_1\cdot\hat{n})(\boldsymbol{\sigma}_2\cdot\hat{n})+C(s,t)\\&\times[(\boldsymbol{\sigma}_1\cdot\hat{p})(\boldsymbol{\sigma}_2\cdot\hat{p})+(\boldsymbol{\sigma}_1\cdot\hat{q})(\boldsymbol{\sigma}_2\cdot\hat{q})]\\&+D(s,t)[(\boldsymbol{\sigma}_1\cdot\hat{p})(\boldsymbol{\sigma}_2\cdot\hat{p})-(\boldsymbol{\sigma}_1\cdot\hat{q})(\boldsymbol{\sigma}_2\cdot\hat{q})],\end{aligned} \quad (4.3)$$

where $\hat{p}\equiv(\mathbf{k}_f+\mathbf{k}_i)/|\mathbf{k}_f+\mathbf{k}_i|$ and $\hat{q}\equiv(\mathbf{k}_f-\mathbf{k}_i)/|\mathbf{k}_f-\mathbf{k}_i|$. In Eqs. (4.2) and (4.3) the usual conservation laws have been assumed.[10]

If the only electromagnetic effect to be taken into account were the spin-independent Coulomb force, then we could interpret the amplitude $f(s,t)$ as used in the preceding sections to be just the spin-independent part of the scattering amplitude $M(s,t;\,\sigma_1,\sigma_2)$. In that case the data analysis would proceed in two stages. First, we would attempt to unravel the spin-independent part of the full scattering amplitude $M(s,t;\,\sigma_1,\sigma_2)$. Then the analysis would continue along the lines previously discussed. We would make the following identifications:

$$[M(s,t;\,\sigma_1,\sigma_2)]_{\text{spin-indep.}}\rightarrow f(s,t) \quad (4.4)$$

and

$$[M(s,t;\,\sigma_1,\sigma_2)]_{\text{spin-indep.}}-f_c(s,t)\rightarrow R(s,t), \quad (4.5)$$

so that

$$[M_{\text{nuc}}(s,t;\,\sigma_1,\sigma_2)]_{\text{spin-indep.}}=\{[M(s,t;\,\sigma_1,\sigma_2)]_{\text{spin-indep.}}-f_c(s,t)\}e^{i\phi(s,t)}, \quad (4.6)$$

where $\phi(s,t)$ is given by Eq. (3.3).

[10] L. Wolfenstein, Ann. Rev. Nucl. Sci. 6, 43 (1956).

Of course, this is not actually the case. The electromagnetic force also has a spin-dependent part. In the actual case, the spin-independent part of the nuclear amplitude obtained in Eq. (4.6) requires further correction for the magnetic scattering. This additional correction is also of order α, however, it is not an important effect unless the spin-dependent and spin-independent parts of the nuclear amplitude are of comparable magnitude. At small angles, however, the spin-independent part of the nuclear amplitude dominates.

V. DATA ANALYSIS

In order to obtain the information required to apply these methods to actual physical situations, we must be able to extract the residual amplitude from the actual scattering data. For small-angle scattering the spin complications can usually be ignored. In that case the residual amplitude may be obtained from the elastic cross section, because

$$\sigma_{\rm el}(s,t)=|f_c(s,t)+R(s,t)|^2 \tag{5.1}$$

or

$$\sigma_{\rm el}(s,t)-\sigma_{\rm Coul}(s,t)\equiv\sigma'(s,t)=|R(s,t)|^2+f_c^*(s,t)R(s,t)+f_c(s,t)R^*(s,t)\,. \tag{5.2}$$

The quantity $\sigma'(s,t)$ may be regarded as an observable since $\sigma_{\rm Coul}(s,t)$ is well known. Similarly the amplitude $f_c(s,t)$ in Eq. (5.2) is completely known. The only unknown then in Eq. (5.2) to be determined from the data is $R(s,t)$. The residual amplitude, for fixed s, considered as a function of t is slowly varying relative to $f_c(s,t)$ in the angular region where $f_c(s,t)$ and $R(s,t)$ are comparable in magnitude, that is in the Coulomb interference region. It is only in this region that the phase of the residual amplitude can be obtained. If we can observe $\sigma'(s,t_1)$ and $\sigma'(s,t_2)$ at values of t_1 and t_2 sufficiently close together that we can ignore the variation of R with t, then Eq. (5.2) serves to identify both the magnitude and phase of $R(s,t)$ in the region around t_1 and t_2. Such a procedure has been employed in the analysis[3] of p-α and p-D scattering at 40 MeV.

At very high energies, π-p and p-p elastic-scattering data are particularly simple to analyze. Beyond the Coulomb interference region, a plot of $\ln\sigma=\ln\sigma'$ versus t is linear. That indicates that $\ln|R(s,t)|=a+bt$ for small t. Since $|R(s,t)|$ is thus known in the Coulomb interference region, the experimental data for $\sigma'(s,t)$ gives the phase of $R(s,t)$.

A study of the available high-energy data in the Coulomb interference region indicates that at small t $\arg R(s,t)$, like $\ln|R(s,t)|$, is linear in t. Therefore, in this case, it is especially easy to parametrize $R(s,t)$ as

$$R(s,t)\equiv e^{\frac{1}{2}(a+bt)}\,, \tag{5.3}$$

where a and b may be complex functions of s. The form for $R(s,t)$ in Eq. (5.3) is a very convenient form for use in Eq. (3.3). In a manner similar to the example of Sec. I, Eq. (1.16) to Eq. (1.18), we can obtain $\phi(s,t)$ from Eq. (5.3). The result is

$$\mathrm{Re}\,\phi(s,t)=-\eta(s)\left[\ln\left(\frac{|b|\,t(\pi)}{2\gamma}\right)+\frac{\mathrm{Re}\,b}{2}t\right]+O(t^2) \tag{5.4}$$

and

$$\mathrm{Im}\,\phi(s,t)=-\eta(s)\left[\tan^{-1}\left(\frac{\mathrm{Im}b}{\mathrm{Re}b}\right)+\frac{\mathrm{Im}b}{2}t\right]+O(t^2)\,, \tag{5.5}$$

where $\ln\gamma\equiv0.5772\cdots$ and $t(\pi)$ is given after Eq. (3.4). These results depend on the assumption that

$$\exp[-|b|\,t(\pi)]\ll1\,.$$

If we parametrize $f_N(s,t)$ in the same way as Eq. (5.3), namely,

$$f_N(s,t)\equiv\exp\tfrac{1}{2}(a_N+b_Nt)\,, \tag{5.6}$$

then Eq. (3.2) and Eqs. (5.4)–(5.5) imply

$$\mathrm{Re}\,a_N=\mathrm{Re}\,a+2\eta(s)\tan^{-1}(\mathrm{Im}b/\mathrm{Re}\,b)\,, \tag{5.7}$$

$$\mathrm{Im}\,a_N=\mathrm{Im}\,a-2\eta(s)\ln\left(\frac{t(\pi)}{2\gamma}|b|\right), \tag{5.8}$$

$$\mathrm{Re}\,b_N=\mathrm{Re}\,b+\eta(s)\,\mathrm{Im}\,b\,, \tag{5.9}$$

$$\mathrm{Im}\,b_N=\mathrm{Im}\,b-\eta(s)\,\mathrm{Re}\,b\,. \tag{5.10}$$

From Eqs. (5.7) and (5.8) we see that in the forward direction the magnitude of the nuclear amplitude differs somewhat from $|R(s,0)|$. On the other hand the phase of the nuclear amplitude in the forward direction can differ markedly from the phase of $R(s,0)$. We note that $t(\pi)\rightarrow s$ as $s\rightarrow\infty$ so that the correction in Eq. (5.8) can increase as $\ln s$ as $s\rightarrow\infty$.