# Pion-Nucleon Scattering in the $P_{11}$ State*

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The $P_{11}$ pion-nucleon scattering amplitude is studied in order to determine the dynamical status of the nucleon. A simple one-channel static model is constructed in which a good fit to the low-energy data is obtained with a Castillejo-Dalitz-Dyson (CDD) pole. If there is, in fact, such a pole, with the strength indicated by our model, then one must conclude that other channels play a significant, although not overwhelming, part in forming the nucleon. It is also possible to reconcile all the available information with a model containing no CDD poles in which the nucleon is formed to a good approximation in the pion-nucleon channel alone. A few ways by which one may hope to choose between the two models are discussed. It is shown how the success of certain "reciprocal bootstrap" calculations can be understood, whether or not a CDD pole is required. The large inelasticity and phase shift in the $P_{11}$ pion-nucleon channel at energies of 400 to 600 MeV are probably due to coupling with the $\pi\pi N$ channel, with all three particles in relative $s$ waves. This latter channel seems to be significant for the properties of the 1400-MeV (Roper) resonance, but probably plays a small role in the formation of the nucleon.

## 1. INTRODUCTION

THERE have been numerous pion-nucleon phase-shift analyses carried out recently.[1–6] All of them find a large increasing phase shift in the $P_{11}$ ($I=J=\frac{1}{2}$, $p$ wave) channel over the range of pion kinetic energies 300 to 700 MeV (Fig. 1). In fact, a large width resonance having the same quantum numbers as the nucleon and a mass of about 1400 MeV, often referred to as the Roper resonance, appears to be consistent with all the phase-shift analyses except perhaps for Cence's. Evidence for such a resonance has also been obtained by different methods in which it was actually possible to see a "bump."[6–8]

The aspect of the amplitude that concerns us here is not so much the fact that the phase shift becomes large at high energy, as that it becomes positive at a low energy—slightly below the three-body inelastic threshold, in fact. Rothleitner and Stech have argued that the nucleon can be regarded as being a pion-nucleon bound state if and only if the position of the zero is above the inelastic threshold.[9] We do not believe that this is a meaningful distinction, but we show that the single-channel $D$ function probably contains a Castillejo-Dalitz-Dyson (CDD) pole[10] at the position of the zero. Without this CDD pole one would not obtain the nucleon at the correct position in a one-channel calculation. These conclusions are consistent with a recent argument of Lyth,[11] which allows one to conclude that the elastic channel forces together with elastic unitarity necessarily give rise to a pole appearing either as a bound state or as a low-lying resonance having approximately the correct residue, whether or not a CDD pole is required.

Ever since Chew first suggested that there is a reciprocal bootstrap relationship between the $N$ and the $\Delta$ in pion-nucleon scattering,[12] there have been numerous dynamical calculations of varying complexity performed to obtain these two particles as dynamical states.[13–18] Some have even tried to include inelastic channels.[19–22] Balázs[13] and Doolen *et al.*,[18] have shown how to calculate the nucleon position and residue from a knowledge of the long-range forces and a self-consistent technique for generating the short-range forces. Both obtain the nucleon pole position fairly well, but they find the $P_{11}$ phase shift decreasing and negative, in blatant disagreement with the phase-shift analyses. We shall argue that this discrepancy may be attributed

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[1] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).
[2] P. Auvil, A. Donnachie, A. T. Lea, and C.Lovelace, Phys. Letters **12**, 76 (1964).
[3] B. H. Bransden, R. G. Moorhouse, and P. J. O'Donnell, Phys. Letters **11**, 339 (1964); Phys. Rev. **139**, B1566 (1965).
[4] C. Lovelace, Invited paper at the Royal Society Meeting, February, 1965 (unpublished).
[5] J. Cence, Phys. Letters **20**, 306 (1966).
[6] P. Bareyre, C. Brickman, A. V. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965); P. Bareyre, C. Brickman, G. Valladas, G. Villet, J. Bizard, and J. Seguinot, *ibid.* **8**, 137 (1964).
[7] S. L. Adelman, Phys. Rev. Letters **13**, 486 (1964).
[8] G. Belletini, G. Cocconi, A. N. Diddens, E. Lillethun, J. P. Scanlon, A. M. Shapiro, and A. M. Wetherell, Phys. Letters **18**, 167 (1965).
[9] J. Rothleitner and B. Stech, Z. Physik **180**, 375 (1964).
[10] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[11] D. H. Lyth, University of Birmingham Report, 1966 (unpublished).
[12] G. F. Chew, Phys. Rev. Letters **9**, 223 (1965).
[13] Louis A. P. Balázs, Phys. Rev. **128**, 1935 (1962).
[14] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).
[15] A. W. Martin and K. C. Wali, Nuovo Cimento **31**, 1324 (1964).
[16] V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963).
[17] P. Narayanaswamy and L. K. Pande, Phys. Rev. **136**, B1760 (1964).
[18] G. D. Doolen, T. Kanki, and A. Tubis, Phys. Rev. **142**, 1072 (1966).
[19] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962); **127**, 297 (1962).
[20] J. S. Ball, W. R. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1963).
[21] F. T. Meiere, Phys. Rev. **136**, B1196 (1964).
[22] H. Goldberg and E. L. Lomon, Phys. Rev. **131**, 1290 (1963).

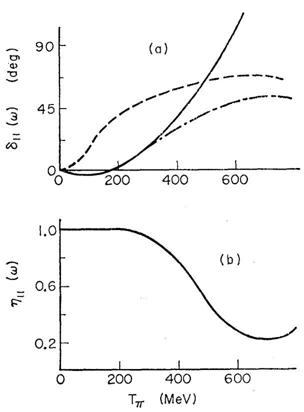


FIG. 1(a). Phase shift $\delta_{11}(\omega)$ as given by Roper *et al.* (solid line) and as calculated from (3.9) (dashed line) and (3.15) (dashed-dotted line). (b). Inelasticity factor $\eta_{11}(\omega)$ as given by Roper *et al.*

to the CDD pole, which they have not included, and that their success in locating the nucleon pole position does not mean that inelastic channels play no role in determining this position.

The pion-nucleon $p$ waves at low energy are of particular interest, because of the availability of good experimental information. Also, the large mass ratio allows some very convenient approximations (static model) to be made. These two facts, together with Chew's simple model, have led many persons to regard this problem as the "hydrogen atom of bootstrap theory." From this point of view out conclusions are rather disappointing, in that we find that other channels are probably required to calculate the nucleon mass, although an approximate one-channel calculation does appear to be possible for the $\Delta$ resonance. The residues of the $N$ and $\Delta$ poles (i.e., the coupling constants) can be approximately obtained from single-channel considerations.

## 2. THE STATIC-MODEL BOOTSTRAP

In this section we shall briefly review the simplest static-model bootstrap for pion-nucleon $p$-wave scattering as given by Chew.[12] Apart from the intrinsic interest of the model itself, our purpose in presenting it here is to have the equations and notation at hand when we argue that such a simple interpretation of the nucleon might not be correct.

The amplitude for $p$-wave pion-nucleon scattering with isotopic spin $I$ and total angular momentum $J$ is defined as

$$a_{2I,2J}(\omega)=\frac{1}{\rho(\omega)}e^{i\delta_{2I,2J}(\omega)}\sin\delta_{2I,2J}(\omega)\,, \tag{2.1}$$

where $\omega$ is related to the center-of-mass energy $s$ by

$$\omega=W-M \quad\text{and}\quad W=s^{1/2}. \tag{2.2}$$

$M$ is the nucleon mass, and the pion mass is set equal to unity. The phase-space factor is defined to be

$$\rho(\omega)=(\omega^2-1)^{3/2}. \tag{2.3}$$

Expression (2.3) is not the correct phase-space factor for removing all the kinematical singularities that the amplitude is known to contain. However, in the static limit, for which

$$1/M\ll 1 \quad\text{and}\quad \omega/M\ll 1\,, \tag{2.4}$$

it is satisfactory. Another way of expressing this is to say that the resulting kinematical singularities at $\omega=-M$ and $\omega=-2M-1$ will be very distant and give a slowly varying contribution over the energy range of interest. We should also remark that in the static limit, the connection with the $S_{11}$ amplitude, which is a consequence of the MacDowell symmetry, is lost.

The amplitude $a_{11}(\omega)$ contains the nucleon pole and $a_{33}(\omega)$ contains the $\Delta$ pole. The conventional definition of residues is

$$-\gamma_{11}=\text{residue of } q^{-3}e^{i\delta_{11}(\omega)}\sin\delta_{11}(\omega)\,, \tag{2.5a}$$

$$-\gamma_{33}=\text{residue of } q^{-3}\tan\delta_{33}(\omega)\,, \tag{2.5b}$$

where

$$q=[(W-M)^2-1]^{1/2}[(W+M)^2-1]^{1/2}/2W. \tag{2.6}$$

The experimental values[23,24] are

$$\gamma_{11}=0.246\pm0.006 \quad\text{and}\quad \gamma_{33}=0.12\pm0.01. \tag{2.7}$$

The position of the $\Delta$ pole is

$$\omega_{33}=2.17. \tag{2.8}$$

One proceeds to calculate the amplitude from the known Born terms for single-particle exchanges. From the work of Frautschi and Walecka,[25] or Abers and Zemach,[14] we can read off all the relevant formulas. For example, let us consider the Born term corresponding to $N$ exchange in the $P_{11}$ amplitude,

$$a_{11}{}^N(\omega)=\frac{-g_{\pi NN}{}^2}{4\pi}\frac{1}{4Wq}\frac{1}{\rho}[(E+M)(W-M)Q_1(x_s{}^N)+(E-M)(W+M)Q_0(x_s{}^N)], \tag{2.9}$$

where

$$x_s{}^N=1-(s-M^2-2)/2q^2\,, \tag{2.10a}$$

$$E\pm M=[(W\pm M)^2-1]/2W\,, \tag{2.10b}$$

$$\gamma_{11}=(3g_{\pi NN}{}^2/4\pi)(1/4M^2). \tag{2.10c}$$

In the static limit (2.9) becomes

$$a_{11}{}^N(\omega)\approx-\frac{\gamma_{11}}{9}\frac{M}{q^2}[Q_0(x_s{}^N)-4Q_2(x_s{}^N)], \tag{2.11a}$$

with

$$x_s{}^N\approx-M\omega/q^2. \tag{2.11b}$$

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[23] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).
[24] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, Phys. Rev. **117**, 226 (1960).
[25] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

We observe that there is a short cut located approximately at $-1/M \leq \omega \leq 1/M$, which actually degenerates to a point in the full static limit. If we write

$$a_{11}(\omega)=N_{11}(\omega)D_{11}^{-1}(\omega) \quad (2.12)$$

and assume that $D_{11}(\omega)$ is approximately linear over the short cut, then we find that the nucleon-exchange contribution to the $N$ function is

$$N_{11}^{N}(\omega)\approx(\gamma_{11}/9)[MQ_0(M\omega)D_{11}(0) + Q_1(M\omega)D_{11}'(0)]. \quad (2.13)$$

The second term is very small away from the cut, of course, but it does give a modest contribution to the nucleon-pole residue.

An analogous argument can be made for $\Delta$ exchange. However, since we need this term only away from its short cut, and since we normalize at the $\Delta$ pole,

$$D_{11}(-\omega_{33})=1, \quad (2.14)$$

we can write

$$N_{11}^{\Delta}(\omega)=\frac{16}{9}\frac{\gamma_{33}}{\omega+\omega_{33}}. \quad (2.15)$$

If we suppose that other forces are negligible,

$$N_{11}(\omega)=N_{11}^{N}(\omega)+N_{11}^{\Delta}(\omega). \quad (2.16)$$

(We shall defer a discussion of the importance of other forces to Sec. 4.)

The dispersion relation for $D_{11}(\omega)$, with elastic unitarity, requires a cutoff $\Lambda$:

$$D_{11}(\omega)=1-\frac{\omega+\omega_{33}}{\pi}\int_1^{\Lambda}\frac{\rho(\omega')N_{11}(\omega')}{(\omega'+\omega_{33})(\omega'-\omega)}d\omega'. \quad (2.17)$$

For the choice $\Lambda=14$, we will have

$$D_{11}(0)=0, \quad (2.18)$$

thereby forcing the nucleon pole to appear at the correct position. One can show that at low energies the $D$ function given by (2.17) is approximately straight, so that one has

$$D_{11}(\omega)\approx-\omega/\omega_{33}. \quad (2.19)$$

Using this expression, one calculates for the nucleon pole residue

$$\gamma_{11}=-N_{11}(0)/D_{11}'(0)=2\gamma_{33}, \quad (2.20)$$

which is in remarkably good agreement with experiment. The corresponding calculation carried out for the amplitude $a_{33}(\omega)$ also yields the relation (2.20).

Before going any further, we should pause to make an important observation. The key fact about scattering of particles with a large mass ratio—as is discussed by Doolen *et al.*,[18] for example—is that a partial-wave expansion in the $s$ channel is rapidly convergent on the nearby portions of the left-hand cuts. This means that crossing symmetry will relate *amplitudes* on the left and the right and not simply discontinuities of amplitudes. It is this fact that makes our whole analysis possible. One way to visualize how this comes about is to study the Mandelstam diagram for pion-nucleon scattering (Fig. 2). The partial-wave expansion is convergent for $s>0$, with the convergence best very near to the $s$-channel threshold. It is therefore clear that there must be rapid convergence in the low-energy parts of the $u$-channel physical region. This is the justification of the static-model crossing relations which we write down in the next section.

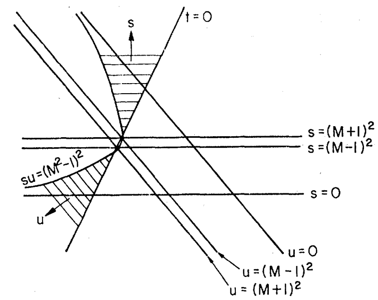


FIG. 2. Mandelstam diagram for pion-nucleon scattering, displaying the $s$- and $u$-channel physical regions.

## 3. A SIMPLE MODEL AND ITS IMPLICATIONS

In the static limit, crossing for the $p$-wave pion-nucleon amplitudes is given by

$$\begin{bmatrix}a_{11}(\omega)\\a_{13}(\omega)\\a_{31}(\omega)\\a_{33}(\omega)\end{bmatrix}=\begin{bmatrix}1/9 & -4/9 & -4/9 & 16/9\\-2/9 & -1/9 & 2/9 & 4/9\\-2/9 & 3/9 & -1/9 & 4/9\\4/9 & 2/9 & 2/9 & 1/9\end{bmatrix}\begin{bmatrix}a_{11}(-\omega)\\a_{13}(-\omega)\\a_{31}(-\omega)\\a_{33}(-\omega)\end{bmatrix}. \quad (3.1)$$

Neglecting $a_{13}(\omega)$ and $a_{31}(\omega)$, which are known to be very small at low energies, leaves

$$\begin{pmatrix}a_{11}(\omega)\\a_{33}(\omega)\end{pmatrix}=\begin{pmatrix}1/9 & 16/9\\4/9 & 1/9\end{pmatrix}\begin{pmatrix}a_{11}(-\omega)\\a_{33}(-\omega)\end{pmatrix}. \quad (3.2)$$

Now it is possible to show that there is no solution of (3.2) for which $a_{11}(\omega)$ and $a_{33}(\omega)$ will both satisfy elastic unitarity. Similarly, the only solutions of (3.1) satisfying elastic unitarity have all four amplitudes equal and even, but this is unacceptable, of course. (The proof of these statements is tedious and not especially illuminating. It depends critically on the particular choice of the phase-space factor.) We therefore find it convenient to approximate (3.2) by

$$\begin{pmatrix}a_{11}(\omega)\\a_{33}(\omega)\end{pmatrix}=\begin{pmatrix}0 & 2\\\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}a_{11}(-\omega)\\a_{33}(-\omega)\end{pmatrix}, \quad (3.3)$$

or, in other words,

$$a_{11}(\omega)=2a_{33}(-\omega). \quad (3.4)$$

In order to solve (3.4) together with elastic unitarity, we first introduce the functions

$$R_{\pm}(\omega)=(\rho(\omega)/\pi)\ln[(\omega^2-1)^{1/2}\mp\omega]. \quad (3.5)$$

The virtue of these functions is that

$$\begin{aligned}\mathrm{Im}R_+(\omega)&=0 \quad \text{for} \quad \omega\leq 1\\ &=\rho(\omega) \quad \text{for} \quad \omega\geq 1,\end{aligned} \quad (3.6a)$$

$$\begin{aligned}\mathrm{Im}R_-(\omega)&=\rho(\omega) \quad \text{for} \quad \omega\leq -1\\ &=0 \quad \text{for} \quad \omega\geq -1,\end{aligned} \quad (3.6b)$$

and

$$R_+(\omega)+R_-(\omega)=i\rho(\omega). \quad (3.7)$$

It is then apparent that the most general solution of (3.4), satisfying elastic unitarity for both $a_{11}(\omega)$ and $a_{33}(\omega)$, may be written in the form

$$a_{11}^{-1}(\omega)=-R_+(\omega)-\tfrac{1}{2}R_-(\omega)+F(\omega), \quad (3.8)$$

where $F(\omega)$ is meromorphic except for $t$-channel force cuts. We remark that an amplitude constructed on the basis of (3.8) differs from a similar one calculated in the $ND^{-1}$ framework, with any of the usual approximations to the left-hand cut, in that it satisfies an approximate crossing condition exactly instead of satisfying an exact one approximately. Of course, to achieve this desirable situation, we had to use static-model approximations, which limit the region for which such an expression is expected to be reliable to small $\omega$. Formula (3.8) may appear pathological in that it determines the high-energy form of the phase-shift independent of other details of the forces, at least if $F(\omega)$ grows less rapidly than $R_{\pm}(\omega)$. We therefore repeat that (3.8) is intended to be a suitable formalism only for low-energy approximations.

Since the Born terms corresponding to $t$-channel exchanges are quite weak compared with the $u$-channel ones (see Sec. 4), it should be a fairly good approximation not to include any $t$-channel force cuts in $F(\omega)$. Furthermore, since the other branch points which $F(\omega)$ should contain (inelastic thresholds, kinematical singularities, etc.) occur for rather large values of $\omega$, one may hope to obtain a reasonable representation of $a_{11}(\omega)$ at low energies by making a simple expansion of $F(\omega)$. Unfortunately, the expressions $R_{\pm}(\omega)$ defined in (3.5) could be modified by the addition of an arbitrary cubic polynomial without changing either their analytic properties or their asymptotic behavior. Therefore, there is no simple *a priori* justification for representing $F(\omega)$ by anything simpler than a cubic polynomial. This would require more parameters than we have available, however. Thus, even though it is completely unwarranted at this stage, we shall nevertheless examine what a linear approximation to $F(\omega)$ produces. We can then determine $F(\omega)$ by assuming that the position and residue of the $\Delta$ resonance are given. In this way we find

$$a_{11}^{-1}(\omega)=-\tfrac{1}{2}i\rho(\omega)-\tfrac{1}{2}[R_+(\omega)-R_+(-\omega_{33})]+1.3(\omega+\omega_{33}). \quad (3.9)$$

The phase shift corresponding to this expression is plotted in Fig. 1. We notice that it starts positive and retreats to zero without passing through 90°. Nevertheless, it is clear that the amplitude does contain a resonance at $\omega\approx 2$. In short, this amplitude contains no bound states, a resonance and (on the basis of Levinson's theorem) no CDD poles. The value of the phase shift at large energies, while it is certainly not expected to bear any relationship to reality, does allow us to count the number of CDD poles contained in the expression we have written down.

In order to make a connection with the formulas of Sec. 2, let us force $a_{11}(\omega)$ to contain the nucleon pole at the correct position without introducing a CDD pole. This can be done by adding a quadratic term to (3.9), a prescription more or less equivalent to choosing the cutoff in (2.17). One obtains

$$a_{11}^{-1}(\omega)=-\tfrac{1}{2}i\rho(\omega)-\tfrac{1}{2}[R_+(\omega)-R_+(-\omega_{33})]+1.3(\omega+\omega_{33})-0.79(\omega+\omega_{33})^2. \quad (3.10)$$

Equation (3.10) gives $\gamma_{11}=0.44$, which is about twice the experimental value. The corresponding phase shift is sharply decreasing, reaching $-40°$ for $E_\pi=200$ MeV and approaching $-\pi$ asymptotically. This might be rather puzzling in that the approximations of Sec. 2 gave a residue in agreement with experiment. However, Coulter and Shaw[26] in doing the corresponding calculation (no CDD poles, cutoff chosen to give the nucleon pole position, elastic unitarity), but with $t$-channel forces and relativistic factors included, found $\gamma_{11}=0.48$ and an amplitude with behavior similar to (3.10) as well. This reinforces our confidence in the interpretation we have given to (3.10), and allows us to conclude that the remarkable success of the nucleon-residue calculation with the approximations of Sec. 2 is fortuitous to a large extent.

Lyth has recently made an interesting argument[11] which is worth mentioning at this point. He argued that since a partial-wave amplitude may be written in the form

$$A_l(\omega)=\frac{1}{\pi}\int_L\frac{\mathrm{Im}A_l(\omega')}{\omega'-\omega}d\omega'+\frac{1}{\pi}\int_R\frac{\mathrm{Im}A_l(\omega')}{\omega'-\omega}d\omega', \quad (3.11)$$

and since unitarity implies that above threshold

$$|A_l(\omega)|\leq 1/\rho_l(\omega), \quad (3.12)$$

there must be a cancellation between the two integrals of (3.11) if either of them vanishes more slowly at infinity than $1/\rho_l(\omega)$. The estimate of (3.12) is made somewhat more stringent by the inclusion of inelasticity. In the case of (3.9) or (3.10) for example, the force term may be approximately represented by $2\gamma_{33}/(\omega+\omega_{33})$. This term by itself begins to exceed the unitarity bound for $\omega=3$. Therefore, in order to provide a cancellation for $\omega\geq 3$, the second integral must have appreciable

[26] P. Coulter and G. Shaw, Phys. Rev. **141**, 1419 (1966).

contributions from $\omega' \leq 3$, since $\text{Im}a_{11}(\omega)$ is positive definite on the right-hand cut. Indeed a resonance or bound state with residue $\approx -2\gamma_{33}$ is required. (This is just the estimate of the residue that we obtained in Sec. 2.) Both (3.9) and (3.10) contain a state below $\omega=3$ as required. The agreement of the residue with the estimate given above is within a factor of 2 in each case. This is about all that can be expected because of contributions from neglected parts of both the left- and right-hand cuts.

Equations (3.9) and (3.10) are not at all close to the experimental amplitude, which is perhaps not surprising in view of the procedure by which they were obtained. There is a striking feature of the experimental amplitude that provides at least a clue as to how one might proceed. Namely, the phase-shift changes sign from negative to positive at an energy near to and perhaps slightly below the inelastic threshold ($\omega=2$). Clearly the addition of a cubic term or any higher order polynomial to (3.10) cannot provide a sign change of this type. This behavior can be represented in the form (3.8) only if $F(\omega)$ contains a pole at the corresponding location. This pole may be a CDD pole from the standpoint of a one-channel calculation, a possibility about which we shall have a good deal more to say and which we shall also make more precise. Two more undetermined parameters have now been introduced into the problem. Since there is no evidence for additional zeros at low energy in either $a_{11}(\omega)$ or $a_{33}(\omega)$, we suppose there is just one. Then if we wish to leave the position and residue of the $\Delta$ pole unaltered, it is reasonable to try adding a term of the form

$$\mu(\omega+\omega_{33})^2/(\omega-\lambda) \tag{3.13}$$

to (3.9). In principle, one could determine $\mu$ and $\lambda$ from the position of the zero and the slope of $\delta_{11}(\omega)$ there. In this way one might obtain an amplitude containing the nucleon pole with nearly the correct position and residue. However, as a practical matter, it is more convenient to determine $\mu$ and $\lambda$ from the nucleon position and residue, which are known to considerably greater accuracy. We find

$$\mu=1.5 \quad \text{and} \quad \lambda=1.9. \tag{3.14}$$

These values are very close to those implied by Roper's calculated phase shift—in fact, there is no discernible discrepancy. Putting all the pieces together, we have obtained the expression

$$\begin{aligned} a_{11}^{-1}(\omega) = & -\tfrac{1}{2}i\rho(\omega) - \tfrac{1}{2}[R_+(\omega) - R_+(-\omega_{33})] \\ & + 1.3(\omega+\omega_{33}) + 1.5(\omega+\omega_{33})^2/(\omega-1.9). \end{aligned} \tag{3.15}$$

The phase-shift corresponding to (3.15) is compared with Roper's phase shift in Fig. 1. We observe that it retreats to zero at high energy, and therefore the amplitude, in the form in which we have written it, contains one bound-state pole and one CDD pole.

The fact that (3.15) agrees well with the experimental phase at low energies—in particular that the position of the zero was correctly determined—strongly suggests that the possible quadratic and cubic terms that we have omitted are in fact quite weak. As a rough estimate, they would not be expected to change (3.15) by more than about 20% for $|\omega| \leq 3$, if we require that reasonable agreement with the experimental situation be maintained. As far as we can tell it was just a lucky coincidence that these terms were not needed. It would, of course, be much more satisfying if a more fundamental explanation of this circumstance could be provided. With this observation as the justification, we may now regard (3.9) as giving a qualitative indication of how the amplitude would be expected to behave if the pole of (3.15) were a CDD pole which we omitted from a calculation while maintaining the position and residue of the $\Delta$ pole. A comparison of (3.9) and (3.15) then gives a rough indication of the importance of other inelastic channels in this case. Of course, the fact that in the form (3.15) the pole is a CDD pole is not sufficient grounds for concluding that a model with a more realistic behavior at high energies would necessarily contain one.

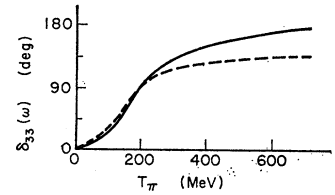


FIG. 3. Phase shift $\delta_{33}(\omega)$ as given by Roper *et al.* (solid line) and as calculated from (3.15) (dashed line).

So far in this section we have shown that there is a reasonably accurate low-energy representation of $a_{11}(\omega)$ containing a CDD pole. Furthermore, if the CDD pole term is not included, the nucleon pole becomes a resonance. In the following section we shall investigate whether different interpretations may be possible in other models. But first, since we have the formulas at hand anyway, let us see what can be said about $a_{33}(\omega)$.

The phase shift $\delta_{33}(\omega)$ determined by (3.4) and (3.15) is compared with that given by the phase-shift analyses in Fig. 3. Notice that the phase shift $\delta_{33}(\omega)$ that we obtain from either (3.15) or (3.19) (derived below) approaches $\pi$ very slowly from below:

$$\delta_{33}(\omega) \underset{\omega\to\infty}{\sim} \pi - \pi/\ln\omega. \tag{3.16}$$

As Levinson's theorem is usually stated,[27,28]

$$N_{\text{CDD}} - N_B = (1/\pi)[\delta(\infty) - \delta(\omega_0)], \tag{3.17}$$

one would infer that our $a_{33}(\omega)$ must contain one CDD pole, since it approaches $\pi$ and has no bound-state poles. However, a more careful investigation shows that if $\delta(\omega)$ approaches a multiple of $\pi$ sufficiently slowly from below, (3.17) implies that there is one more CDD pole

[27] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).

[28] R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

than is actually the case. We show in an Appendix that for a phase shift behaving as in (3.16), there is in fact no CDD pole. The pole in (3.15), which acts as a CDD pole of the $D$ function associated with $a_{11}(\omega)$, is a zero of the $N$ function associated with $a_{33}(\omega)$.

In analogy with (3.9), we fit $F(\omega)$ in (3.8) by a linear function chosen so as to give the nucleon pole with the correct position and residue:

$$a_{33}^{-1}(\omega)=-i\rho(\omega)-R_{-}(\omega)+1.5+9.2\omega. \quad (3.18)$$

This expression has a resonance at $\omega=4.0$.

As in (3.9), we can offer no fundamental reason for expecting a linear fit to $F(\omega)$ to give meaningful results. There we did succeed in showing by comparison with experiment that terms more singular at infinity were not required. If that circumstance persists in this case, then (3.18) can be regarded as providing evidence that the force due to the short cut arising from nucleon exchange in elastic pion-nucleon scattering is sufficient to form a resonance. This is not surprising, since the Lyth argument applied here predicts a state with residue approximately $\frac{1}{2}\gamma_{11}$ below $\omega=3.1$. At any rate the position of the resulting resonance in (3.18) should probably not be taken too seriously. This case does not differ drastically from (3.9), in which the nucleon appeared as a resonance.

One can continue as we did for the nucleon channel, and, in complete analogy with (3.10), add a quadratic term to (3.18) so as to get the $\Delta$ resonance at the correct position, while maintaining the nucleon pole position and residue. This procedure gives

$$a_{33}^{-1}(\omega)=-i\rho(\omega)-R_{-}(\omega)+1.5-9.2\omega-3.9\omega^2; \quad (3.19)$$

(3.19) has a $\Delta$ residue of about 0.075. That this is smaller than 0.12 is complementary to the fact that the value $\gamma_{33}=0.12$ used in (3.10) gave a nucleon residue which was too large. However, if one adjusts for the difference between $\rho(\omega)$ and $q^3(\omega)$ at the position of the $\Delta$ resonance, using the definition (2.5b), one finds that (3.19) actually gives $\gamma_{33}=0.11$, which is really very good. The point we are making is that deviations from static-model kinematics at the energy of the $\Delta$ resonance require $\gamma_{33}$ to be about 50% larger when used as the residue of a force pole than when used as the residue of a resonance pole in conjunction with the phase-space factor (2.3). This fact indicates a limitation of the model (3.4). This being the case, it would probably be stretching a point to maintain that (3.19) is better than (3.10).

The 33 phase shift corresponding to (3.19) is identical with that for (3.15), within a few degrees, at all energies. This is a reflection of the fact that the CDD pole is (3.15) plays a very small role in determining the 33 phase shift.

## 4. CAN THE NUCLEON BE CONSIDERED A PION-NUCLEON BOUND STATE?

The principal concern is to determine whether the nucleon can be regarded as being formed in the pion-nucleon channel, at least in some approximate sense. In order to state as clearly as possible what this would mean, we require that in order to be so regarded it should be approximately calculable in a one-channel model without CDD poles, if we use elastic unitarity and the forces due to exchanges in crossed channels. Other criteria could undoubtedly be devised, but this choice seems to us to be a reasonable one. For a discussion of the significance of a CDD pole in a one-channel calculation, the reader is referred to Refs. 29–31.

Rothleitner and Stech claim to have "proved" that the nucleon can be a bound state of the pion-nucleon channel if and only if the zero of the real part of the phase shift is at a higher energy than the first inelastic threshold.[10] Their work has two shortcomings which, in our opinion, vitiate each half of their conclusion.

The first shortcoming is that they make assumptions about asymptotics which, for the zero below the inelastic threshold, preclude the possibility of the $N$ function's developing a zero instead of the $D$ function's requiring a CDD pole. It is easy in potential theory, for instance, to give examples for which such a zero arises in the $N$ function. For example, a potential giving rise to a bound state and a low-energy resonance is in this category. We observe, however, that a potential for the $P_{11}$ state based on the first Born approximation to $\Delta$ exchange alone is not of the type we have just suggested. This is known from the large number of relativistic $ND^{-1}$ calculations which have been performed with just this input. It is quite unlikely that inclusion of the higher order Born approximations to $\Delta$ exchange would make an appreciable difference.

The second shortcoming in the analysis of Rothleitner and Stech is connected with their definition of the conditions under which the nucleon may be regarded as a pion-nucleon bound state. They suppose that a calculation is performed in which in addition to the information we would require as input, one also knows the $R$ factor

$$R=\frac{\sigma_{\text{tot}}}{\sigma_{\text{el}}}=\frac{2-2\eta\cos 2\delta}{\eta^2+1-2\eta\cos 2\delta}. \quad (4.1)$$

If the phase shift has a zero at a point above the inelastic threshold for which the inelasticity is small ($\eta$ is near to 1), then $R$ will be sharply peaked there. When such an $R$ is included in the appropriate $ND^{-1}$ equations,[32] it will cause the $D$ function to have a sharply peaked behavior at the corresponding point. Therefore in this case the $R$ factor is introducing much the same information about inelasticity into the problem as a CDD pole below the inelastic threshold would. By formulating our criterion for a bound state in terms of

[29] J. B. Hartle and C. E. Jones, Phys. Rev. **140**, B90 (1965); Ann. Phys. (N.Y.) **38**, 348 (1966).
[30] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 270 (1965).
[31] J. Finkelstein, Phys. Rev. **140**, B175 (1965).
[32] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

elastic unitarity only, we avoid having to make such an artificial distinction.

If the zero of $a_{11}(\omega)$ does correspond to a CDD pole, then, as we showed in a preceding section, this CDD pole (or other channels presumably) is required in order to obtain an approximate dynamical understanding of the formation of the nucleon. This statement depends only on the low-energy behavior of the amplitudes in (3.9) and (3.15) and should therefore be reliable. Perhaps we should pause here a moment to clarify our use of the word "approximate." The criterion we have just applied is based on the pole position; since (3.9) contains a broad resonance at $\omega \approx 2$, this is unambiguous, at least if one takes (3.9) seriously. Another approach one might consider makes use of the pole position as the strength of the coupling of the crossed-channel pole is increased. It turns out that if $\gamma_{33}$ is increased by 50%, then the pole in (3.9) is brought to the correct nucleon position.

A very similar statement can be made in terms of an $N/D$ formulation. Assuming that the pole of (3.12) is a CDD pole, $D(\omega)$, normalized to 1 at $\omega=-\omega_{33}$, can be written in the form

$$D(\omega)=1+\mu\frac{\omega+\omega_{33}}{\omega-\lambda}-\frac{\omega+\omega_{33}}{\pi}\times\int_1^\Lambda \frac{\rho(\omega')N(\omega')}{(\omega'+\omega_{33})(\omega'-\omega)}d\omega'. \quad (4.2)$$

If we further assume that for low energies

$$N(\omega)\approx\frac{16}{9}\frac{\gamma_{33}}{\omega+\omega_{33}}, \quad (4.3)$$

then we can determine the pole parameters of (4.2) by comparison with (3.12). This procedure yields

$$\lambda=1.9 \quad \text{and} \quad \mu=0.32. \quad (4.4)$$

With these parameters the pole term in (4.2) has the value $-\frac{1}{3}$ at $\omega=0$. Therefore if the cutoff is chosen so that $D(0)=0$, the integral term will be $\frac{2}{3}$ at the nucleon pole. This might be interpreted to mean that the forces in the pion-nucleon channel alone provide $\frac{2}{3}$ of the total binding. Such a statement is undoubtedly model-dependent to a certain degree. Still, it is probably fair to say that the pion-nucleon channel provides more than half the binding even if there is a CDD pole. Indeed the Lyth argument seems to guarantee at least that much by itself.

Notice that the amplitude determined by (4.2) and (4.3) contains one more parameter (the cutoff) than the formula (3.15) to which it is supposed to bear a strong resemblance. This is because (3.15) happened to agree with the experimental values very well without the inclusion of quadratic or cubic terms. Our attitude toward (3.15) here is that it provides a convenient low-energy representation of the amplitude which allows us to determine the CDD pole parameters for use in an $N/D$ calculation.

By choosing the cutoff in (4.2) so that $D(0)=0$, and then making a linear approximation to the integral as in (2.19), we obtain

$$D(\omega)=1+0.32(\omega+\omega_{33})/(\omega-1.9)-0.29(\omega+\omega_{33}). \quad (4.5)$$

The resulting amplitude has a nucleon pole residue $\gamma_{11}=0.15$, which is somewhat small, but perhaps adequate in view of all the approximations that have been made. In any case, it gives a measure of the extent to which (3.15) agrees with a corresponding amplitude computed by $N/D$ techniques.

Although we have already encountered some ambiguity of interpretation in the above discussion, it is clear that if there is a CDD pole its residue is sufficiently large for it to be playing an important role in the dynamics. The really difficult question is whether the zero of the amplitude does in fact correspond to a CDD pole. Even though (3.12) gives a definite answer, the formulas of Sec. 3 are certainly not reliable for settling this question. The existence of a CDD pole is closely tied up with questions of asymptotic behavior, whereas Sec. 3 employed low-energy approximations. We could, for example, add to (3.12) a term such as

$$[\alpha(\omega+\omega_{33})^2/(\omega-\beta)]; \quad 1/\alpha\gg 1, \quad \beta\gg 1, \quad (4.6)$$

without appreciably changing the low-energy behavior of the amplitude. The modified amplitude will no longer contain a CDD pole, since the phase shift will retreat back through zero at $\beta$ and end up at $-\pi$. This demonstrates that no study of low-energy behavior alone can allow one to decide whether or not there is a CDD pole.

It is rather unlikely that phase-shift analyses carried out to higher energies than have been done to date would help to decide whether there is a CDD pole, at least in the foreseeable future. Such an analysis would have to be very accurate up to energies beyond the resonance region (several BeV), where the great multiplicity of contributing partial waves would create serious practical problems. Inelasticity would further compound the difficulties of such a program.

As has already been suggested, an amplitude written in the form $N/D$ can have a zero in two inequivalent ways. Either the forces are such that the $N$ function has a zero, or, as a consequence of the effect of inelastic channels (or an elementary particle), the $D$ function has a CDD pole. This distinction is well-defined once the $D$ function is required to satisfy a singly subtracted dispersion relation. Strictly speaking, in any realistic calculation only a finite number of channels can be explicitly included. The others may be represented by the strip approximation method, or any other suitable technique that may be developed. Such calculations put some of the effects that "ideally" belong in the $D$ function into the $N$ functions, and in general one would expect that the particular way in which this is done could affect the asymptotics of the calculated $D$

functions. Thus, we would expect that there will always be ambiguities which may make it difficult in practice to decide whether any given zero is an $N$-function zero or a $D$-function CDD pole. We are asking the question for $a_{11}(\omega)$ calculated only in a one-channel model with elastic unitarity. The difficult questions of asymptotics remain, however, and we cannot really expect to find a definitive answer. Yet we know that there are cases for which the answer is obvious (e.g., to obtain the $\Lambda$ as a "resonance" of the pion-nucleon system would obviously require a CDD pole). So we discuss here some approaches by which one might attempt to decide which case applies to the amplitude $a_{11}(\omega)$.

We emphasize from the outset that the suggestion that the $N$ function has a zero at low energy is a radical one in that it runs directly counter to the usual picture that forces at low energy are dominated by the short cut associated with $\Delta$ exchange. Nucleon exchange gives a very small contribution and is certainly not the culprit for such a crime. Unfortunately, there is considerable disagreement in the literature over the strengths of the various other forces. We shall concentrate our attention on the values given by Abers and Zemach[14] (AZ) and Donnachie, Hamilton, and Lea[33] (DHL). The latter values were used by Lyth in the numerical parts of his work.

The first discrepancy is in the strength of the $\Delta$-exchange Born term. At threshold AZ find a value for this term of $8.8\times10^{-2}$, whereas DHL obtain $2.5\times10^{-2}$. These numbers are not exactly comparable in that AZ keep the complete Born term whereas DHL use only the portion corresponding to the nearby short cut. Using the static model of Sec. 2, we calculate the value

$$(16/9)\gamma_{33}/(\omega_{33}+1)=6.7\times10^{-2}. \qquad (4.7)$$

It is our opinion that corrections due to relativistic phase space and the faraway cut can reconcile this number with the value given by AZ, whereas the value used by DHL appears to be wrong. For the threshold value of the $\rho$ exchange Born term, AZ give approximately 0.01 whereas DHL find 0.03. Again these numbers are not exactly comparable, because DHL retain only the nearby portions of the cuts. A more important difference, however, is that AZ neglect the magnetic coupling of the $\rho$ and make only a very rough estimate of the electric coupling. DHL, on the other hand, make a rather careful study of the relevant nucleon form factors. We conclude that the AZ estimate of the $\Delta$ and DHL estimate of the $\rho$ couplings are the best choice. This gives a $\rho$ force about one-third the $\Delta$ force. DHL have also estimated that exchange of two pions in an $I=J=0$ configuration provides a force about one-half as great as $\rho$ exchange. Coulter and Shaw[26] used the correct $\Delta$ coupling and allowed the strength of the $\rho$ coupling to vary over a range of reasonable values. Their work shows that the correct long-range forces by themselves cannot produce the zero. We conclude that if there is a zero in the $N$ function additional forces of considerable strength have to be provided by distant singularities.

An interesting approach has been taken recently by Atkinson and Halpern.[34] They show that the couplings of degenerate $SU(6)$ symmetry imply that other channels (e.g., $\pi\Delta$) play a more important role than $\pi N$ in forming the nucleon, and that a CDD pole is consequently to be expected. These arguments are not conclusive because of the difficulty in estimating the importance of symmetry breaking. It is possible, however, that an approach that can estimate the symmetry breaking, such as the current algebra method combined with the methods of Atkinson and Halpern, may eventually provide the most reliable statement of whether a CDD pole is expected to occur.

If the zero arises as a zero of the $N$ function, it is necessary that the short-range forces have important repulsive components. Such a possibility has been suggested by Chew[35] in connection with the mechanism of "Pomeranchuk repulsion." Chew pointed out that the $J=0$ contribution of the force due to exchange of Pomeranchuk Regge pole gives a strong repulsion which is effectively of longer range than one would estimate simply on the basis of the distance of the nearest singularities. This mechanism arises in the context of the strip approximation, in which the effect of thresholds above the strip boundary is included in the forces. Therefore the strong repulsion obtained in this way includes the effect of both distant right and left singularities. We have seen in Sec. 3 that the long-range parts of the forces corresponding to single-particle exchanges produce a reasonance but probably not a bound state. Therefore, in addition to the repulsion that Pomeranchuk exchange may provide, we need an additional short-range attraction to provide the rest of the binding. This attraction might arise from the long cut associated with $\Delta$ exchange. An attempt at a calculation to establish this point would be inconclusive because of the need for a cutoff. What one can say is that such a combination of ingredients could possibly provide a consistent model for the behavior of the $P_{11}$ amplitude in the elastic region.

On the basis of the discussion in this section we can suggest two different possible models for the formation of the nucleon (assumed not to be elementary). The first is that low-lying channels other than pion-nucleon provide substantial forces for its formation, and that the effect of these forces can be introduced into a one-channel calculation only by the inclusion of a CDD pole. In this model, the forces in the pion-nucleon channel alone are sufficient to bind a resonance. The second

[33] A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964); Ann. Phys. (N. Y.) **17**, 1 (1962).

[34] D. Atkinson and M. B. Halpern, Phys. Rev. **150**, 1377 (1966). The author is grateful to them for informing him of their results prior to publication.

[35] G. F. Chew, Phys. Rev. **140**, B1427 (1965).

possibility is that other low-lying channels are not important, and that the forces in the pion-nucleon channel alone are sufficient to give a deeply bound state. In this case, in addition to this single-channel attraction, there is important repulsion arising principally from the influence of the whole array of closed channels. This repulsion can be represented by Pomeranchuk exchange in the strip approximation, and it causes the phase shift to turn positive. Although both these possibilities are consistent with all the available information, we tend to regard the former as the simpler and in some ways more attractive alternative. As has already been suggested, a refinement of the methods of Atkinson and Halpern may provide the best hope for making a choice.

## 5. RESOLUTION OF A PARADOX

We have argued that the nucleon might not be calculable as a bound state of the pion-nucleon channel without the inclusion of a CDD pole. Several authors, however, seem to have done exactly this. True, the phase shifts they obtain are qualitatively wrong, but still they seem to have calculated the nucleon pole. A good example of such a calculation is the recent one of Doolen *et al.*,[18] in which the assertion is made that, given the long-range forces and by use of elastic unitarity, the nucleon pole position is found to 5% accuracy, and the residue to 20% accuracy.

We shall demonstrate that although their calculation has a certain validity, it does not imply that the nucleon is formed in the pion-nucleon channel. Rather, they have shown that the $P_{11}$ amplitude which is most consistent with crossing symmetry, given the $P_{33}$ amplitude, contains the nucleon pole. Our contention is that this is a consequence of relativity, and nothing more, since the correct $P_{33}$ amplitude already knows about the nucleon pole in its crossed channel.

The argument we are making is most easily understood in the static limit, for which $N$ and $\Delta$ exchange are represented by poles. In this model, crossing is given by an equation such as (3.1), or (3.4), let us say. If we now impose crossing symmetry we have (3.8). The important point is that in this way one is effectively assuming knowledge of the complete crossed-channel amplitude and not only of the left-hand cut discontinuities. This is possible because of the large nucleon-pion mass ratio, as was emphasized earlier. Suppose we next represent $F(\omega)$ in terms of distant singularities (Balázs poles) and choose the parameters introduced in this way so as to fit the experimental $P_{33}$ amplitude as well as possible. The expression so obtained is constrained not to have the CDD pole which we included in (3.15), and hence must break down for $\omega \geq 1.9$. It is clear, however, that the more flexible the parametrization introduced into $F(\omega)$, the closer to $\omega = 1.9$ the amplitude obtained will be approximately correct. In particular, the nucleon pole position and residue will be found quite accurately. This argument is essentially unchanged when rephrased in terms of a relativistic calculation not containing the approximations to crossing that are implicit in this discussion. In the relativistic form this is the method of Doolen *et al.*

We therefore conclude that Doolen *et al.*, calculate the nucleon pole parameters as a consequence of the way in which they impose crossing symmetry and not because the nucleon is formed in the pion-nucleon channel. Furthermore, if they had allowed for a CDD pole, they could have obtained better results with the same number of parameters—e.g., one CDD pole and one Balázs pole. These parameters would be no more "undetermined" than the ones they use, in that they could be calculated by the same procedure as they use for the Balázs poles alone—namely, fitting to the requirements of crossing.

Once one clearly understands the argument we have just given, he may wish to re-examine other calculations (e.g., that of Sec. 2) of bound state or resonance poles to decide which ones are true calculations of "dynamical" states. Admittedly, most examples are not as easy to see through as the one we have just discussed.

## 6. THE ROLE OF THE FIRST INELASTIC CHANNEL

The inelasticity parameter [Fig. 1(b)] $\eta_{11}$ shows a rapid decrease to a value of about 0.2 at 1400 MeV (center of mass), indicating that coupling to inelastic channels is quite appreciable. All the analyses are in agreement on this point. Strictly speaking, the channel responsible for the inelasticity is the three-body channel $\pi\pi N$. However, the quantum numbers of the $P_{11}$ channel are such that the three particles can be in relative $s$ waves. This configuration is favored kinematically at low energy and also by the strong attraction between pions in the $I=J=0$ configuration at low energies.[36] The identification of this channel is further confirmed by the fact that no other pion-nucleon partial wave has anywhere near as much inelasticity at these energies. Also, Kirz *et al.*,[37] have been able to identify the importance of this configuration experimentally.

It is worthwhile to investigate what effect strong forces in this inelastic channel can have on the phase shift $\delta_{11}(\omega)$. Coulter and Shaw[20] have performed calculations including the experimental inelasticity by means of a Frye-Warnock calculation.[38,39] They find that the resulting phase shift is modified only slightly—tending to become less negative in a calculation without a CDD pole and more positive in a calculation with one. We wish to argue now that the channel consisting of $\pi\pi N$ in relative $s$ waves is responsible for the large inelasticity

[36] R. H. Dalitz and R. G. Moorhouse, Phys. Letters **14**, 159 (1965).
[37] J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **130**, 2481 (1963).
[38] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).
[39] P. Coulter, A. Scotti, and G. Shaw, Phys. Rev. **136**, B1399 (1964).

and the change in the phase shift attributable to inelasticity in a Frye-Warnock calculation, but that by itself it may not be responsible for the CDD pole, if there is one.

Since the three-body configuration $\pi\pi N$ is rather awkward for us to discuss, we would like to represent it by a two-body configuration which might be more or less equivalent. Since it is known that the $s$-wave $\pi$-$N$ forces are very weak, whereas the $s$-wave $\pi$-$\pi$ forces are very strong, the natural two-body configuration to consider is $\sigma N$; $\sigma$ represents the $\pi\pi$ $I=J=0$ configuration at a mass of about 400 MeV. As a matter of fact, Brown and Singer have proposed that there is in fact a true resonance (i.e., a pole of a scattering amplitude).[40] Although much evidence has been cited in its favor,[41–45] the effect is noticeably absent in certain experiments such as the recent one on $K_{e4}$ decays.[46] The possibility that there is an important enhancement without an actual resonance is difficult to exclude, particularly in view of the theoretical suggestion of Chew[47] that the $\pi\pi$ phase shift may be *decreasing* through $\frac{1}{2}\pi$.

In this two-channel model ($\pi N$ and $\sigma N$) the inelastic threshold is at $\omega=2.8$, which just coincides with the energy at which inelasticity begins to set in sharply (Fig. 1). Corresponding to the two channels there will be two eigenamplitudes, one of which vanishes at the inelastic threshold, whereas the other one need not do so. Furthermore, since in the case of two channels it is easily seen that the sum of the two eigenphases equals the sum of the two phases for scattering in each of the elastic channels, the nonvanishing eigenphase must just equal the phase $\delta_{11}(\omega)$ at the inelastic threshold, i.e., must be about 20° there. Now since this is the eigenamplitude that contains the nucleon pole, the same arguments we gave in the single-channel case concerning the possible existence of a CDD pole should apply here as well. Admittedly some care should be taken in discussing eigenamplitudes, because they contain cuts associated with the diagonalization. Also the one which we have said contains the nucleon pole is not an analytic continuation of $a_{11}(\omega)$ below the inelastic threshold. We do not believe that these difficulties will affect our conclusions, although it is a possibility we cannot definitely exclude. There does not appear to be any simple argument with which to settle this question conclusively.

[40] L. M. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).
[41] F. S. Crawford, R. A. Grossman, L. J. Lloyd, L. R. Price, and E. C. Fowler, Phys. Rev. Letters **11**, 564 (1963).
[42] R. Del Fabbro, M. DePretis, R. Jones, G. Marini, A. Odian, G. Stoppini, L. Tau, and R. Visentin, Phys. Rev. Letters **12**, 674 (1964).
[43] V. V. Anisovich and L. G. Dakhno, Phys. Letters **10**, 221 (1964).
[44] D. L. Lind, B. C. Barish, R. J. Kurz, P. M. Ogden, and V. Perez-Mendez, Phys. Rev. **138**, B1509 (1965).
[45] M. Olsson and G. B. Yodh, Phys. Rev. Letters **10**, 353 (1963).
[46] R. W. Birge, R. P. Ely, U. Camerini, *et al.*, Phys. Rev. **135**, B416 (1964).
[47] G. F. Chew, Phys. Rev. Letters **16**, 60 (1966).

If the other eigenphase (the one that vanishes at the inelastic threshold) contains a reasonance pole, then this will serve very nicely to explain why the phase shift calculated by Coulter and Shaw with inelasticity included becomes larger at high energy than the one calculated without inelasticity. The point of this discussion is to emphasize that if the $\sigma N$ channel introduces a resonance into the problem, it will not start "pulling" the phase shift to appreciably more positive values below the inelastic threshold. On the other hand, the large value reached by the phase shift at higher energy suggests that there probably is such a resonance. Whether or not this resonance is actually made in the $\sigma N$ channel is beyond our ability to answer. The fact that such a resonance is $s$-wave would appear to be an argument against it, although the mechanism of "Pomeranchuk repulsion" suggested by Chew[35] might provide enough of a barrier to hold it together, especially since it is a very broad resonance. On the other hand, if the phase shift continues up through $\pi$, as could very well happen, then with the same arguments we used for the nucleon itself, we could argue that closed channels may be playing an important role in making the resonance. If this is the case, its $s$-wave aspect is less of a problem. In either case there is a strong attraction provided by nucleon exchange in $\sigma N$ scattering, which undoubtedly is an important ingredient in the whole picture.

To summarize, we believe that the $\pi N$ and $\sigma N$ channels are both important contributions to the complete dynamics of the $P_{11}$ wave. However, the possibility of calculating the nucleon is not likely to be substantially affected by including only the $\sigma N$ channel.

## 7. INVESTIGATION OF A TWO-CHANNEL MODEL

We have argued in the preceding section that coupled $p$-wave channels could not possibly account for the large inelasticity in the $P_{11}$ channel. Nevertheless it may be worthwhile to investigate whether the presence of a bound state or resonance formed in the $\pi\Delta$ channel could possibly account for the general features of the phase shift.

A calculation was therefore performed along the following lines. Static-model kinematics was used, although the nondegeneracy of the $\pi N$ and $\pi\Delta$ thresholds was maintained. For the forces the following Born terms were kept [corresponding to Figs. 4(a)-4(d), respectively]:

$$\frac{16}{9}\frac{\gamma_{33}}{\omega+\omega_{33}},\quad \frac{4}{9}\frac{(\gamma_{11}\gamma_{33})^{1/2}}{\omega-\omega_{33}},\quad \frac{10}{9}\frac{(\gamma_{33}G)^{1/2}}{\omega},\quad \frac{4}{9}\frac{G}{\omega-\omega_{33}}. \tag{7.1}$$

The terms shown in Fig. 4(e) and 4(f) were not included because their static-model crossing elements are 1/9 and 1/36, respectively; $G$ represents the square of the $\pi\Delta\Delta$ coupling constant.

The value for $G$ is not known experimentally, although any reasonable symmetry scheme will predict it to be comparable to $\gamma_{11}$ and $\gamma_{33}$ (if it makes any prediction at all). Rather than work with a value given by $SU(6)$, for example, we decided to search through a range of values for $G$ between 0.0 and 1.0. There is one further unknown feature about the forces we are including: the relative sign between the two off-diagonal force terms. We therefore investigated both values of this sign. (An over-all sign for the off-diagonal terms is of no consequence in this model.) Finally, since we wished the nucleon to merge from the calculations, we chose a cutoff in each case to ensure that the nucleon would have the correct mass. The reason that we chose to work with one cutoff for all four $D$-function integrals is that only in this way would there be few enough parameters to enable us to reach any conclusions. Of course, there is a good deal of arbitrariness in this prescription, especially since one of the Born terms we include [Fig. 4(b)] involves exchange of a different spin from the other three. For most values of the parameters this is the least important term, however.

Notice that two of the pole terms in (7.1) occur in the physical region of the $\pi N$ channel. This situation arises from treating the unstable $\Delta$ as a stable particle that gives rise to a real threshold, while neglecting the $\pi\pi N$ channel. We must, therefore, decide how the corresponding $D$-function integrations are to be performed. A typical integral that arises in this way is

$$\frac{1}{\pi}\frac{4}{9}(\gamma_{11}\gamma_{33})^{1/2}\int_1^\Lambda \frac{(\omega'^2-1)^{3/2}}{\omega'-\omega_{33}}\frac{d\omega'}{\omega'-\omega}. \quad (7.2)$$

Of the three possibilities at $\omega'=\omega_{33}$ (principal value, $\pm i\epsilon$ prescription), only the principal-value integration gives a result satisfying Hermitian analyticity. This is a natural requirement to impose when the $\pi\Delta$ threshold is taken to be real. When constructing the $N$ function from the Born terms we will also need to know the real part of $D$ at the pole positions. The determination of these values involves integration across a double pole. Although more awkward from a calculational point of view, such a principal-value integral is perfectly well defined.

It was expected that making $G$ sufficiently large would form a second state. While this state could not cross the position of the force pole at $\omega_{33}$, it could be brought quite close to it. We were rather surprised, therefore, to find no second state arising as $G$ was increased. In fact for $G \geq 0.08$ it was even impossible to maintain the nucleon pole position. (This effect set in for $G\approx 0.20$ when the two off-diagonal terms had opposite signs.) The explanation for this puzzling behavior is that by increasing $G$ we also increased the coupling between the channels, which resulted in an increased attraction in the first channel as well. Therefore to maintain the nucleon pole position a smaller cutoff was required. But with this smaller cutoff there was less binding available for a second state, even though $G$ was larger. By the time $G$ had reached 0.08, the cutoff had become so small ($\Lambda=10$) that the results were already quite meaningless. For all reasonable cutoff values, the phase shift in the $\pi N$ channel always was found to be negative and decreasing.

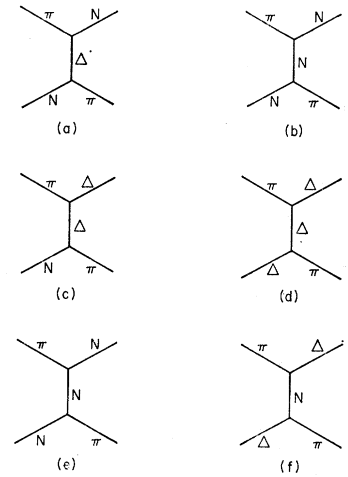


FIG. 4. Born terms for a two-channel calculation.

In order to make sure that our ideas were correct, we repeated the calculations, neglecting the Born term of Fig. 4(c). In this way the interchannel coupling was made small and independent of $G$. As $G$ was then increased from 0.0 to 1.0 the approach of the second state to the force pole at $\omega=\omega_{33}$ could be followed by observing that Re Det$D(\omega_{33})$ went from $-2.46$ to $-0.53$. But even for the latter value the $\pi N$ phase shift still was about $-10°$ at $\omega_{33}$. The phase shift made a very sudden rise to a value above 90° in the vicinity of the second state. This behavior is plotted in Fig. 5.

These calculations suggest that the behavior of the $P_{11}$ amplitude is not likely to be understood on the basis of $p$-wave channels alone. The spacing between the nucleon and the Roper resonance is too small if the latter's width is due mainly to $\pi N$ decay. The simplest model with a chance of success would seem to require $\pi N$, $\pi\Delta$, and $\sigma N$ operating in conjunction, the latter two channels being mainly responsible for the position and width of the Roper resonance—whose coupling to the $\pi N$ channel is relatively weak.

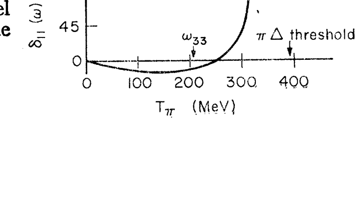


FIG. 5. Phase shift resulting from a two-channel calculation described in the text.

## CONCLUSION

We have shown that the pion-nucleon $P_{11}$ amplitude very likely contains a CDD pole at a position near to the $\pi\pi N$ inelastic threshold, where the phase shift has been experimentally determined to pass through zero, although the possibility that there is no CDD pole has not been rigorously excluded. If there is a CDD pole, it has sufficient strength to play an appreciable role in the dynamics of the $P_{11}$ amplitude, indicating the importance of inelastic channels. By removing this pole in a suitable way we obtained an estimate of the amplitude that one should calculate by using elastic-channel forces only. We found that the amplitude constructed in this way contains the nucleon pole as a low-lying resonance having approximately the correct residue. A two-channel calculation ($\pi N$ and $\pi\Delta$) showed that a strongly coupled $p$-wave channel could not produce the Roper resonance with a sufficiently large width while maintaining the correct nucleon position. We also argued that the CDD pole is probably not introduced by adding the $s$-wave $\sigma N$ channel. It therefore appears that at the very least the three channels $\pi N$, $\pi\Delta$, and $\sigma N$ are required to calculate both the nucleon bound state and the Roper resonance accurately.

## ACKNOWLEDGMENTS

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## APPENDIX A

In Sec. 3 it was stated that if an amplitude containing no bound states has a phase shift asymptotically behaving according to

$$\delta(\omega) \underset{\omega\to\infty}{\sim} \pi-\pi/\ln\omega\,, \tag{A1}$$

then the amplitude contains no CDD poles. This is true in spite of the fact that Levinson's theorem, as it is usually stated, would imply that there is one CDD pole.

Let us first introduce the Omnès function $D(\omega)$, normalized to unity at $\omega=0$:

$$D(\omega)=\exp\left\{-\frac{\omega}{\pi}\int_{\omega_0}^{\infty}\frac{\delta(\omega')}{\omega'(\omega'-\omega)}d\omega'\right\}. \tag{A2}$$

A criterion for deciding whether $D(\omega)$ is the correct $D$ function or CDD poles are required is whether or not $D(\omega)$ defined by (A2) satisfies the once-subtracted dispersion relation

$$D(\omega)=1+\frac{\omega}{\pi}\int_{\omega_0}^{\infty}\frac{\mathrm{Im}D(\omega')}{\omega'(\omega'-\omega)}d\omega'. \tag{A3}$$

A sufficient condition for $D(\omega)$ to satisfy (A3) is that

$$\lim_{|\omega|\to\infty}\frac{D(\omega)}{\omega}=0\,, \tag{A4}$$

since this will ensure that in the equation

$$\frac{D(\omega)-1}{\omega}=\frac{1}{2\pi i}\oint\frac{D(\omega')-1}{\omega'(\omega'-\omega)}d\omega' \tag{A5}$$

the contribution from the circle at infinity, which arises as the contour is enlarged, vanishes. Therefore, we need only show that $D(\omega)$, specified by (A1) and (A2), satisfies (A4). It is easy to show from (A1) and (A2) that if one neglects factors weaker than logarithms,

$$D(\omega)\underset{|\omega|\to\infty}{\sim}\frac{\omega}{\ln\omega}. \tag{A6}$$

This completes the proof.