baryons is even the decay width is

$$\Gamma_+(B_{s'}\to B_s+\pi)=\frac{g^2}{4\pi}\frac{2T+1}{2s'+1}\frac{p^{2(s'-s)+3}\sum_\lambda(\epsilon_{s\lambda}{}^+)^2}{M_{s'}(E_s+M_s)}\times\Big(\prod_{k=1}^{n}C(s+k-1,1,s+k;\lambda 0)\Big)^2. \quad (5.5)$$

If the relative γ parity is odd the width is

$$\Gamma_-(B_{s'}\to B_s+\pi)=\frac{g^2}{4\pi}\frac{2T+1}{2s'+1}\frac{p^{2(s'-s)+1}(E_s+M_s)}{M_{s'}}\times\sum_\lambda(\epsilon_{s\lambda}{}^-)^2\Big(\prod_{k=1}^{n}C(s+k-1,1,s+k;\lambda 0)\Big)^2. \quad (5.6)$$

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Theory of Proton-Proton Bremsstrahlung

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The proton-proton bremsstrahlung cross section is derived to first order in the electromagnetic interaction and the nucleon-nucleon t matrix. The cross section is expressed in terms of the nucleon-nucleon phase shifts and the quasi-phase parameters needed to describe t off the energy shell. The formulas are given in a form suitable for calculation.

I. INTRODUCTION

ELASTIC nucleon-nucleon scattering experiments give information only about on-energy-shell matrix elements of the nucleon-nucleon interaction. Off-energy-shell elements are needed in any fundamental calculation of a nuclear system containing three or more nucleons; that is, in any calculation which tries to understand the many-body system in terms of the two-body interaction. In practice either the off-energy-shell effects are ignored if they are believed to be small or else they are calculated from a nucleon-nucleon potential which has been fitted to the elastic-scattering data. In this latter case the potential is a device for extrapolating matrix elements off the energy shell.

It is clearly desirable to have measurements which give information on the off-energy-shell matrix elements and which therefore can be used to judge various potential models. The process of proton-proton bremsstrahlung,

$$p+p\to p+p+\gamma,$$

seems well suited to this purpose since it is essentially a two-body inelastic event, the γ ray being only weakly coupled to the system.

In an earlier paper[1] the theory of p-p bremsstrahlung was outlined and some preliminary results were given. Since then the measurements have been made at a number of laboratories[2–6] and more are being planned. Also a number of other calculations have been published,[7–10] some of which are based on this earlier work.[7,8,10] In this paper we present a detailed discussion of the theory. More recent calculations based on some potential models are given in a forthcoming paper[11] and a more phenomenological analysis will be published shortly.[12]

In the typical bremsstrahlung experiment the two scattered protons are detected in coincidence telescopes arranged at angles θ_1 and θ_2 on either side of the incident beam (see Fig. 1). Elastic scattering events are eliminated by making the angle between the two telescopes less than 90°. The measurement of the energy of either scattered proton determines the direction and energy

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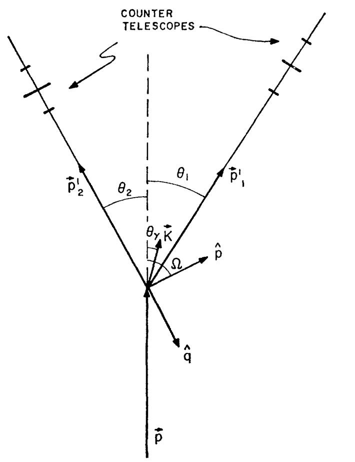

FIG. 1. Kinematics of proton-proton bremsstrahlung. Here $\mathbf{p}$ is the momentum of the incident proton in the laboratory, $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are the laboratory momenta of the scattered protons, and $\mathbf{K}$ is the momentum of the photon. The unit vector $\hat{n}$ is into the paper and $\hat{p}$ and $\hat{q}$ are perpendicular unit vectors in the scattering plane.

of the γ ray and so by measuring the energy of both protons accidental coincidences can be largely eliminated. In practice then at least five out of nine final-state parameters are measured and these suffice to completely determine the kinematics of the event.

II. FORMULATION

Our problem is to calculate the cross section for scattering from an initial state,[13] $|i\rangle=|\mathbf{p}_1,\mathbf{p}_2\rangle$, of two protons of momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ to a final state, $|f\rangle=|\mathbf{p}_1',\mathbf{p}_2',\mathbf{K}\rangle$ of two protons of momenta $\mathbf{p}_1'$ and $\mathbf{p}_2'$ and a γ ray of momentum $\mathbf{K}$. The Hamiltonian for this free system is

$$H_0=K_1+K_2+K_\gamma,$$

where K_1, K_2, and K_γ are the kinetic-energy operators for protons 1 and 2 and the photon, respectively. The states $|i\rangle$ and $|f\rangle$ are eigenstates of H_0 with the same energy E. Letting $e(p)=\frac{1}{2}p^2/m$, we have

$$E_i=E=e(p_1)+e(p_2) \tag{2.1}$$

and

$$E_f=E=e(p_1')+e(p_2')+K=E'+K, \tag{2.2}$$

respectively, for the initial and final states.

The total Hamiltonian is

$$H=K_1+K_2+K_\gamma+V_N+V_{\mathrm{em}},$$

where V_N is the nuclear potential between the protons and V_{em} is the electromagnetic coupling of the γ ray to the protons. Further we write

$$H=H_N+K_\gamma+V_{\mathrm{em}},$$

where

$$H_N=K_1+K_2+V_N$$

is the Hamiltonian for free nucleon-nucleon scattering.

The operator for transitions between eigenstates of H_0 is given exactly by

$$T(E)=(V_N+V_{\mathrm{em}})+(V_N+V_{\mathrm{em}})\times[E+i\epsilon-H_N-K_\gamma-V_{\mathrm{em}}]^{-1}(V_N+V_{\mathrm{em}}),$$

with the usual understanding that $\epsilon\to 0^+$ after the appropriate integrations have been performed. Using the operator identity $(A-B)^{-1}=A^{-1}+(A-B)^{-1}BA^{-1}$ and keeping only terms to first order in V_{em} one obtains

$$T=V_N+V_{\mathrm{em}}+V_NG_NV_N+V_{\mathrm{em}}G_NV_N+V_NG_NV_{\mathrm{em}}+V_NG_NV_{\mathrm{em}}G_NV_N,$$

where $G_N=[E+i\epsilon-H_N-K_\gamma]^{-1}$. Next we take the matrix elements of T for the photon states; i.e., between an initial state with no photon and a final state with one photon of momentum $\mathbf{K}$. Assuming that V_N is diagonal in the photon states it can be shown that

$$\langle\mathbf{K}|T(E)|0\rangle=t(E)\delta^3(\mathbf{K})+\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle+\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle G_0(E)t(E)+t(E')G_0(E')\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle+t(E')G_0(E')\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle G_0(E)t(E). \tag{2.3}$$

Here $G_0(E)=[E+i\epsilon-K_1-K_2]^{-1}$ is the Green's function for the free nucleon-nucleon system and

$$t(E)=V_N+V_NG_0(E)t(E)$$

is the transition operator for nucleon-nucleon scattering.

The first term in Eq. (2.3) describes normal elastic nucleon-nucleon scattering without photon emission; the second term describes photon emission without nuclear scattering and is not kinematically allowed. The third and fourth terms describe photon emission after and before nuclear scattering, respectively, and are the terms we shall calculate. The last term describes photon emission between two nuclear interactions and has been shown[14] to be small relative to the single scattering terms.

Taking, then,

$$\langle\mathbf{K}|T(E)|0\rangle=\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle G_0(E)t(E)+t(E')G_0(E')\langle\mathbf{K}|V_{\mathrm{em}}|0\rangle,$$

we get for the complete matrix element between initial and final proton states

$$\langle f|T|i\rangle\equiv\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{K}|T(E)|\mathbf{p}_1,\mathbf{p}_2,0\rangle$$
$$=\int d^3k_1d^3k_2\{\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{K}|V_{\mathrm{em}}|\mathbf{k}_1,\mathbf{k}_2,0\rangle\times[E+i\epsilon-e(k_1)-e(k_2)]^{-1}\langle\mathbf{k}_1,\mathbf{k}_2|t(E)|\mathbf{p}_1,\mathbf{p}_2\rangle+\langle\mathbf{p}_1',\mathbf{p}_2'|t(E')|\mathbf{k}_1,\mathbf{k}_2\rangle[E+i\epsilon-e(k_1)-e(k_2)]^{-1}\times\langle\mathbf{k}_1,\mathbf{k}_2,\mathbf{K}|V_{\mathrm{em}}|\mathbf{p}_1,\mathbf{p}_2,0\rangle\}, \tag{2.4}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the intermediate momenta of protons 1 and 2.

[13] We use a system of units in which $\hbar=c=1$. Also our momentum states are normalized so that $\langle\mathbf{r}|\mathbf{p}\rangle=(2\pi)^{-3/2}\exp(i\mathbf{p}\cdot\mathbf{r})$.

[14] M. I. Sobel, thesis, Harvard University, 1963 (unpublished); M. I. Sobel (to be published).

The contribution to $V_{\rm em}$ from meson currents in the proton-proton system is unknown. We must therefore neglect such contributions and include in $V_{\rm em}$ only the coupling of the electromagnetic field to the proton currents. Thus we take

$$\langle \mathbf{k}_1,\mathbf{k}_2,\mathbf{K}|V_{\rm em}|\mathbf{p}_1,\mathbf{p}_2,0\rangle = -(\epsilon/2\pi m\sqrt{K})[\mathfrak{A}(\mathbf{p}_1)\delta^3(\mathbf{p}_2-\mathbf{k}_2)\delta^3(\mathbf{p}_1-\mathbf{K}-\mathbf{k}_1) + \mathfrak{A}(\mathbf{p}_2)\delta^3(\mathbf{p}_1-\mathbf{k}_1)\delta^3(\mathbf{p}_2-\mathbf{K}-\mathbf{k}_2)], \quad (2.5)$$

where

$$\mathfrak{A}(\mathbf{p}_i)=[\mathbf{p}_i\cdot\hat{e}-i\tfrac{1}{2}\mu\boldsymbol{\sigma}_i\cdot(\mathbf{K}\times\hat{e})]. \quad (2.6)$$

Here $\epsilon=1/\sqrt{137}$ is the charge and $\mu=2.79$ is the magnetic moment of the proton, $\hat{e}$ is the polarization of the photon, and $\boldsymbol{\sigma}_i$ is the Pauli spin operator of the ith nucleon. Since the photon is real we have $\mathbf{K}\cdot\hat{e}=0$.

Using Eq. (2.5) together with the similar expression for $\langle \mathbf{p}_1',\mathbf{p}_2',\mathbf{K}|V_{\rm em}|\mathbf{k}_1,\mathbf{k}_2,0\rangle$ in Eq. (2.4) we get

$$\langle f|T|i\rangle = -(\epsilon/2\pi m\sqrt{K})\{\mathfrak{A}(\mathbf{p}_1')[E-e(\mathbf{p}_1'+\mathbf{K})-e(\mathbf{p}_2')]^{-1}\langle \mathbf{p}_1'+\mathbf{K},\mathbf{p}_2'|t(E)|\mathbf{p}_1,\mathbf{p}_2\rangle + \langle \mathbf{p}_1',\mathbf{p}_2'|t(E)|\mathbf{p}_1-\mathbf{K},\mathbf{p}_2\rangle[E'-e(\mathbf{p}_1-\mathbf{K})-e(\mathbf{p}_2)]^{-1}\mathfrak{A}(\mathbf{p}_1) + \mathfrak{A}(\mathbf{p}_2')[E-e(\mathbf{p}_2'+\mathbf{K})-e(\mathbf{p}_1')]^{-1} \times\langle \mathbf{p}_1',\mathbf{p}_2'+\mathbf{K}|t(E)|\mathbf{p}_1,\mathbf{p}_2\rangle + \langle \mathbf{p}_1',\mathbf{p}_2'|t(E')|\mathbf{p}_1,\mathbf{p}_2-\mathbf{K}\rangle[E'-e(\mathbf{p}_1)-e(\mathbf{p}_2-\mathbf{K})]^{-1}\mathfrak{A}(\mathbf{p}_2)\}. \quad (2.7)$$

The four terms in brackets will, for reference purposes, be designated a, b, c, d, in the order in which they are written. Each term is the amplitude for the photon to be emitted by the proton of momentum $\mathbf{p}_1'$, $\mathbf{p}_1$, $\mathbf{p}_2'$, $\mathbf{p}_2$, respectively.

Because of the identity of the two protons, $\langle f|T|i\rangle$ must be antisymmetrized. This is done automatically if we use the properly antisymmetrized form for the matrix elements of $t(E)$.

The energy denominators as written in Eq. (2.7) are non-relativistic. Thus for the first term we have, using Eq. (2.2),

$$\Delta E_a=E-e(\mathbf{p}_1'+\mathbf{K})-e(\mathbf{p}_2')=K-\mathbf{K}\cdot\mathbf{p}_1'/m-\tfrac{1}{2}K^2/m.$$

At the energies normally considered $\frac{1}{2}K^2/m$ is very much smaller than K and can be neglected. Then ΔE_a is very close to the relativistic form

$$\Delta E_a=(K[p_1'^2+m^2]^{1/2}-\mathbf{K}\cdot\mathbf{p}_1')/m.$$

Corresponding expressions for the relativistic forms of the energy denominators in all four cases are given in Table I. In actual calculations the relativistic denominators are used.

We next consider the matrix elements of the t matrices that appear in Eq. (2.7). Actually these are matrix elements in momentum space and matrices in spin space. Thus, for example, in the first term in (2.7) we have the spin matrix $\langle \mathbf{p}_1'+\mathbf{K},\mathbf{p}_2'|t(E)|\mathbf{p}_1,\mathbf{p}_2\rangle$. This matrix can be written in terms of the off-energy-shell center-of-mass scattering matrix, $M_a(\mathbf{k}_a',\mathbf{k}_a)$, which is a matrix in spin space, each element of which is a function of the initial and final relative center-of-mass momenta, $\mathbf{k}_a$ and $\mathbf{k}_a'$. Since these matrices are off the energy shell $|\mathbf{k}_a|$ is not equal to $|\mathbf{k}_a'|$. The energy E corresponds to the initial state only. The relation between $t(E)$ and M is

$$\langle \mathbf{p}_1'+\mathbf{K},\mathbf{p}_2'|t(E)|\mathbf{p}_1,\mathbf{p}_2\rangle=\delta^3(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2) \times\langle \mathbf{k}_a'|t(e(k_a))|\mathbf{k}_a\rangle=-(2\pi^2 m)^{-1}M_a(\mathbf{k}_a',\mathbf{k}_a) \times\delta^3(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2). \quad (2.8)$$

We shall discuss the calculation of M in detail in the next section. The t matrices in the other terms in Eq. (2.7) have a similar expression in terms of M_x ($x=a$, b, c, d) with the same delta function of momentum.

Then we have

$$\langle f|T|i\rangle=(\epsilon/4\pi^3 m^2\sqrt{K}) \times\mathfrak{M}\delta^3(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2), \quad (2.9)$$

where

$$\mathfrak{M}=\{\mathfrak{A}(\mathbf{p}_1')(M_a/\Delta E_a)+(M_b/\Delta E_b)\mathfrak{A}(\mathbf{p}_1) +\mathfrak{A}(\mathbf{p}_2')(M_c/\Delta E_c)+(M_d/\Delta E_d)\mathfrak{A}(\mathbf{p}_2)\}. \quad (2.10)$$

To calculate the cross section we need $|\langle f|T|i\rangle|^2$ averaged over initial proton spins and summed over final proton spins and the photon polarization. This is given by

$$(\epsilon^2/16\pi^6 m^4 K)\langle\tfrac{1}{4}\,\mathrm{tr}(\mathfrak{M}^\dagger\mathfrak{M})\rangle\delta^3(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2)$$

where $\langle\cdots\rangle$ here indicates the sum over proton polarization. In the laboratory system with $\mathbf{p}_2=0$ and $\mathbf{p}_1$ the momentum of the incident beam, the cross-section then is

$$d\sigma=(\epsilon^2/\pi^2 m^3 p_1 K)\langle\tfrac{1}{4}\,\mathrm{tr}(\mathfrak{M}^\dagger\mathfrak{M})\rangle \times\delta^3(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2) \times\delta(E_f-E_i)d^3p_1'd^3p_2'd^3K. \quad (2.11)$$

In the experiments under consideration five of the nine parameters describing the final state are measured and the other four are determined by the conservation laws. Thus we must integrate over four variables in Eq. (2.11). For example, if we measure the solid angles of the two final protons and the angle θ_γ between the photon and the incident beam, then the kinematics is determined. In the case in which all the particles are

TABLE I. Relativistic expressions for the energy denominators in Eq. (2.10).

$\Delta E_a=([p_1'^2+m^2]^{1/2}K-\mathbf{K}\cdot\mathbf{p}_1')/m$
$\Delta E_b=-([p_1^2+m^2]^{1/2}K-\mathbf{K}\cdot\mathbf{p}_1)/m$
$\Delta E_c=([p_2'^2+m^2]^{1/2}K-\mathbf{K}\cdot\mathbf{p}_2')/m$
$\Delta E_d=-([p_2^2+m^2]^{1/2}K-\mathbf{K}\cdot\mathbf{p}_2)/m$

coplanar this cross section is

$$d\sigma/d\Omega_1 d\Omega_2 d\theta_\gamma = (\epsilon^2/\pi^2 m^3 p_1 K)\langle \tfrac{1}{4}\,\mathrm{tr}(\mathfrak{M}^\dagger \mathfrak{M})\rangle \mathfrak{F}, \quad (2.12)$$

where $\mathfrak{F}$ is the phase-space factor:

$$\mathfrak{F} = \left[\int \delta^3(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{K} - \mathbf{p}_1 - \mathbf{p}_2) \times \delta(E_f - E_i) p_1'^2 dp_1' p_2'^2 dp_2' \times K^2 dK \sin\theta_\gamma d\phi_\gamma\right]_{\phi_\gamma=0}$$
$$= p_1'^2 p_2'^2 K |\sin(\theta_1+\theta_2) + \beta_2 \sin(\theta_\gamma - \theta_1) - \beta_1 \sin(\theta_2+\theta_\gamma)|^{-1}. \quad (2.13)$$

Here β_i is the velocity of proton i ($i=1, 2$) and θ_i is the angle between $\mathbf{p}_i'$ and $\mathbf{p}_1$. In the usual experiment θ_1 and θ_2 are on opposite sides of the incident beam and are both taken to be positive; the angle θ_γ between $\mathbf{K}$ and $\mathbf{p}_1$ is taken to be positive when K is on the same side as $\mathbf{p}_1'$ and is taken to be negative when it is on the same side as $\mathbf{p}_2'$.

In all experiments to date it has been the solid angles of both protons and e_1', the energy of proton one, that are actually measured. The data can be easily transformed into the form given by Eq. (2.12) or they can be presented directly in the form $d\sigma/d\Omega_1 d\Omega_2 de_1'$. When all the particles are coplanar this cross section is

$$d\sigma/d\Omega_1 d\Omega_2 de_1' = (\epsilon^2/\pi^2 m^3 p_1 K)\langle \tfrac{1}{4}\,\mathrm{tr}(\mathfrak{M}^\dagger \mathfrak{M})\rangle \mathfrak{F}', \quad (2.14)$$

where

$$\mathfrak{F}' = p_1'^2 p_2'^2 \beta_1^{-1} |\beta_2 - \cos(\theta_2+\theta_\gamma)|^{-1}. \quad (2.15)$$

In I the cross section was calculated in the form given by Eq. (2.14) and was plotted as a function of e_1' for various proton angles. There are two disadvantages in expressing the cross section in this form: (1) the solution of the kinematic equations is double-valued, that is, there are two values of e_2' for each value of e_1'; (2) the cross section has kinematic singularities when plotted as a function of e_1'. These singularities occur at both ends of the allowed range of e_1'. Neither of these problems occurs when the cross section is expressed in the form given by Eq. (2.12); this cross section is well-behaved when plotted as a function of θ_γ over the entire range from 0° to 180°.

III. EVALUATION OF THE M_x OFF THE ENERGY SHELL

In order to calculate $\mathfrak{M}$ we must evaluate the four scattering matrices, M_x ($x=a, b, c, d$), which appear in Eq. (2.10). The matrices M_a and M_c are easier to handle than the other two and we begin our discussion with them. From Eq. (2.8) we have

$$M_y = -2\pi^2 m \langle \mathbf{k}_y' | t(e_y) | \mathbf{k}_y\rangle, \quad (y=a, c),$$

where e_y is the energy of the initial state. This means that (neglecting spins for the moment) M_y has the form

$$M_y = -(m/4\pi)\int \exp(-i\mathbf{k}_y'\cdot\mathbf{r}) V_N(\mathbf{r}) \chi_{\mathbf{k}_y}(\mathbf{r}) d^3r,$$

where $\chi_{\mathbf{k}_y}$ is the exact scattering state which goes asymptotically as

$$\chi_{\mathbf{k}_y} \underset{r\to\infty}{\longrightarrow} \exp(i\mathbf{k}_y\cdot\mathbf{r}) + M_y r^{-1}\exp(ik_y r).$$

When spin is taken into account, M_y is a 4×4 matrix in spin space. It is customary to take the incident center-of-mass direction as the axis of quantization which defines the representation of M_y. In this representation the on-energy-shell ($|\mathbf{k}_y| = |\mathbf{k}_y'|$) matrix elements of M_y have a well-known expansion in terms of phase shifts $\delta(k_y)$, at the (center-of-mass) energy $\frac{1}{2}k_y^2/m$ and Legendre polynomials of the angle θ_y between $\mathbf{k}_y$ and $\mathbf{k}_y'$. Sobel has shown[15] that off the energy shell these elements have the same expression as is given for the on-energy-shell elements by Stapp *et al.*[16] in their Table III, if off the energy shell the α_{lj}'s have the following form[17] (using the Blatt-Biedenharn parametrization[18]):

$$\begin{aligned}
\alpha_{l,l\pm1} &= 2i(\cos^2\epsilon_{l\pm1} e^{i\delta_{l,l\pm1}}\Delta_{l,l\pm1}^{\pm} + \sin^2\epsilon_{l\pm1} e^{i\delta_{l\pm2,l\pm1}} \times \Delta_{l\pm2,l\pm1}^{\pm}),\\
\alpha_+^{l+1} &= i\sin 2\epsilon_{l+1}(e^{i\delta_{l,l+1}}\Delta_{l,l+1}^{+} - e^{i\delta_{l+2,l+1}}\Delta_{l+2,l+1}^{+}),\\
\alpha_-^{l-1} &= i\sin 2\epsilon_{l-1}(e^{i\delta_{l-2,l-1}}\Delta_{l-2,l-1}^{-} - e^{i\delta_{l,l-1}}\Delta_{l,l-1}^{-}), \quad (3.1)\\
\alpha_l &= 2ie^{i\delta_l}\Delta_l,\\
\alpha_{l,l} &= 2ie^{i\delta_{l,l}}\Delta_{l,l}.
\end{aligned}$$

Here the phase shifts, $\delta(k_y)$, depend on the momentum of the incident state, whereas the quasi-phase parameters $\Delta(k_y',k_y)$ are functions of both the initial and final momenta. On the energy shell $\Delta(k_y,k_y)$ is equal to $\sin\delta(k_y)$, where δ is the phase shift which appears in the exponential factor that multiplies Δ. Off the energy shell the Δ's are given, in a potential model, by integrals over the potential, as described by Sobel.[15] However, this parametrization of M_y does not depend on a potential model. It is based on general considerations of unitarity, time-reversal invariance, and conservation of angular momentum and parity.[19] Thus M_y can be calculated from any assumed set of phases and quasiphases.

In cases b and d we have to evaluate

$$M_z = -2\pi^2 m\langle \mathbf{k}_z' | t(e_z') | \mathbf{k}_z\rangle, \quad (z=b, d),$$

[15] M. I. Sobel, Phys. Rev. **138**, B1517 (1965).

[16] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[17] In the formulas in Ref. 16 (Table III) α_-^{l+1} is used wherever α^{l+1} appears explicitly and α_+^{l-1} is used wherever α^{l-1} appears.

[18] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

[19] M. I. Sobel (to be published).

where e_z' is the energy of the final state. To do this we use time-reversal invariance which gives for the spin and momentum matrix elements of $t(e_z')$ the relation[20]

$$\begin{aligned}\langle\mu',\mathbf{k}_z'|t(e_z')|\mathbf{k}_z,\mu\rangle &= (-1)^{\mu-\mu'}\langle-\mu, -\mathbf{k}_z|t(e_z')|-\mathbf{k}_z', -\mu'\rangle\\ &= (-1)^{\mu-\mu'}\langle-\mu, \mathbf{k}_z|t(e_z')|\mathbf{k}_z', -\mu'\rangle, \qquad (3.2)\end{aligned}$$

the last equality following from parity conservation. Now the spin matrix elements of

$$\tilde{M}_z = -2\pi^2 m\langle\mathbf{k}_z|t(e_z')|\mathbf{k}_z'\rangle, \quad (z=b, d), \qquad (3.3)$$

have phase expansions identical to that of M_y but in terms now of $\delta(k_z')$ and $\Delta(k_z,k_z')$ and with $\mathbf{k}_z'$ as the axis of quantization. Then the relation

$$\langle\mu'|M_z|\mu\rangle = (-1)^{\mu-\mu'}\langle-\mu|\tilde{M}_z|-\mu'\rangle, \quad (z=b, d) \qquad (3.4)$$

enables us to calculate the matrix elements of M_z in terms of phases and quasiphases.

On the energy shell the scattering matrix M has a well-known expansion in terms of the Pauli spin operators of the two protons and the five Wolfenstein parameters,[21] A, B, C, E, F. Off the energy shell we obtain a similar expansion with some important differences. In cases a and c, if we define $\hat{n}$ to be a unit vector in the direction $\mathbf{k}_y\times\mathbf{k}_y'$, $\hat{p}$ to be an arbitrary unit vector in the scattering plane and $\hat{q}=\hat{n}\times\hat{p}$, M_y can be written in the form

$$\begin{aligned}M_y = A_y + B_y\boldsymbol{\sigma}_1\cdot\hat{n}\boldsymbol{\sigma}_2\cdot\hat{n} + C_y(\boldsymbol{\sigma}_1\cdot\hat{n}+\boldsymbol{\sigma}_2\cdot\hat{n})\\ + E_y\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{q} + F_y\boldsymbol{\sigma}_1\cdot\hat{p}\boldsymbol{\sigma}_2\cdot\hat{p}\\ + G_y(\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{p}+\boldsymbol{\sigma}_1\cdot\hat{p}\boldsymbol{\sigma}_2\cdot\hat{q}), \qquad (3.5)\end{aligned}$$

where the six amplitudes A_y, B_y, C_y, E_y, F_y, and G_y are functions of k_y, k_y', and θ_y. These amplitudes can be expressed in terms of the singlet-triplet matrix elements of M. Dropping the subscript y these relations are, for cases a and c:

$$\begin{aligned}A &= \tfrac{1}{4}(2M_{11}+M_{00}+M_{ss}),\\ B &= \tfrac{1}{4}(M_{00}-M_{ss}-2M_{1-1}),\\ C &= i8^{-1/2}(M_{01}-M_{10}),\\ F+E &= \tfrac{1}{2}(M_{11}+M_{1-1}-M_{ss}),\\ F-E &= \tfrac{1}{2}[\sqrt{2}(M_{01}+M_{10})\sin 2\Omega \\ &\qquad +(M_{11}-M_{1-1}-M_{00})\cos 2\Omega], \qquad (3.6)\\ G &= \tfrac{1}{4}[\sqrt{2}(M_{01}+M_{10})\cos 2\Omega\\ &\qquad -(M_{11}-M_{1-1}-M_{00})\sin 2\Omega].\end{aligned}$$

Here Ω is the angle between $\hat{p}$ and the axis of quantization. In cases a and c the axis of quantization is along $\mathbf{k}_y$ which is also the direction of the incident laboratory momentum $\mathbf{p}_1$ (see Fig. 1). We use the convention that Ω is positive when $\hat{p}$ is on the same side of $\mathbf{p}_1$ as $\mathbf{p}_1'$ and is negative when it is on the opposite side.

On the energy shell M has an expansion identical to that given by Eqs. (3.5) and (3.6). However, on the energy shell it is customary to take $\hat{p}$ in the direction $\mathbf{k}_y+\mathbf{k}_y'$ so that $\Omega=\frac{1}{2}\theta_y$. But time-reversal invariance implies that on the energy shell[21]

$$\sqrt{2}(M_{01}+M_{10})\cos\theta_y = (M_{11}-M_{1-1}-M_{00})\sin\theta_y,$$

so that $G=0$. No such relation exists off the energy shell and so there is no direction of $\hat{p}$ for which G vanishes.

Some care must be taken in applying Eq. (3.6) to cases b and d. We want $\tilde{M}_z$ expanded in terms of the Pauli matrices and the same unit vectors, $\hat{n}$, $\hat{p}$, and $\hat{q}$, as M_y. But since $\tilde{M}_z$ is calculated with initial momentum $\mathbf{k}_z'$, so that spin is quantized along $\mathbf{k}_z'$, the angle which appears in (3.6) is no longer Ω. Applying Eqs. (3.2) and (3.6) together with the standard symmetry relations[16]

$$M_{-1-1}=M_{11}, \quad M_{-11}=M_{1-1}, \quad M_{01}=-M_{0-1},$$
$$M_{10}=-M_{-10},$$

it can be shown that M_z $(z=b, d)$ has the expansion given in Eq. (3.5) with G replaced by $-G$. The amplitudes A_z, $\cdots$ G_z are in turn given by expressions similar to Eq. (3.6) in terms of the matrix elements of $\tilde{M}_z$ and the angle $\tilde{\Omega}$ (instead of Ω) between $\hat{p}$ and the axis of quantization, which is now $\mathbf{k}_z'$. In terms of Ω and the angle α between $\mathbf{k}_z'$ and $\mathbf{p}_1$ we have $\tilde{\Omega}=\alpha-\Omega$.

To summarize, M_a and M_c are calculated in terms of the singlet-triplet matrix elements by using Eqs. (3.5) and (3.6). These matrix elements, in turn, are given by the standard formulas in terms of the angle θ_y between $\mathbf{k}_y'$ and $\mathbf{k}_y$ and the α_{lj} given in Eq. (3.1). We note that k_y is the same in both cases a and c, so that the phase shifts are the same in both cases; however, the scattering angles and the quasiphases are different since the $\mathbf{k}_y'$ are different in the two cases. The matrices M_b and M_d are given by the same expressions with the sign of G reversed, Ω replaced by $\tilde{\Omega}$, and k_z and k_z' interchanged everywhere. This means, in particular, that the phases are calculated for the momenta $\mathbf{k}_z'$, which are the same in these two cases.

IV. EVALUATION OF $\langle\frac{1}{4}\,\mathrm{tr}\,(\mathfrak{M}^\dagger\mathfrak{M})\rangle$

The scattering matrix, M_x, can be put in the form of Eq. (3.5) for all four cases so that $\mathfrak{M}$, Eq. (2.10), can be written in terms of the amplitudes A_x, $\cdots$, G_x. To do this most conveniently it is useful to pick the unit vectors $\hat{n}$, $\hat{p}$, $\hat{q}$ to be the same for all four terms. This can be done easily in the case of coplanar scattering for then $\mathbf{k}_x\times\mathbf{k}_x'$, the direction of $\hat{n}$, is the same for all terms and all we have to do is to pick $\hat{p}$ to be the same also.

While the choice of $\hat{p}$ is arbitrary, the sum over photon polarization is considerably simplified if $\hat{p}$ is

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[20] J. J. Sakurai, *Invariance Principles and Elementary Particles* (Princeton University Press, Princeton, New Jersey, 1964).

[21] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

TABLE II. The coefficients X_i and Y_i in Eq. (4.4). Only the nonzero coefficients are listed. In this table the amplitudes $A_x, \cdots G_x$ are understood to be divided by the corresponding energy denominator ΔE_x from Table I. Here $\xi=\frac{1}{2}\mu K$.

i	O_i	X_i
1	1	$A_a\mathbf{p}_1'\cdot\hat{q}+A_b\mathbf{p}_1\cdot\hat{q}+A_c\mathbf{p}_2'\cdot\hat{q}-i\xi(C_a+C_b+C_c+C_d)$
2	$\boldsymbol{\sigma}_1\cdot\hat{n}$	$C_a\mathbf{p}_1'\cdot\hat{q}+C_b\mathbf{p}_1\cdot\hat{q}+C_c\mathbf{p}_2'\cdot\hat{q}-i\xi(A_a+A_b+B_c+B_d)$
3	$\boldsymbol{\sigma}_2\cdot\hat{n}$	$C_a\mathbf{p}_1'\cdot\hat{q}+C_b\mathbf{p}_1\cdot\hat{q}+C_c\mathbf{p}_2'\cdot\hat{q}-i\xi(B_a+B_b+A_c+A_d)$
4	$\boldsymbol{\sigma}_1\cdot\hat{n}\boldsymbol{\sigma}_2\cdot\hat{n}$	$B_a\mathbf{p}_1'\cdot\hat{q}+B_b\mathbf{p}_1\cdot\hat{q}+B_c\mathbf{p}_2'\cdot\hat{q}-i\xi(C_a+C_b+C_c+C_d)$
5	$\boldsymbol{\sigma}_1\cdot\hat{p}\boldsymbol{\sigma}_2\cdot\hat{q}$	$G_a\mathbf{p}_1'\cdot\hat{q}+G_b\mathbf{p}_1\cdot\hat{q}+G_c\mathbf{p}_2'\cdot\hat{q}-\xi(E_a-E_b-F_c+F_d)$
6	$\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{p}$	$G_a\mathbf{p}_1'\cdot\hat{q}+G_b\mathbf{p}_1\cdot\hat{q}+G_c\mathbf{p}_2'\cdot\hat{q}+\xi(F_a-F_b-E_c+E_d)$
7	$\boldsymbol{\sigma}_1\cdot\hat{p}\boldsymbol{\sigma}_2\cdot\hat{p}$	$F_a\mathbf{p}_1'\cdot\hat{q}+F_b\mathbf{p}_1\cdot\hat{q}+F_c\mathbf{p}_2'\cdot\hat{q}-\xi(G_a-G_b+G_c-G_d)$
8	$\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{q}$	$E_a\mathbf{p}_1'\cdot\hat{q}+E_b\mathbf{p}_1\cdot\hat{q}+E_c\mathbf{p}_2'\cdot\hat{q}+\xi(G_a-G_b+G_c-G_d)$
		Y_i
9	$\boldsymbol{\sigma}_1\cdot\hat{}$	$i\xi(A_a+A_b+E_c+E_d)$
10	$\boldsymbol{\sigma}_2\cdot\hat{q}$	$i\xi(E_a+E_b+A_c+A_d)$
11	$\boldsymbol{\sigma}_1\cdot\hat{p}$	$i\xi(i(C_a-C_b)+G_c+G_d)$
12	$\boldsymbol{\sigma}_2\cdot\hat{p}$	$i\xi(G_a+G_b+i(C_c-C_d))$
13	$\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{n}$	$i\xi(C_a+C_b-i(G_c-G_d))$
14	$\boldsymbol{\sigma}_1\cdot\hat{n}\boldsymbol{\sigma}_2\cdot\hat{}$	$i\xi(-i(G_a-G_b)+C_c+C_d)$
15	$\boldsymbol{\sigma}_1\cdot\hat{n}\boldsymbol{\sigma}_2\cdot\hat{p}$	$\xi(F_a-F_b-B_c+B_d)$
16	$\boldsymbol{\sigma}_1\cdot\hat{p}\boldsymbol{\sigma}_2\cdot\hat{n}$	$-\xi(B_a-B_d-F_c+F_d)$

taken to be in the direction of the photon momentum, $\mathbf{K}$. For then the polarization vector $\hat{e}$ must lie in the $\hat{n}$-$\hat{q}$ plane so that

$$\hat{e}=\hat{q}\cos\phi+\hat{n}\sin\phi\,,$$
$$\mathbf{K}\times\hat{e}=K(\hat{n}\cos\phi-\hat{q}\sin\phi)\,, \tag{4.1}$$

where ϕ is the angle between $\hat{e}$ and $\hat{q}$. To sum over photon polarization we must integrate $d\phi/2\pi$ from 0 to 2π. That is,

$$\langle\tfrac{1}{4}\operatorname{tr}(\mathfrak{M}^\dagger\mathfrak{M})\rangle=(2\pi)^{-1}\int\tfrac{1}{4}\operatorname{tr}(\mathfrak{M}^\dagger\mathfrak{M})d\phi\,. \tag{4.2}$$

Using Eq. (4.1) we write Eq. (2.6) in the form

$$\mathfrak{A}(\mathbf{p}_x)=\mathbf{p}_x\cdot\hat{q}\cos\phi-i\tfrac{1}{2}K\mu(\boldsymbol{\sigma}_x\cdot\hat{n}\cos\phi-\boldsymbol{\sigma}_x\cdot\hat{q}\sin\phi)\,,$$
$$(x=a,\,b,\,c,\,d)\,. \tag{4.3}$$

In this notation we have $\mathbf{p}_a=\mathbf{p}_1'$, $\mathbf{p}_b=\mathbf{p}_1$, $\mathbf{p}_c=\mathbf{p}_2'$, $\mathbf{p}_d=\mathbf{p}_2=0$, $\boldsymbol{\sigma}_a=\boldsymbol{\sigma}_b=\boldsymbol{\sigma}_1$, and $\boldsymbol{\sigma}_c=\boldsymbol{\sigma}_d=\boldsymbol{\sigma}_2$. Putting Eqs. (3.5) and (4.3) into Eq. (2.10) we find that $\mathfrak{M}$ can be written in the form

$$\mathfrak{M}=\sin\phi\sum_{i=1}^{16}X_iO_i+\cos\phi\sum_{i=1}^{16}Y_iO_i\,, \tag{4.4}$$

where the O_i are the sixteen independent spin operators

$$1,\ \boldsymbol{\sigma}_1\cdot\hat{p},\ \boldsymbol{\sigma}_1\cdot\hat{n},\ \boldsymbol{\sigma}_1\cdot\hat{q},\ \boldsymbol{\sigma}_2\cdot\hat{p},\ \boldsymbol{\sigma}_2\cdot\hat{n},\ \boldsymbol{\sigma}_2\cdot\hat{q}\,,$$

and the nine independent products of these.

The coefficients X_i and Y_i are linear combinations of the amplitudes $A_x, \cdots, G_x$, divided by their approximate energy denominator. The expressions for these coefficients are given in Table II, where for simplicity we have written A_x to mean $A_x/\Delta E_x$, etc. Note that for each operator O_i either X_i or Y_i is zero. This is a consequence of the parity invariance of the electromagnetic interaction.

The O_i satisfy the relation

$$\tfrac{1}{4}\operatorname{tr}(O_i\,O_j)=\delta_{ij}\,,$$

so we have

$$\langle\tfrac{1}{4}\operatorname{tr}(\mathfrak{M}^\dagger\mathfrak{M})\rangle=\tfrac{1}{2}\sum[|X_i|^2+|Y_i|^2]\,, \tag{4.5}$$

the factor of $\frac{1}{2}$ coming from the sum over polarization. The bremsstrahlung cross section is given by Eq. (4.5) together with either Eqs. (2.12) or (2.14).

V. CONCLUSION

In this paper the proton-proton bremsstrahlung cross section has been expressed directly in terms of the phase shifts and quasi-phase parameters of the proton-proton interaction. The phase shifts are known from the extensive measurements of elastic p-p scattering, but there are no measurements of the quasiphase. It is hoped that investigation of the bremsstrahlung cross section will provide valuable information about these important parameters.

The only approximations made here (apart from treating the electromagnetic interaction only to first order) were the neglect of meson currents and the neglect of the rescattering term in Eq. (2.3). The rescattering term was calculated by Sobel[14] in one instance and was found to be very much smaller than the two main terms. In any event this term could be calculated, should it prove necessary, albeit with about an order of magnitude more work. However, there is at present no completely satisfactory way to estimate the contribution of the meson currents. This problem deserves further investigation.

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