It is easily checked that these equations also guarantee

$$A_{\varphi\varphi}(\nu,k^2)\rightarrow 0 \quad \text{for} \quad m_\pi\rightarrow 0. \tag{3.13}$$

Assuming Eqs. (3.12a, b) to hold for finite pion masses we get the wanted equations

$$\sigma_{N\pi}(\nu,-m_\pi^2)=\frac{m_\pi^4}{R^2q_L}\tilde{a}_2(\nu,-m_\pi^2),$$
$$\sigma_{\pi N}(\nu,0)=\frac{m_\pi^4}{R^2q_L}m^2\nu^2a_1(\nu,0)=\frac{m_\pi^4}{R^2q_L}\tilde{a}_2(\nu,0). \tag{3.14}$$

The integral in (3.9) is now indeed determined by the πN scattering amplitude $\tilde{a}_2(\nu,k^2)$ continued from $k^2=-m_\pi^2$ to $k^2=0$ leaving m_π, which enters in any calculation of $\tilde{a}_2(\nu,k^2)$ as a parameter, at its physical value. In the limit $m_\pi\rightarrow 0$ we get, because of (3.13) and (3.8),

$$g_A(0)=C(=1) \tag{3.15}$$

which, of course, is a consequence of the assumed current conservation.

The k^2 dependence of $\tilde{a}_2(\nu,k^2)$ is presumably weak[14] so that the abovementioned approximation should be meaningful.

In summarizing, we restate that to derive the Adler-Weisberger relation the only assumptions needed are Gell-Mann's scaling condition and the assumption of no subtraction in $M_1(\nu,k^2)$. The PCAC condition in a form proposed by Nambu then provides a tool to relate the absorptive part of $M_1(\nu,k^2)$ to the πN cross section $k^2=0$. Covariant Schwinger terms antisymmetric in isospin indices necessitate subtractions in various amplitudes but do not alter these conclusions as long as they do not influence $M_1(\nu,k^2)$.[11]

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[14] S. L. Adler, Phys. Rev. 140, 736 (1965).

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Nondynamical Formalism and Tests of Time-Reversal Invariance*

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In this paper we consider those consequences of time-reversal invariance (or, briefly, T invariance) which are independent of dynamics. A general prescription is developed to find all those tests of T invariance which are valid for arbitrary values of the form factors. Our considerations are independent of whether or not any of the other usual conservation laws (such as parity conservation) hold. The spins of the particles may be arbitrary. The results and methods of earlier papers dealing with the nondynamical properties of particle reactions are used. For elastic processes it is shown that T invariance eliminates some of the product sets and curtails others, but that no entire subclasses are eliminated. The restrictions on observables imposed by T invariance are of two types: There are so-called "mirror relations" between pairs of observable components, and there are relations which are not of the mirror type and involve a number of observable components. It is shown that the number of relations of the latter type is always nonzero whenever T invariance is not implied by other assumed conservation laws. Therefore, the mirror relations do not form a complete set of tests of T invariance. They do not even form a sufficient set of such tests, as they can be satisfied by a symmetric as well as an antisymmetric M matrix. A proof is given that in any non-mirror-type relation no particle is unpolarized in all the observable components which appear in the relation. It is also shown that the only reaction in which all mirror relations follow from parity conservation alone is the reaction involving two spin-$\frac{1}{2}$ and two spin-0 particles. A number of examples of elastic reactions are worked out in detail. For inelastic reactions the results are less interesting, since the restrictions imposed by T invariance can all be written as mirror-type relations between observable components of the direct and time-reversed reactions.

I. INTRODUCTION

In view of some recent experiments, the universal validity of time-reversal invariance has been repeatedly questioned. The purpose of this paper is to give a comprehensive discussion of tests of time reversal invariance which are valid regardless of what the dynamical details of the interactions are. Furthermore, our results will be relevant regardless of whether or not other conservation laws hold.

In a series of papers we have recently developed a general formalism for the study of the nondynamical structure of particle reactions of arbitrary spins and we will further develop this formalism to consider

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consequences of time-reversal invariance. In Sec. II, the properties of the M matrix under time reversal invariance are listed, and the modifications are discussed which these bring about in the factorization of the M matrix, and in the class and subclass structure of the observables of elastic reactions. It is then shown that the restrictions of time-reversal invariance result not only in the rather obvious mirror-type relations between observables, but in addition, also bring about other, more involved relations among the observables. In Sec. III, a general formula is developed giving all nondynamical time-reversal invariance tests for any reaction involving particles of arbitrary spins. Some examples are worked out in Sec. IV. Finally, Sec. V deals with the rather trivial case of inelastic reactions.

II. THE STRUCTURE OF THE M MATRIX AND THE OBSERVABLES UNDER TIME REVERSAL

As in our previous papers, we will base our description of elastic particle reactions on the M matrix which we write as[1,2]

$$M=\sum_{J,r} a_J{}^{(r)} S_{[J]}:T_{[J]}{}^r, \qquad (2.1)$$

where the a's are the form factors or invariant amplitudes, the S's are the spin tensors, and the T's are the momentum tensors composed of

$$\hat{l}\equiv\frac{\mathbf{q}'-\mathbf{q}}{|\mathbf{q}'-\mathbf{q}|},\quad \hat{m}\equiv\frac{\mathbf{q}'\times\mathbf{q}}{|\mathbf{q}'\times\mathbf{q}|},\quad \hat{n}\equiv\hat{l}\times\hat{m}, \qquad (2.2)$$

where $\mathbf{q}$ and $\mathbf{q}'$ are two noncollinear momenta in the reaction, which go over into (-1) times each other under time-reversal [such as the momentum of the incoming and outgoing s_1 in reaction (3.1)]. The index J denotes the rank of the tensor, and the index r labels the particular set of l's, m's and n's contained in the T.

The observable components L can then be written as

$$L=\mathrm{Tr}(M\mathcal{S}_I M^\dagger \mathcal{S}_F)$$
$$=\sum_{J_1,r_1}\sum_{J_2,r_2} a_{J_1}{}^{r_1} a_{J_2}{}^{r_2*}\,\mathrm{Tr}(S_{[J_1]}:T_{[J_1]}{}^{r_1} S_{[J_I]}:T_{[J_I]}{}^{r_I} \times S_{[J_2]}:T_{[J_2]}{}^{r_2} S_{[J_F]}:T_{[J_F]}{}^{r_F}). \qquad (2.3)$$

TABLE I. Transformation properties of the quantities appearing in the M matrix, under space reflection and time reversal.

Quantity	l	m	n	$S_{[J]}$
Space reflection	$-l$	$+m$	$-n$	$+S_{[J]}$
Time reversal	$+l$	$-m$	$-n$	$(-1)^J S_{[J]}$

[1] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Ann. Phys. (to be published); P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, University of California Radiation Laboratory Report No. UCRL-14222 (unpublished).

[2] M. J. Moravcsik, in *Recent Developments in Particle Physics* (Gordon and Breach, Science Publishers, Inc., New York, to be published).

TABLE II. Product sets and subclasses for rotationally invariant and T-invariant observable components. The symbols ξ and υ denote "even" and "odd," respectively. The symbol l_M denotes the total number of l's appearing in one term of M and one term of $M^\dagger$ together. Similarly m_M and n_M denote the number of m's and n's, respectively, in this combination. The symbol l_S denotes the total number of l's contained in one term of $\mathcal{S}_I$ and in one term of $\mathcal{S}_F$ together, and similar definitions hold for m_S and n_S.

	l_S	m_S	n_S	l_M m_M n_M	ξ ξ ξ	ξ υ ξ	ξ υ υ	ξ ξ υ
	(ξ	ξ	ξ)	$++$ $--$	Re	...	...	...
Re Im	(υ	υ	ξ)		...	...	...	Im Re
	(ξ	υ	ξ)	$++$ $--$	...	Re	...	...
Re Im	(ξ	υ	υ)		...	...	Re Im	...
Re Im	(υ	ξ	ξ)		...	...	Im Re	...
	(υ	ξ	υ)	$++$ $--$	...	Im	...	...
Re Im	(ξ	ξ	υ)		...	...	...	Re Im
	(υ	υ	υ)	$++$ $--$	Im	...	...	...

The restrictions on M and L under time-reversal invariance (henceforth called T invariance) for elastic processes can be discussed easily once the properties of l, m, n, and $S_{[J]}$ are known under this transformation. These are given in Table I.

Let us now write the M matrix as

$$M=M'+M'', \qquad (2.4)$$

where M' does not change sign under time reversal, and M'' does. Let us furthermore denote by $\lambda(M)$, $\mu(M)$, and $\nu(M)$ the number of l's, m's, and n's, respectively, that appear in each term of M. Then, from Table I, we see that for M' we must have

$$\mu(M)+\nu(M)+J=\xi, \qquad (2.5)$$

and for M'',

$$\mu(M)+\nu(M)+J=\upsilon, \qquad (2.6)$$

where ξ denotes "even" and υ denotes "odd." The last term on the left-hand sides of Eq. (2.5) and (2.6) arises from the transformation property of $S_{[J]}$ under time reversal.

On the other hand, we also have

$$\lambda(M)+\mu(M)+\nu(M)=J, \qquad (2.7)$$

and hence

$$\lambda(M)+\mu(M)+\nu(M)+J=\xi, \qquad (2.8)$$

TABLE III. Product sets and subclasses for observable components of rotationally invariant, parity-conserving, and T-invariant reactions. For notation, see Table II.

l_S	m_S	n_S	l_M / m_M / n_M	ξ ξ ξ	ξ υ ξ
ξ	ξ	ξ	$++$ $--$	Re	...
υ	ξ	υ	$++$ $--$	...	Im
ξ	υ	ξ	$++$ $--$	...	Re
υ	υ	υ	$++$ $--$	Im	...

so that Eqs. (2.5) and (2.6) are equivalent to

$$\lambda(M)=\xi, \tag{2.9}$$

and

$$\lambda(M)=\upsilon, \tag{2.10}$$

respectively.

Similar results hold also for $M^\dagger$. If we now define, as in Ref. 3

$$l_M\equiv\lambda(M)+\lambda(M^\dagger), \tag{2.11}$$

$$m_M\equiv\mu(M)+\mu(M^\dagger), \tag{2.12}$$

$$n_M\equiv\nu(M)+\nu(M^\dagger), \tag{2.13}$$

then we get from Eq. (2.9), for an observable component generated by a T-invariant M matrix

$$l_M=\xi. \tag{2.14}$$

Thus we get a modification of the subclass and product set tables as compared to Tables II and III in Ref. 3. These modified tables are shown in Tables II and III. It can be seen that neither in the purely rotational-invariant case nor in the rotation- and reflection-invariant case are any complete subclasses eliminated, although in both cases complete product sets are wiped out as T invariance is imposed. It should be added, however, that Tables II and III do not indicate the total extent of the restrictions imposed by T invariance, since Eq. (2.9) is only a sufficient but not a necessary condition for Eq. (2.14) to hold. In other words, in addition to wiping out certain product sets, Eq. (2.9) will also restrict the remaining product sets to consist of bilinear products of form factors both of which pertain to a term in the M matrix which has $\lambda(M)=\xi$.

The factorization of the M matrix for a T-invariant case is formally the same[3] as in the parity-conserving case. Let us assume that reaction 1 can be factorized into reactions 2 and 3. The corresponding M matrices are denoted by M_1, M_2, and M_3, respectively. Then, if rotation invariance alone is assumed, we have

$$M_1(=)M_2\otimes M_3, \tag{2.15}$$

where (=) denotes "nondynamical equality," i.e., the two sides are equal in every respect except for the values of the form factors, and $\otimes$ denotes an outer product in spin space.

In order to emphasize the similarity between reflection invariance and T invariance, we will introduce the notation $(M^a)_i$ for the M matrix of reaction i in the case when rotation and reflection invariance holds, with $a=+$ when the product of intrinsic parities of all particles in the reaction is $+1$, and $a=-$ when this product is -1. Then we have

$$(M^a)_1(=)\sum (M^b)_2\otimes(M^c)_3, \tag{2.16}$$

where the sum goes over all possible combinations of b and c such that

$$bc=a. \tag{2.17}$$

The actual number of combinations is two for each value of a. If now rotation invariance and T invariance are assumed, we can define $(M_a)_i$ as the M matrix of reaction i in this case, where $a=+$ if the M matrix does not change sign under time reversal, and $a=-$ if it does. We can then write

$$(M_a)_1(=)\sum (M_b)_2\otimes(M_c)_3, \tag{2.18}$$

with the restriction on b and c given by Eq. (2.17). Again, the sum in Eq. (2.18) contains two terms.

If now rotation, reflection, and T invariance are assumed, then we have

$$(M_b{}^a)_1(=)\sum (M_d{}^c)_2\otimes(M_f{}^e)_3, \tag{2.19}$$

where the sum goes over all values of c, d, e, and f subject to

$$ce=a, \tag{2.20}$$

$$df=b. \tag{2.21}$$

There are therefore four terms in the sum of Eq. (2.19).

Correspondingly, we can also factorize the observable components. We use the notation

$$(L)_i\equiv\mathrm{Tr}(M_i\mathcal{S}_IM_i{}^\dagger\mathcal{S}_F), \tag{2.22}$$

$$(L^{ab})_i\equiv\mathrm{Tr}((M^a)_i\mathcal{S}_I(M^b)_i\mathcal{S}_F), \tag{2.23}$$

$$(L_{ab})_i\equiv\mathrm{Tr}((M_a)_i\mathcal{S}_I(M_b)_i\mathcal{S}_F), \tag{2.24}$$

$$(L_{cd}{}^{ab})_i\equiv\mathrm{Tr}((M_c{}^a)_i\mathcal{S}_I(M_d{}^b)_i\mathcal{S}_F). \tag{2.25}$$

We can then write

$$L_1(=)L_2L_3, \tag{2.26}$$

$$(L^{ab})_1(=)\sum (L^{cd})_2(L^{ef})_3, \tag{2.27}$$

with

$$a=ce,\quad b=df; \tag{2.28}$$

that is, the sum has four terms; and

$$(L_{ab})_1(=)\sum (L_{cd})_2(L_{ef})_3, \tag{2.29}$$

[3] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Ann. Phys. (to be published).

with restrictions given again by Eq. (2.28), i.e., again with four terms in the sum.

Finally

$$(L_{cd}{}^{ab})_1(=)\sum (L_{gh}{}^{ef})_2(L_{kl}{}^{ij})_3, \tag{2.30}$$

where

$$a=ei,\quad b=fj, \tag{2.31}$$

$$c=gk,\quad d=hl, \tag{2.32}$$

so that the sum now has 16 terms.

Now let us investigate the type of linearly independent relations among observable components that arise when T invariance is imposed. A very obvious type of relation that appears is

$$L(a,b;c,d)=\pm L(c,d;a,b), \tag{2.33}$$

as it will be seen later, where in $L(x,y;z,w)$ we denote by x the spin state of the first initial particle, by y the spin state of the second initial particle, by z the spin state of the first final particle, and by w the spin state of the second final particle. We will restrict ourselves, for the time being, to the discussion of reactions involving four particles only.

We will call Eq. (2.33) a mirror relation. It is clear that mirror relations must hold if T invariance holds. It is, however, not *a priori* clear (a) whether a mirror relation might not hold, owing to another invariance principle, even in the absence of T invariance; and, (b) whether mirror relations are the only restrictions on the observables brought by T invariance. In fact, we will demonstrate, that, (a) some mirror relations can, under special circumstances, also hold when T invariance does not, (b) there are other linearly independent relations beside the mirror ones in the presence of T invariance.

In order to show this, we must recall the formulas giving the number of form factors N under the various conservation laws. These are derived in Ref. 3, and will only be quoted here. The subscript zero refers to rotation invariance only, P to rotation and reflection invariance, T to rotation and T invariance, and $P+T$ to rotation, reflection, and T invariance. Using the definition

$$x\equiv(2s_1+1)^2(2s_2+1)^2, \tag{2.34}$$

where s_1 and s_2 are the spin of the two particles participating in the (elastic) reaction, we have

$$N_0=x, \tag{2.35}$$

$$N_P=\begin{cases}\frac{1}{2}x & (\alpha)\\ \frac{1}{2}x+\frac{1}{2} & (\beta)\end{cases}, \tag{2.36}$$

$$N_T=\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{x}, \tag{2.37}$$

$$N_{P+T}=\begin{cases}\frac{1}{2}[\frac{1}{2}x+\sqrt{x}] & (\alpha)\\ \frac{1}{2}[\frac{1}{2}x+\sqrt{x}+\frac{1}{2}] & (\beta)\end{cases}, \tag{2.38}$$

where (α) denotes the case when fermions are also participating in the reaction, and (β) the case when only bosons are present.

The number of observable components Q have also been derived before. We have

$$Q_0=x^2, \tag{2.39}$$

$$Q_P=\begin{cases}\frac{1}{2}x^2 & (\alpha)\\ \frac{1}{2}x^2+\frac{1}{2} & (\beta)\end{cases}, \tag{2.40}$$

$$Q_T=x^2, \tag{2.41}$$

$$Q_{P+T}=\begin{cases}\frac{1}{2}x^2 & (\alpha)\\ \frac{1}{2}x^2+\frac{1}{2} & (\beta)\end{cases}. \tag{2.42}$$

Let us furthermore call B the number of bilinear combinations of form factors (i.e., the total number of products in all product sets). We have, by squaring the appropriate N's,

$$B_0=x^2, \tag{2.43}$$

$$B_P=\begin{cases}\frac{1}{4}x^2 & (\alpha)\\ \frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{4} & (\beta)\end{cases}, \tag{2.44}$$

$$B_T=\tfrac{1}{4}x^2+\tfrac{1}{2}x\sqrt{x}+\tfrac{1}{4}x, \tag{2.45}$$

$$B_{P+T}=\begin{cases}\frac{1}{16}x^2+\frac{1}{4}x\sqrt{x}+\frac{1}{4}x & (\alpha)\\ \frac{1}{16}x^2+\frac{1}{4}x\sqrt{x}+\frac{3}{8}x+\frac{1}{4}\sqrt{x}+\frac{1}{16} & (\beta)\end{cases}. \tag{2.46}$$

The quantity Q-B gives the number of linearly independent relationships that must be present among the observable components. These are

$$\Delta_0\equiv Q_0-B_0=0, \tag{2.47}$$

$$\Delta_P\equiv Q_P-B_P=\begin{cases}\frac{1}{4}x^2 & (\alpha)\\ \frac{1}{4}(x-1)^2 & (\beta)\end{cases}, \tag{2.48}$$

$$\Delta_T\equiv Q_T-B_T=\tfrac{3}{4}x^2-\tfrac{1}{2}x\sqrt{x}-\tfrac{1}{4}x, \tag{2.49}$$

$$\Delta_{P+T}\equiv Q_{P+T}-B_{P+T}$$
$$=\begin{cases}\frac{7}{16}x^2-\frac{1}{4}x\sqrt{x}-\frac{1}{4}x & (\alpha)\\ \frac{7}{16}x^2-\frac{1}{4}x\sqrt{x}-\frac{3}{8}x-\frac{1}{4}\sqrt{x}+\frac{7}{16} & (\beta)\end{cases}. \tag{2.50}$$

For all Δ's we can show that

$$\Delta\geq 0, \tag{2.51}$$

where, except for Δ_0, the equality holds only for $x=1$ (i.e., when all particles are spinless).

Now we turn to the calculation of the number of linearly independent mirror relations. Any observable component has a mirror image observable component unless it is of the form $L(a,b;a,b)$. This latter type of observable component will be called self-mirrored. The number of self-mirrored observable components is clearly x, and they clearly may all be nonzero under parity conservation. Thus the number of pairs of observable components between which linearly independent mirror relations might exist is one-half times the difference between the total number of observable components and the number of self-mirrored observable components

$$V_0=\tfrac{1}{2}x^2-\tfrac{1}{2}x, \quad (2.52)$$

$$V_P=\begin{cases}\tfrac{1}{4}x^2-\tfrac{1}{2}x & (\alpha)\\ \tfrac{1}{4}x^2-\tfrac{1}{2}x+\tfrac{1}{4} & (\beta)\end{cases}, \quad (2.53)$$

$$V_T=\tfrac{1}{2}x^2-\tfrac{1}{2}x, \quad (2.54)$$

$$V_{P+T}=\begin{cases}\tfrac{1}{4}x^2-\tfrac{1}{2}x & (\alpha)\\ \tfrac{1}{4}x^2-\tfrac{1}{2}x+\tfrac{1}{4} & (\beta)\end{cases}. \quad (2.55)$$

Let us define $A_i\equiv\Delta_i-V_i$, which gives the number of linearly independent nonmirror relations. This is expected to have significance only for $i=T$ and $i=P+T$. We get

$$A_T=\tfrac{1}{4}x(\sqrt{x}-1)^2, \quad (2.56)$$

and

$$A_{P+T}=\begin{cases}\tfrac{1}{8}x^2+\tfrac{1}{4}(\tfrac{1}{2}x-\sqrt{x})^2 & (\alpha)\\ \tfrac{1}{8}(x-1)^2+\tfrac{1}{4}(\tfrac{1}{2}x-\sqrt{x}+\tfrac{1}{2})^2 & (\beta)\end{cases}. \quad (2.57)$$

All three of these A's are positive for $x>1$, thus proving that there are nonmirror relations among the observable components. In the case of A_T, these clearly are a result of the imposition of T invariance, since previously the number of relations among the observable components was zero. For A_{P+T} this is not so clear. Previous to the imposition of T invariance, the number of linearly independent relations among the observable components here was given by Eq. (2.48). After the imposition of T invariance the number of linearly independent relations is given by Eq. (2.50). Thus T invariance brought about

$$\Delta_{P+T}-\Delta_P=\begin{cases}\tfrac{1}{4}x^2-(\tfrac{1}{4}x+\tfrac{1}{2}\sqrt{x})^2\geq 0 & (\alpha)\\ \tfrac{1}{4}(x+1)^2-(\tfrac{1}{4}x+\tfrac{1}{2}\sqrt{x}+\tfrac{1}{4})^2>0 & (\beta)\end{cases} \quad (2.58)$$

new relations. On the other hand, the mirror relations might not be all independent now, since the relations which exist as a result of parity conservation might now make one mirror relation a consequence of another. An example for this will be given in Sec. IV. All in all, therefore, the situation in the parity conserving case is rather complex, and it is not immediately evident whether there are relations required by T invariance in addition to the mirror relations. For the case of only rotational invariance, Eq. (2.56) clearly proves that the mirror relations do not represent all the restrictions that T invariance imposes on the observable components.

For the parity-conserving case the question also arises as to whether mirror relations could not hold even in the absence of T invariance. In this connection we will now show that the only reaction in which mirror relations hold among all observable components as a result of parity conservation is the $0+\tfrac{1}{2}\rightarrow 0+\tfrac{1}{2}$ reaction. Actually, this statement is already plausible from Ref. 3, where we have shown that the only reaction where T invariance cannot be tested if parity is conserved is the $0+\tfrac{1}{2}\rightarrow 0+\tfrac{1}{2}$ reaction. It could still be possible, however, to have another reaction where, although mirror relations held as a result of parity conservation only, T invariance could be checked through a relation of a nonmirror type. We now show that there are no such reactions.

A mirror relation between two observable components would mean that

$$\mathrm{Tr}(M\mathcal{S}_I M^\dagger \mathcal{S}_F)=\pm\mathrm{Tr}(M\mathcal{S}_F M^\dagger \mathcal{S}_I). \quad (2.59)$$

In terms of the so-called four-traces[1] and the bilinear combinations of form factors, this can be written as

$$\sum_{J_1}\sum_{J_2}\sum_{r_1}\sum_{r_2} a_{J_1}{}^{r_1} a_{J_2}{}^{r_2*} X_{J_1J_IJ_2J_F}{}^{r_1r_Ir_2r_F} =\pm\sum_{J_1}\sum_{J_2}\sum_{r_1}\sum_{r_2} a_{J_1}{}^{r_1} a_{J_2}{}^{r_2*} X_{J_1J_FJ_2J_I}{}^{r_1r_Fr_2r_I}, \quad (2.60)$$

$$X_{J_1J_IJ_2J_F}{}^{r_1r_Ir_2r_F}\equiv\mathrm{Tr}(S_{[J_1]}:T_{[J_1]}{}^{r_1}S_{[J_I]}:T_{[J_I]}{}^{r_I} \times S_{[J_2]}:T_{[J_2]}{}^{r_2}S_{[J_F]}:T_{[J_F]}{}^{r_F}).$$

Since the bilinear combinations of form factors are independent of each other, Eq. (2.60) implies

$$X_{J_1J_IJ_2J_F}{}^{r_1r_Ir_2r_F}=\pm X_{J_1J_FJ_2J_I}{}^{r_1r_Fr_2r_I}, \quad (2.61)$$

where the sign on the right-hand side is either + for all J_1, r_1, J_2, and r_2, or − for all J_1, r_1, J_2, and r_2. But we have[1,3]

$$X_{J_1J_IJ_2J_F}{}^{r_1r_Ir_2r_F}=(-1)^{J_1+J_2+J_I+J_F} \times X_{J_1J_FJ_2J_I}{}^{r_1r_Fr_2r_I} \quad (2.62)$$

so that, for a given J_I and J_F, Eq. (2.61) is equivalent to the requirement that J_1+J_2 be always even or always odd.

Now let us look at the subclass structure of observable components as given in Ref. 3, Tables II and III. We see from there that each observable component receives contributions from two product sets, one of which has J_1+J_2 even, and the other J_1+J_2 odd. Equation (2.59) will therefore be satisfied only for those reactions for which for every observable component one of the two product sets is empty. When is this the case?

In order to answer this question, we have to use the formulas giving the number of products in a product set. For an irreducible reaction $0+s\rightarrow 0+s$, with parity conservation, these formulas are as follows[4].

Product set $(\xi\xi\xi)$:

$$\tfrac{1}{16}[(2s+1)^2-1]^2+(3s^2+3s+1) \quad \text{(bosons)}, \quad (2.63)$$

$$\tfrac{1}{16}(2s+1)^4+\tfrac{1}{4}(2s+1) \quad \text{(fermions)}; \quad (2.64)$$

[4] P. L. Csonka and M. J. Moravcsik, J. Natl. Sci. Math. (to be published).

Product set $(\nu\xi\nu)$

$$\tfrac{1}{16}[(2s+1)^2-1]^2+s(s+1) \quad \text{(bosons)}, \tag{2.65}$$

$$\tfrac{1}{16}(2s+1)^4-\tfrac{1}{4}(2s+1)^2 \quad \text{(fermions)}; \tag{2.66}$$

Product set $(\xi\nu\xi)$

$$\tfrac{1}{16}[(2s+1)^2-1]^2+s(s+1) \quad \text{(bosons)}, \tag{2.67}$$

$$\tfrac{1}{16}(2s+1)^4+\tfrac{1}{4}(2s+1)^2 \quad \text{(fermions)}; \tag{2.68}$$

Product set $(\nu\nu\nu)$

$$\tfrac{1}{16}[(2s+1)^2-1]-s(s+1) \quad \text{(bosons)}, \tag{2.69}$$

$$\tfrac{1}{16}(2s+1)^4-\tfrac{1}{4}(2s+1)^2 \quad \text{(fermions)}. \tag{2.70}$$

To two of the subclasses the product sets $(\xi\xi\xi)$ and $(\nu\nu\nu)$ contribute, while to the other two subclasses the product sets $(\xi\nu\xi)$ and $(\nu\xi\nu)$ contribute. For the first two subclasses the product set $(\nu\nu\nu)$ is the smaller one, so we must now determine under what condition that is empty.

For bosons the only integer solution of Eq. (2.69) being zero is for $s=0$. This already proves, therefore, that for bosons the $(\nu\nu\nu)$ product set is never empty except for the trivial reaction $0+0\rightarrow 0+0$. We can thus abandon the boson case and turn to the fermion case.

For fermions in all subclasses, the smaller product sets have the same number of products, given by Eq. (2.70). The only half-integer solution of Eq. (2.70) being zero is $s=\frac{1}{2}$. Thus, we get the result that for the reaction $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$ all mirror relations are required by parity conservation, and this is the only irreducible reaction which has that property.

As a matter of fact, this is altogether the only reaction with that property, because there are no composite reactions with that property at all. This can be seen by remarking that the necessary condition for a composite reaction having that property is that its constituents have that property. This, however, is not a sufficient condition. In order to see that, let us investigate the only possible[2,5] candidate, the reaction $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$. The observable component L_0 for this reaction can be factorized into the observable components L_1 of $\frac{1}{2}+0\rightarrow\frac{1}{2}+0$, and L_2 of $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$, as follows:

$$L_0^{++}(x,y;z,w)=L_1^{++}(x,0;z,0)L_2^{++}(0,y;0,w)+L_1^{--}(x,0;z,0)L_2^{--}(0,y;0,w). \tag{2.71}$$

The necessary and sufficient condition for a parity-conservation-induced mirror relation for the composite reaction is therefore

$$L_1^{++}(x,0;z,0)=\epsilon_1 L_1^{++}(z,0;x,0), \tag{2.72}$$

$$L_2^{++}(0,y;0,w)=\epsilon_2 L_2^{++}(0,w;0,y), \tag{2.73}$$

$$L_1^{--}(x,0;z,0)=\epsilon_2 L_1^{--}(z,0;x,0), \tag{2.74}$$

$$L_2^{--}(0,y;0,w)=\epsilon_4 L_2^{--}(0,w;0,y), \tag{2.75}$$

where each ϵ_i is $+1$ or -1, and we must have

$$\epsilon_1\epsilon_2=\epsilon_3\epsilon_4. \tag{2.76}$$

It is, however, easy to find observable components for which this is not true. For instance, for $x=m$, $z=0$, and $y=w$, we have $\epsilon_1=\epsilon_2=\epsilon_4=+1$ and $\epsilon_3=-1$, which violates Eq. (2.76). Thus, there is no composite reaction for which all mirror relations hold just by parity conservation, in the absence of T invariance.

III. NONDYNAMICAL TESTS OF TIME-REVERSAL INVARIANCE

In this section our aim is to find a complete set as well as sufficient and necessary sets of linearly independent nondynamical tests of time-reversal invariance for elastic-scattering reactions of the type

$$s_1+s_2+\cdots+s_m\rightarrow s_1+s_2+\cdots+s_m, \tag{3.1}$$

where the s_1, s_2, $\cdots$, s_m are particles with spins s_1, s_2, $\cdots$, s_m respectively.[6] The values of the spins are arbitrary.

A certain relation satisfied by elements of the M matrix will be called trivial, if they are satisfied for *any* M matrix (whether time-reversal invariant or not). For example, if M_{ik} is some element of M, then $M_{ik}-M_{ik}=0$ is a trivial relation. Similarly $(M_{ik}-M_{ik})\times M_{lm}=0$ is trivial. Analogously, a certain relation between observable components will be called trivial if it is satisfied for *any* M matrix. For example if $L(S_I,S_F)$ is an observable component, then $L(S_I,S_F)-L(S_I,S_F)=0$ is a trivial relation.

We say that a certain linear combination of observable components is a test of time-reversal invariance (or, briefly "T test"), if it is not trivial and is satisfied whenever time-reversal invariance holds. In other words, a T test is a necessary condition for T invariance, it must be satisfied if T invariance holds, but T invariance does not necessarily follow if a T test is satisfied:

$$(T\text{ invariance})\Rightarrow(\text{a }T\text{ test is satisfied}). \tag{3.2}$$

We call a set of T tests sufficient and necessary if T invariance holds if, and only if, every T test in the set is satisfied.

$$(T\text{ invariance})\leftrightarrow(\text{each }T\text{ test in a sufficient and necessary set is satisfied}). \tag{3.3}$$

We say that a T test is a nondynamical T test (or, briefly, an NDT test), if it does not make use of any knowledge of dynamics. In other words, an NDT test is a T test no matter what the numerical values of form factors are.[5]

[5] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Nuovo Cimento **42**, 743 (1966).

[6] The results to be obtained are analogous to the ones obtained in Ref. 3 to test parity conservation and to determine parities, and which are related to some previous results obtained by using the method of "Bohr rotation." Loosely speaking, the results of the present section in this sense are related to "Bohr rotations and time reversal," which differs from the ordinary Bohr rotations in that it also contains an operation which exchanges the spin states of initial and final particles.

An NDT test will be called linearly independent of a set of other NDT tests, if it cannot be written as a linear combination of those in the set, if the numerical values of form factors are arbitrary.

A set of NDT tests is called (linearly) complete, if all NDT tests can be obtained by linear combination of those T tests which belong to the set.

A sufficient and necessary set of linearly independent NDT tests is a sufficient and necessary set of NDT tests each of which is linearly independent of the others in the set.

The only invariance principle we shall assume to always hold in nature is invariance under rotations in three space.

It was shown in Ref. 5 that if observables measured in connection with an elastic reaction of the type shown by Eq. (3.1) are to be used to construct NDT tests, then the number of ingoing particles must not exceed two. Therefore, in all further considerations we may restrict ourselves to elastic reactions of the type[7]

$$s+s' \to s+s'. \tag{3.4}$$

All such reactions can be decomposed into the two constituent reactions

$$s+0 \to s+0, \tag{3.5a}$$

$$0+s' \to 0+s'. \tag{3.5b}$$

In the first part of this section we shall derive results for reactions of the type shown in Eqs. (3.5), and the corresponding results for reactions shown in Eq. (3.4) will be derived from these.

To obtain NDT tests for reactions (3.5) we label the rows and columns of matrices in spin space by m, where m is the projection of spin along a suitably chosen axis. In the following we chose this axis to be parallel to $\hat{m}$ defined in Eq. (2.2). In this representation T invariance holds if, and only if, the M matrix is symmetric

$$(T \text{ invariance}) \leftrightarrow (M=\tilde{M}). \tag{3.6}$$

Therefore, an NDT test is any linear relation between observable components which holds whenever M is symmetric, independently of further dynamical details. In order to find such relations, we define the $(2s+1)\times(2s+1)$ spin space matrix $S_{\mu'\mu}$ by its matrix elements

$$\langle s,m'|S_{\mu'\mu}|s,m\rangle=\delta_{\mu'm'}\delta_{m\mu}. \tag{3.7}$$

Thus the $S_{\mu'\mu}$ is a matrix, all of whose elements are zero, except the one at the intersection of the μ'th row and μth column. Any $S_{\mu'\mu}$ can be written as

$$S_{\mu'\mu}=\sum_{J=M}^{2s}(-1)^{s-\mu}\langle s,s;\mu',-\mu|J,M\rangle\Omega_{J,M=\mu'-\mu}, \tag{3.8}$$

[7] It was shown in Ref. 5 that a sufficient set of NDT tests always involves observable components which yield polarization information about all those particles in the reaction whose spin is nonzero. Indeed, all sufficient sets to be derived satisfy this theorem.

where the $\langle s,s;\mu',-\mu|J,M\rangle$ are Clebsch-Gordan coefficients with the usual phase convention which makes them real, and the $\Omega_{J,M}$ are $(2s+1)\times(2s+1)$ matrices in spin space. They are nonzero for $2s\geq J\geq 0$, $J\geq M\geq -J$, and their elements are defined by[1]

$$\langle s,m'|\Omega_{J,M}|s,m\rangle=(-1)^{s-m}\langle s,s;m',-m|J,M\rangle. \tag{3.9}$$

The $\Omega_{J,M}$ form a complete set of linearly independent $(2s+1)\times(2s+1)$ matrices in spin space. All elements of $\Omega_{J,M}$ are real, because so are the Clebsch-Gordan coefficients. It also follows from Eq. (3.9) that

$$\Omega_{J,M}{}^{\dagger}=(-1)^M\Omega_{J,-M} \tag{3.10}$$

where the superscript † means Hermitian conjugation.

Substituting Eq. (3.9) into Eq. (3.8) one finds that $S_{\mu'\mu}$ given by Eq. (3.8) does indeed satisfy Eq. (3.7):

$$\begin{aligned}\langle s,m'|S_{\mu'\mu}|s,m\rangle &=\sum_{J=0}^{2s}\sum_{J'=0}^{2s}(-1)^{s-\mu}\langle s,s;\mu',-\mu|J,M\rangle(-1)^{s-m}\\ &\qquad\times\langle s,s;m',-m|J,M\rangle\delta_{M,\mu'-\mu}\delta_{M,m'-m}\\ &=\delta_{\mu'm'}\delta_{m\mu},\end{aligned} \tag{3.11}$$

where we have replaced the lower limit of summation over J and J' by zero. This can be done because the Clebsch-Gordan coefficients vanish anyhow whenever $\mu'-\mu\neq M$. The last line follows from the well known orthogonality property of Clebsch-Gordan coefficients.

Equation (3.8) can be rewritten in terms of the Hermitian matrices

$$\Omega_{J,M}{}^{(+)}=\tfrac{1}{2}[\Omega_{J,M}+(-1)^M\Omega_{J,-M}], \tag{3.12a}$$

$$\Omega_{J,M}{}^{(-)}=\frac{1}{2i}[\Omega_{J,M}-(-1)^M\Omega_{J,-M}]. \tag{3.12b}$$

That these matrices are Hermitian follows from Eq. (3.10). The $\Omega_{J,M}{}^{(+)}$ are symmetric, while the $\Omega_{J,M}{}^{(-)}$ are antisymmetric. Of course, if $M=0$, then $\Omega_{J,M}{}^{(-)}=0$ for all J. We also have, for integer J values,

$$\Omega_{J,-M}{}^{(\pm)}=\pm(-1)^M\Omega_{J,M}{}^{(\pm)}. \tag{3.13}$$

In terms of these Hermitian matrices, Eq. (3.8) becomes

$$S_{\mu'\mu}=\sum_{J=|M|}^{2s}(-1)^{s-\mu}\langle s,s;\mu',-\mu|J,M\rangle\times[\Omega_{J,M}{}^{(+)}+\Omega_{J,M}{}^{(-)}]. \tag{3.14}$$

The $S_{\mu'\mu}$ matrices have been defined in such a manner that for any two matrices A and B we can write

$$\begin{aligned}\mathrm{Tr}(S_{\mu_3\mu_1}AS_{\mu_2\mu_4}B)&=\sum_{\alpha\beta\gamma}(S_{\mu_3\mu_1})_{\alpha\beta}A_{\beta\gamma}(S_{\mu_2\mu_4})_{\gamma\delta}B_{\delta\alpha}\\&=A_{\mu_1\mu_2}B_{\mu_4\mu_3}.\end{aligned} \tag{3.15}$$

If we choose $A=M$, $B=M^{\dagger}$, then

$$\mathrm{Tr}(S_{\mu_3\mu_1}MS_{\mu_2\mu_4}M^{\dagger})=M_{\mu_1\mu_2}M_{\mu_3\mu_4}{}^{*}. \tag{3.16}$$

The symmetry of M implies

$$M=\tilde{M} \Rightarrow (M_{\mu_1\mu_2}-M_{\mu_2\mu_1})M_{\mu_3\mu_4}{}^*=0\,, \text{ for all } \mu_1, \mu_2, \mu_3, \mu_4 \text{ and } (\mu_1\neq\mu_2)\,; \tag{3.17a}$$

$$M_{\mu_1\mu_2}(M_{\mu_3\mu_4}{}^*-M_{\mu_4\mu_3}{}^*)=0\,, \text{ for all } \mu_1, \mu_2, \mu_3, \mu_4 \text{ and } (\mu_3\neq\mu_4)\,. \tag{3.17b}$$

(Of course, when $\mu_1=\mu_2$ then Eq. (17a) becomes trivial. Similarly, Eq. (3.17b) is trivial when $\mu_3=\mu_4$.)

Applying successively Eqs. (3.6), (3.17), (3.16), and (3.14), we obtain

$$\begin{aligned} T \text{ invariance} \Rightarrow \mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}=0\,, &\text{ for all } \mu_1, \mu_2, \mu_3, \mu_4, (\mu_1\neq\mu_2)\,, \\ \mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=0\,, &\text{ for all } \mu_1, \mu_2, \mu_3, \mu_4, (\mu_1\neq\mu_2)\,. \end{aligned} \tag{3.18}$$

Here

$$\begin{aligned} \mathcal{L}_{\mu_1\mu_2\mu_3\mu_4} &\equiv \mathrm{Tr}(S_{\mu_3\mu_1}MS_{\mu_2\mu_4}M^\dagger)-\mathrm{Tr}(S_{\mu_3\mu_2}MS_{\mu_1\mu_4}M^\dagger) \\ &= \sum_{J=0}^{2s}\sum_{J'=0}^{2s}\{\langle s,s;\mu_3,-\mu_1|J,M_{31}\rangle\langle s,s;\mu_2,-\mu_4|J',M_{24}\rangle[L(\Omega_{J,M_{31}}{}^{(+)},\Omega_{J'M_{24}}{}^{(+)}) \\ &\quad -L(\Omega_{J,M_{31}}{}^{(-)},\Omega_{J',M_{24}}{}^{(-)})+iL(\Omega_{J,M_{31}}{}^{(-)},\Omega_{J',M_{24}}{}^{(+)})+iL(\Omega_{J,M_{31}}{}^{(+)},\Omega_{J',M_{24}}{}^{(-)})] \\ &\quad -(-)^{\mu_2-\mu_1}\langle s,s;\mu_3,-\mu_2|J,M_{32}\rangle\langle s,s;\mu_1,-\mu_4|J',M_{14}\rangle[L(\Omega_{J,M_{32}}{}^{(+)},\Omega_{J',M_{14}}{}^{(+)}) \\ &\quad -L(\Omega_{J,M_{32}}{}^{(-)},\Omega_{J',M_{14}}{}^{(-)})+iL(\Omega_{J,M_{32}}{}^{(-)},\Omega_{J',M_{14}}{}^{(+)})+iL(\Omega_{J,M_{32}}{}^{(+)},\Omega_{J',M_{14}}{}^{(-)})]\}(-1)^{2s-\mu_1-\mu_4}. \end{aligned} \tag{3.19a}$$

We have used the notation

$$\begin{aligned} M_{31}&\equiv\mu_3-\mu_1, \text{ etc.}, \\ L(\Omega_{J,M}{}^{(+)},\Omega_{J',M'}{}^{(-)})&\equiv\mathrm{Tr}(\Omega_{J,M}{}^{(+)}M\Omega_{J',M'}{}^{(-)}M)\,, \text{ etc.}, \end{aligned} \tag{3.19b}$$

and $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=0$ is the equation obtained from $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}=0$ by taking the complex conjugate. Equation (3.16) together with the first line of Eq. (3.19a) shows that

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}=-\mathcal{L}_{\mu_2\mu_1\mu_3\mu_4}\,, \tag{3.20a}$$

$$\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}=\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}{}^*\,. \tag{3.20b}$$

Since the $\Omega_{J,M}{}^{(\pm)}$ form a complete set of $(2s+1)\times(2s+1)$ matrices, the density matrix in spin space for any spin-s particle can be written as a linear combination of them. The expectation value of any spin observable in the final state of reaction (3) can then be written[1] as a linear combination of $L(\Omega_{J,M}{}^{(\pm)},\Omega_{J',M'}{}^{(\pm)})$. Accordingly, the $L(\Omega_{J,M}{}^{(\pm)},\Omega_{J',M'}{}^{(\pm)})$ form a complete set of observable components. Then $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}=0$ is a linear relation between observable components (and they are given in the "spherical representation"). Comparison of relations (3.2) and (3.18) shows, that *for any values* $\mu_1\mu_2\mu_3\mu_4(\mu_1\neq\mu_2)$, *the* $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}=0$ *or* $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=0$ *is an NDT test.*

Furthermore, the set of all $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}$ with $\mu_1<\mu_2$ together with the set of all $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*$ with $\mu_1<\mu_2$ is a complete set of NDT tests. This follows from the fact that all bilinear combinations of elements of M implied by the symmetry of M are of the form (3.17), and these are all contained in the set of an $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}$ together with the set of all $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*$ with $\mu_1\neq\mu_2$. This together with Eq. (3.18) proves the statement.

On the other hand, in this set not all $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}$ and $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*$ with $\mu_1<\mu_2$ are linearly independent. In fact,

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}-\mathcal{L}_{\mu_1\mu_2\mu_4\mu_3}+\mathcal{L}_{\mu_3\mu_4\mu_2\mu_1}{}^*-\mathcal{L}_{\mu_3\mu_4\mu_1\mu_2}{}^*\equiv0\,, \tag{3.21a}$$

and also

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*-\mathcal{L}_{\mu_1\mu_2\mu_4\mu_3}{}^*+\mathcal{L}_{\mu_3\mu_4\mu_2\mu_1}-\mathcal{L}_{\mu_3\mu_4\mu_1\mu_2}\equiv0\,. \tag{3.21b}$$

A complete linearly independent set of NDT tests can now easily be written down. We first write down the set of all $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}$ and $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*$ with $\mu_1<\mu_2$. Then, to take into account Eq. (3.21), we divide them in groups of four, those four which are connected by Eq. (3.21). Out of each group we omit one. (If $\mu_3=\mu_4$, then two of the $\mathcal{L}$'s in Eq. (3.21) represent trivial relations; of the remaining two we omit one.)

Next we rewrite this complete set of linearly independent NDT tests in a more manageable form. Our aim is to write as many as possible of the NDT tests in the form

$$M_{\mu_1\mu_2}M_{\mu_3\mu_4}{}^*-M_{\mu_2\mu_1}M_{\mu_4\mu_3}{}^*=0\,. \tag{3.22}$$

As will be shown later, relations of this type are closely related to "mirror relations." We note that if $\mu_3=\mu_4$, then two of the four $\mathcal{L}$'s appearing in each of Eqs. (3.21) are trivial, of the remaining two one only is linearly independent and that can be written in the form (3.22). This can be seen using Eqs. (3.19a) and (3.16). If on the other hand, when $\mu_3\neq\mu_4$ and neither $\mu_1=\mu_3$, $\mu_2=\mu_4$, nor $\mu_1=\mu_4$, $\mu_2=\mu_3$ holds, then none of the four $\mathcal{L}$'s appearing in each of Eqs. (3.21) are trivial and three of them are linearly independent. Four linear combinations of these six linearly independent $\mathfrak{L}$'s can be written in the form of Eq. (3.22), while the remaining two cannot. The first part of this statement can be proven by exhibiting four such linear combinations:

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}+\mathfrak{L}_{\mu_4\mu_3\mu_1\mu_2}{}^*=M_{\mu_1\mu_2}M_{\mu_4\mu_3}{}^* -M_{\mu_2\mu_1}M_{\mu_3\mu_4}{}^*=0\,, \quad (3.23a)$$

$$-\mathfrak{L}_{\mu_1\mu_2\mu_4\mu_3}+\mathfrak{L}_{\mu_4\mu_3\mu_1\mu_2}{}^*=M_{\mu_2\mu_1}M_{\mu_4\mu_3}{}^* -M_{\mu_1\mu_2}M_{\mu_3\mu_4}{}^*=0\,, \quad (3.23b)$$

and their two complex-conjugate relations. The fact that the remaining two cannot be written in such a form follows from the observation, that the set of all $\mathfrak{L}$ and $\mathfrak{L}^*$ for $\mu_1<\mu_2$ is a complete set of NDT tests, and therefore must contain a sufficient set of NDT tests. On the other hand, all equations of the form (3.22) do not require the symmetry of M, in fact, an antisymmetric M would also satisfy them. The additional two conditions are the ones which require that M be its own transpose. These two conditions can be chosen to be $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}=0$ and $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=0$. In the case when $\mu_3\neq\mu_4$ but either $\mu_1=\mu_3$, $\mu_2=\mu_4$, or $\mu_1=\mu_4$, $\mu_2=\mu_3$ holds, then according to (3.20b) the $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}$ is no longer linearly independent of its complex conjugate, nor are the two equations in (3.21). Equations (3.23) are then equal to their own complex conjugates and we are left with one relation which is not of the form of Eq. (3.22). This can be chosen to be $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}=0$.

It will now be shown that the set of all relations of the type (3.22) holds if and only if, the set of all mirror relations hold. To prove it, substitute Eqs. (3.16) and (3.19a) into (3.22), observe that the transpose of $\Omega_{J,M}{}^{(\pm)}$,

$$(\Omega_{J,M}{}^{(\epsilon)})^{\sim}=\epsilon\Omega_{J,M}{}^{(\epsilon)}\,,\quad \epsilon=\pm\,, \quad (3.24)$$

and obtain

$$L(\Omega_{J,M}{}^{(\epsilon)};\Omega_{J',M'}{}^{(\eta)})=\epsilon\eta L(\Omega_{J',M'}{}^{(\eta)};\Omega_{J,M}{}^{(\epsilon)})\,. \quad (3.25)$$

This last equation is, according to the definition given in Sec. II, a mirror relation. Clearly the trivial mirror relations satisfied by self-mirrored observable components correspond to the trivial relation of the type (3.22) when $\mu_1=\mu_2$ and $\mu_3=\mu_4$. As we have just shown, any complete set of linearly independent NDT tests includes at least one nonmirror relation. In fact, the number of linearly independent nonmirror relations is $[s(2s+1)]^2$, as is evident from Eq. (2.56).

We are now in a position to write down a complete set of linearly independent NDT tests in a simple form. We include in the set all mirror relations which we denote by $\{\mathfrak{L}_{\rm mirror}\}$, and $[s(2s+1)]^2$ nonmirror relations. As we know, the latter can be chosen to be the vanishing of $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}=0$, $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=0$ for all $\mu_1<\mu_2$ and $\mu_3<\mu_4$. Instead of choosing the restrictions $\mu_1<\mu_2$, $\mu_3<\mu_4$, we may choose another set of restrictions $|\mu_1|\leq|\mu_2|$, $|\mu_3|\leq|\mu_4|$, if $|\mu_1|=|\mu_2|$, then $\mu_1<0$, $\mu_2>0$, if $|\mu_3|=|\mu_4|$, then $\mu_3<0$, $\mu_4>0$. The set of these conditions we shall denote by $\{\mathfrak{L}\}$. Assuming all mirror relations to hold, as expressed by Eq. (3.25) and making use of the well-known property of the Clebsch-Gordan coefficients that

$$\langle s,s;\mu_1,\mu_2|J,M\rangle=\langle s,s;-\mu_2,-\mu_1|J,-M\rangle\,, \quad (3.26)$$

we find that

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}{}^*=\mathfrak{L}_{\mu_3\mu_4\mu_1\mu_2}\,, \quad (3.27a)$$

which together with (3.20a) implies also

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}=-\mathfrak{L}_{\mu_1\mu_2\mu_4\mu_3}\,. \quad (3.27b)$$

Instead of including into our complete set the set of conditions $\{\mathfrak{L}\}$, we now choose to impose the equivalent set of conditions $\{\mathfrak{L}(\gamma)\}$, $\gamma=\pm$, defined as the vanishing of all $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma)$, where $\gamma=\pm$, $|\mu_1|\leq|\mu_2|$, $|\mu_3|\leq|\mu_4|$ and if $|\mu_1|=|\mu_2|$, then $\mu_1<0$, $\mu_2>0$, if $|\mu_3|=|\mu_4|$, then $\mu_3<0$, $\mu_4>0$

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}{}^{(+)}\equiv\tfrac{1}{2}(\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}+\mathfrak{L}_{\mu_3\mu_4\mu_1\mu_2})=\mathrm{Re}\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}\,, \quad (3.28a)$$

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}{}^{(-)}\equiv\frac{1}{2i}(\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}-\mathfrak{L}_{\mu_3\mu_4\mu_1\mu_2}) =\mathrm{Im}\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}\,. \quad (3.28b)$$

Of course, if $\mu_1=\mu_3$, $\mu_2=\mu_4$, then

$$\mathfrak{L}_{\mu_1\mu_2\mu_1\mu_2}(-)=0 \quad (3.28c)$$

and if $\mu_1=\mu_4$, $\mu_2=\mu_3$, then

$$\mathfrak{L}_{\mu_1\mu_2\mu_2\mu_1}(+)=0\,. \quad (3.28d)$$

Finally, we can define yet another set of nonmirror relations, namely $\{\mathfrak{L}(\gamma,\delta)\}$, $\gamma=\pm$, $\delta=\pm$, defined as the set of conditions requiring the vanishing of all $\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$, where $\gamma=\pm$, $\delta=\pm$, $|\mu_1|\geqslant|\mu_2|$, $|\mu_3|\geqslant|\mu_4|$, $\mu_1<0$, if $|\mu_3|=|\mu_4|$, and/or $|\mu_1|=|\mu_2|$, then either $\mu_3<0$ or $\mu_3=0<\mu_4$

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)\equiv\tfrac{1}{2}[\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma) +\gamma\cdot\delta\cdot(-1)^{\mu_1+\mu_4-\mu_2-\mu_3}\mathfrak{L}_{-\mu_1,-\mu_2,-\mu_3,-\mu_4}(\gamma)]\,. \quad (3.29)$$

The advantage of using the set $\{\mathfrak{L}(\gamma,\delta)\}$ is, that each condition in it is in general a linear combination of roughly half as many terms as in the set $\{\mathfrak{L}(\gamma)\}$. This follows from a property of the Clebsch-Gordan coefficients:

$$\langle s,s;\mu_1,\mu_2|J,M\rangle=(-1)^{2s-J} \times\langle s,s;-\mu_1,-\mu_2|J,-M\rangle\,. \quad (3.30)$$

Substituting Eqs. (3.30) and (3.28) into Eq. (3.29) we find

$$\mathfrak{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta) =\sum_{J=|M_{31}|}^{2s}\sum_{J'=|M_{24}|}^{2s}C(s;\mu_3\mu_1\mu_2\mu_4;J,J') \times L^{(\gamma)}(\Omega_{J,M_{31}};\Omega_{J',M_{24}})+(-1)^{1+\mu_2-\mu_1} \times\sum_{J=|M_{32}|}^{2s}\sum_{J'=|M_{14}|}^{2s}C(s;\mu_3\mu_2\mu_1\mu_4;J,J') \times L^{(\gamma)}(\Omega_{J,M_{32}};\Omega_{J',M_{14}}) \quad (3.31a)$$

where

$J+J'$ can take only even integer values if $\delta=+$
$J+J'$ can take only odd integer values if $\delta=-$

and the following notation is used:

$$M_{ab}\equiv\mu_a-\mu_b,$$
$$C(s;\mu_a\mu_b\mu_c\mu_d;J,J')\equiv\langle s,s;\mu_a-\mu_b|J,M_{ab}\rangle \times\langle s,s;\mu_c-\mu_d|J',M_{cd}\rangle,$$
$$L^{(+)}(\Omega_{J,M_{ab}},\Omega_{J',M_{cd}})\equiv L(\Omega_{J,M_{ab}}{}^{(+)};\Omega_{J',M_{cd}}{}^{(+)}) -L(\Omega_{J,M_{ab}}{}^{(-)};\Omega_{J',M_{cd}}{}^{(-)}), \quad (3.31b)$$
$$L^{(-)}(\Omega_{J,M_{ab}};\Omega_{J',M_{cd}})\equiv L(\Omega_{J,M_{ab}}{}^{(-)};\Omega_{J',M_{cd}}{}^{(+)}) +L(\Omega_{J,M_{ab}}{}^{(+)};\Omega_{J',M_{cd}}{}^{(-)}).$$

If $\mu_1=-\mu_2$ then $\mathcal{L}_{-\mu_1,-\mu_2,-\mu_3,-\mu_4}(\gamma)$ does not belong to our complete set of linearly independent NDT tests. This is so because it does not satisfy the condition that if $|\mu_1|=|\mu_2|$ then $\mu_1<\mu_2$ (if $\mu_1<\mu_2$, then $-\mu_1>-\mu_2$). Therefore, it is a linear combination of $\mathcal{L}_{\mu_1-\mu_2,\mu_3\mu_4}(\gamma)$ and the other conditions belonging to our complete set. Since we are interested in counting linearly independent conditions, we must not count the two conditions $\mathcal{L}_{\mu_1,-\mu_1,\mu_3,\mu_4}(\gamma,\delta)$ for $\delta=+$ and $\delta=-$ separately. Alternatively, we may count the two conditions given by $\delta=+$ and $\delta=-$ separately, but then we have to restrict the values of μ_3 by requiring for example $\mu_3<0$. Indeed, the $\mathcal{L}_{\mu_1,-\mu_1,\mu_3,\mu_4}(\gamma,\delta)$ for $\mu_3>0$ are then linear combinations of those in which $\mu_3<0$, because

$$\mathcal{L}_{\mu_1,-\mu_1,\mu_3\mu_4}(\gamma,\delta)+\gamma\cdot\delta\cdot(-1)^{\mu_1+\mu_4-\mu_2-\mu_3} \times\mathcal{L}_{\mu_1,-\mu_1,-\mu_3,-\mu_4}(\gamma,\delta)\equiv0. \quad (3.29')$$

Similarly, if $\mu_3=-\mu_4$, we have to impose the condition $\mu_3<0$, because we have

$$\mathcal{L}_{\mu_1,\mu_2,\mu_3,-\mu_3}(\gamma,\delta)+\gamma\cdot\delta\cdot(-1)^{\mu_1+\mu_4-\mu_2-\mu_3} \times\mathcal{L}_{-\mu_1,-\mu_2,\mu_3,-\mu_3}(\gamma,\delta)\equiv0. \quad (3.29'')$$

Finally, if $\mu_1=-\mu_2$, and *at the same time* $\mu_3=-\mu_4$, then it follows from Eqs. (3.20a) and (3.27b) that

$$\mathcal{L}_{\mu_1,-\mu_1,\mu_3,-\mu_3}(\gamma,-)\equiv0. \quad (3.29''')$$

In conclusion we can say that a complete set of linearly independent NDT tests, $\{\mathcal{L}\}_{\text{complete}}$, is obtained by adding to the set of all mirror relations $\{\mathcal{L}_{\text{mirror}}\}$, a set of $[s(2s+1)]^2$ nonmirror relations, $\{\mathcal{L}(\gamma,\delta)\}$.

$$\{\mathcal{L}\}_{\text{complete}}=\{\mathcal{L}_{\text{mirror}}\}+\{\mathcal{L}(\gamma,\delta)\}. \quad (3.32)$$

The set $\{\mathcal{L}_{\text{mirror}}\}$ is given by the set of all equations of the form (3.25) and the set of conditions $\{\mathcal{L}(\gamma,\delta)\}$ is obtained by requiring that $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$ vanish for $\gamma=\pm$, $\delta=\pm$, and for all those values of μ_1, μ_2, μ_3, and μ_4 which satisfy the conditions that $|\mu_1|\leq|\mu_2|$, $|\mu_3|\leq|\mu_4|$, $\mu_1<0$, and if $|\mu_3|=|\mu_4|$ and/or $|\mu_1|=|\mu_2|$, then, in these special cases, we have either $\mu_3<0$, or we have $\mu_3=0<\mu_4$. The expression for $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$ is given by Eqs. (3.31). Note that the $C(s;\mu_a\mu_b\mu_c\mu_d;J,J')$ coefficients appearing in Eqs. (3.31) are the same for $\gamma=+$ and $\gamma=-$. This helps when carrying out actual calculations. To illustrate the use of these formulas, the case when $s=\frac{1}{2}$ is worked out in detail in the next section.

It is now easy to write down a *sufficient and necessary set of linearly independent NDT tests*. We claim that the set of all $\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}$ is such a set, i.e.,

$$T \text{ invariance} \Longleftrightarrow \mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}=0 \text{ for all } \mu_1\neq\mu_2. \quad (3.33)$$

To prove this, we only have to use Eqs. (3.19a), (3.16) and (3.6) and show that

$$M=\tilde{M} \Longleftrightarrow (M_{\mu_1\mu_2}-M_{\mu_2\mu_1})M_{\mu_1\mu_2}{}^* \text{ for all } \mu_1\neq\mu_2. \quad (3.34)$$

First of all, it is clear from the foregoing that in relation (3.34) the $\Rightarrow$ holds. Furthermore, if either $M_{\mu_1\mu_2}\neq0$ or $M_{\mu_2\mu_1}\neq0$, then relation (3.34) implies that $M_{\mu_1\mu_2}=M_{\mu_2\mu_1}$. If on the other hand, both $M_{\mu_1\mu_2}=0$ and $M_{\mu_2\mu_1}=0$, then again $M_{\mu_1\mu_2}=M_{\mu_2\mu_1}$. On the other hand, if any one of the relations is omitted, for example $\mathcal{L}_{\mu_1'\mu_2'\mu_1'\mu_2'}$, then T invariance need not hold because $M_{\mu_1'\mu_2'}\neq M_{\mu_2'\mu_1'}$ may be true if $M_{\mu_1'\mu_2'}=0$. This concludes the proof.

Proceeding just as we have done after Eq. (3.26), one can show that[8]

$$\{\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}=0 \text{ for all } \mu_1<\mu_2\} \Longleftrightarrow \mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}(+,\delta), \quad (3.35a)$$

for $\delta=\pm1$ for all $\mu_1\neq\mu_2$

$$\text{which satisfy } |\mu_1|\geq|\mu_2|, \quad \mu_1<0, \quad (3.35b)$$

(when $|\mu_1|=|\mu_2|$, then $\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}(+-)\equiv0$ and only $\delta=+$ has to be tested) where $\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}(+\delta)$ is given by Eq. (3.31a), if in it we substitute $\mu_3=\mu_1$, $\mu_4=\mu_2$. Therefore, a sufficient set of linearly independent NDT tests is given by all $\mathcal{L}_{\mu_1\mu_2\mu_1\mu_2}(+,\delta)$ for which μ_1, μ_2 satisfy Eq. (3.35b).

We remark that this set contains at least one NDT test which involves measurement of the most "complicated" observable component, namely, $L(\Omega_{J=2s,M=2s},\Omega_{J'=2s,M'=2s})$. Therefore, one cannot establish T invariance using this set without making any assumption about dynamics, and get away without measuring the most complicated polarization tensors possible. To prove this statement, choose $\mu_1=-s$, $\mu_2=s$ and observe that according to Eq. (3.31a), the expression $\mathcal{L}_{-s,+s,-s,+s}(++)$ involves $L^{(+)}(\Omega_{J=2s,M=2s},\Omega_{J'=2s,M'=2s})$. It can also be seen, that all other observable components appearing in $\mathcal{L}_{-s,+s,-s,+s}(++)$ have $M=M'\neq2s$, and, consequently, cannot cancel this observable component.

We are now in a position to prove the following simple theorem. *Any nonmirror relation is a linear*

[8] When neither $\mu_1=\mu_3$, $\mu_2=\mu_4$ nor $\mu_1=\mu_4$, $\mu_2=\mu_3$ holds, then to obtain Eq. (3.27a) it was necessary to assume the validity of all mirror relations. However, in the present case $\mu_1=\mu_3$, $\mu_2=\mu_4$ and Eq. (3.27a) now follows just by using Eq. (3.26), *without* assuming anything about mirror relations.

combination of observable components which have the property that some spin information is given about all the particles with nonzero spin before scattering by at least one observable component appearing in the relation. The same is true about all particles after scattering.

To see this, we recall that according to Eq. (3.19a), any nonmirror relation can be written in the form

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}\equiv \mathrm{Tr}(S_{\mu_3\mu_1}MS_{\mu_2\mu_4}M^\dagger)-\mathrm{Tr}(S_{\mu_3\mu_2}MS_{\mu_1\mu_4}M^\dagger)=0\,,$$

where $\mu_1\neq\mu_2$, $\mu_3\neq\mu_4$. If $\mu_2-\mu_1=0$, then $\mu_3-\mu_2$ is certainly not, and vice versa. Similarly, at most only one of $\mu_2-\mu_4$ and $\mu_1-\mu_4$ can be zero. First of all assume the case when $\mu_3-\mu_1\neq 0$, $\mu_2-\mu_4\neq 0$. (The proof is similar in the other three cases, when (a) $\mu_3-\mu_1\neq 0$, $\mu_2-\mu_4=0$, (b) $\mu_3-\mu_1=0$, $\mu_2-\mu_4\neq 0$, (c) $\mu_3-\mu_1=0$, $\mu_2-\mu_4=0$.) Equation (3.14) then shows that the first term in the above equation can be written as a linear combination of observable components: $L(\Omega_{J,M}{}^{(\pm)}, \Omega_{J',M'}{}^{(\pm)})$ where $J\neq 0$ and $J'\neq 0$. Accordingly, all of these observable components contain some polarization information about the particle of spin s before scattering. They also contain polarization information about this particle after scattering. The second term may similarly be written as a linear combination of observable components $L(\Omega_{J'',M''}{}^{(\pm)},\Omega_{J''',M'''}{}^{(\pm)})$ where $M''\equiv\mu_3-\mu_2\neq\mu_3-\mu_1\equiv M$, $M'\equiv\mu_2-\mu_4\neq\mu_3-\mu_4\equiv M'''$. Therefore, the $L(\Omega_{J'',M''}{}^{(\pm)},\Omega_{J''',M'''}{}^{(\pm)})$ are all different from all $L(\Omega_{J,M}{}^{(\pm)},\Omega_{J',M'}{}^{(\pm)})$, and cannot cancel any of them. This concludes the proof.

Theorem 3 of Ref. 5, referring to T invariance, is an immediate consequence of the foregoing. In our present language it states that any sufficient and necessary set of NDT tests must include tests in which observable components appear giving some polarization information about all participating particles. This is now obvious, since we know that for reactions of the type (3.5) any sufficient and necessary set of linearly independent NDT tests must include nonmirror relations. [In view of what is said below it is clear that this is true also for reactions of the type (3.4).]

We mention in passing that the "observable effects of invariance principles" given in Ref. 9, Sec. B, to test T invariance in fermion-fermion scattering experiments are all what we here call NDT tests. They form a complete linearly independent set of NDT tests for those experiments which are discussed in that section. Of course, they do not form a sufficient set since they do not include nonmirror experiments. This is so because they refer to experiments in which one particle is unpolarized, and the polarization state of another one is not measured, in other words, no spin information is obtained about two of the particles.

Turning now to the composite reaction (3.4), we factorize its M matrix according to Eq. (2.15). A complete set of linearly independent NDT tests can then be obtained by "multiplying together" the complete sets of linearly independent NDT tests of the two composite reactions, or relations closely related to them. This fact is of great help when actually writing down NDT tests. The underlying principle is easy to understand, and all proofs are simple generalizations of the ones given above for the simpler reactions (3.5). Therefore, we shall not go into details.

Let us define

$$\mathcal{L}'_{\mu_1\mu_2\mu_3\mu_4}\equiv\mathrm{Tr}(S_{\mu_3\mu_1}MS_{\mu_2\mu_4}M^\dagger)+\mathrm{Tr}(S_{\mu_3\mu_2}MS_{\mu_1\mu_4}M^\dagger)\,. \quad (3.36)$$

Observe that from Eq. (3.16) it follows that

$$(M=-\tilde{M})<=>\begin{pmatrix}\mathcal{L}'_{\mu_1\mu_3\mu_4\mu_5}=0 \text{ for all } \mu_1,\,\mu_2,\,\mu_3,\,\mu_4 \quad (\mu_1\neq\mu_2)\\ \mathcal{L}'_{\mu_1\mu_2\mu_3\mu_4}{}^*=0 \text{ for all } \mu_1,\,\mu_2,\,\mu_3,\,\mu_4 \quad (\mu_1\neq\mu_2)\end{pmatrix}. \quad (3.37)$$

A sufficient linearly independent set of linear relations between observable components for $M=-\tilde{M}$ to hold is given by the set of all mirror relations together with the set $\{\mathcal{L}'(\gamma,\delta)\}$, the set[10] of $\mathcal{L}'_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)=0$ for which the μ satisfy the conditions stated before [Eq. (3.29a)]

$$\{M=-\tilde{M}\}<=>\{\mathcal{L}_{\mathrm{mirror}}\}+\{\mathcal{L}'(\gamma,\delta)\}\,. \quad (3.38)$$

Let us introduce the convention that whenever we write some quantities in round brackets with a subscript 1, 2, or 3 outside the bracket, then the quantities refer to reactions (3.4), (3.5a) and (3.5b), respectively. Thus $(M)_1$ is the M matrix of reaction (3.4), $(S_{\mu'\mu})_2$ is a matrix in the spin space of particle s in reaction (3.5a), etc.

Any $\mathcal{L}$ belonging to $\{\mathcal{L}\}$ can be written schematically as

$$\mathcal{L}^\alpha=\sum C^\alpha(J_1M_1\epsilon_1;J_2M_2\epsilon_2)L(\Omega_{J_1M_1}{}^{(\epsilon_1)};\Omega_{J_2M_2}{}^{(\epsilon_2)})\,,\quad \epsilon_1,\,\epsilon_2=\pm \quad (3.39a)$$

where $\mathcal{L}^\alpha$ is some $\mathcal{L}$ belonging to $\{\mathcal{L}\}$. Similarly, any $\mathcal{L}'^\beta$ belonging to $\{\mathcal{L}'\}$ has the form

$$\mathcal{L}'^\beta=\sum C'^\beta(J_3M_3\epsilon_3,J_4M_4\epsilon_4)\times L(\Omega_{J_3M_3}{}^{(\epsilon_3)};\Omega_{J_4M_4}{}^{(\epsilon_4)})\,. \quad (3.39b)$$

The coefficients

$$C^\alpha(J_1M_1\epsilon_1;J_2M_2\epsilon_2)\,,\quad C'^\beta(J_3M_3\epsilon_3;J_4M_4\epsilon_4)$$

are defined by Eqs. (3.39). The summation runs over all values of all arguments of the coefficient C.

[9] P. L. Csonka, Rev. Mod. Phys. **37**, 177 (1965).
[10] The definition of $\mathcal{L}'_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$ is analogous to the definition of $\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$.

The symmetry of M_1 clearly implies the following relations:

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4;\mu_5\mu_6\mu_7\mu_8} \equiv [(M_{\mu_1\mu_2})_2(M_{\mu_5\mu_6})_3 - (M_{\mu_2\mu_1})_2(M_{\mu_6\mu_5})_3](M_{\mu_3\mu_4}{}^*)_2(M_{\mu_7\mu_8}{}^*)_3 = 0. \quad (3.40)$$

According to Eq. (3.16), these relations can be written as

$$\mathrm{Tr}(S_{\mu_3\mu_1}MS_{\mu_2\mu_4}M^\dagger)_2\,\mathrm{Tr}(S_{\mu_7\mu_5}MS_{\mu_6\mu_8}M^\dagger)_3 - \mathrm{Tr}(S_{\mu_3\mu_2}MS_{\mu_1\mu_4}M^\dagger)_2\,\mathrm{Tr}(S_{\mu_7\mu_6}MS_{\mu_5\mu_8}M^\dagger)_3 = 0. \quad (3.41)$$

Any one of these relations, for example the ρth one, can be written as

$$\mathcal{L}^{\rho}{}_{\mu_1\mu_2\mu_3\mu_4;\mu_5\mu_6\mu_7\mu_8} = \sum C^\rho(J_1M_1\epsilon_1,J_2M_2\epsilon_2;J_3M_3\epsilon_3,J_4M_4\epsilon_4) \times L(\Omega_{J_1M_1}{}^{\epsilon_1},\Omega_{J_2M_2}{}^{\epsilon_2};\Omega_{J_3M_3}{}^{\epsilon_3},\Omega_{J_4M_4}{}^{\epsilon_4}), \quad (3.42)$$

where the summation goes over all arguments of C. Every one of the Eqs. (3.42) is an NDT test for reaction (3.4).

It is easy to prove that the set of all $\mathcal{L}_{\mu_1\cdots\mu_8}{}^{\rho}$ is the same as the set of all $\mathcal{L}_{\mu_1\cdots\mu_8}{}^{\alpha\beta}$ and the set of all $\mathcal{L}'_{\mu_1\cdots\mu_8}{}^{\alpha\beta}$, where

$$\begin{aligned}\mathcal{L}_{\mu_1\cdots\mu_8}{}^{\alpha\beta} &= \sum C^\alpha(J_1M_1\epsilon_1;J_2M_2\epsilon_2)\\ &\quad\times C^\beta(J_3M_3\epsilon_3;J_4M_4\epsilon_4)\\ &\quad\times L(\Omega_{J_1M_1}{}^{\epsilon_1},\Omega_{J_2M_2}{}^{\epsilon_2};\Omega_{J_3M_3}{}^{\epsilon_3},\Omega_{J_4M_4}{}^{\epsilon_4}),\\ \mathcal{L}'_{\mu_1\cdots\mu_8}{}^{\alpha\beta} &\equiv \sum C'^\alpha(J_1M_1\epsilon_1;J_2M_2\epsilon_2)\\ &\quad\times C'^\beta(J_3M_3\epsilon_3;J_4M_4\epsilon_4)\\ &\quad\times L(\Omega_{J_1M_1}{}^{\epsilon_1},\Omega_{J_2M_2}{}^{\epsilon_2};\Omega_{J_3M_3}{}^{\epsilon_3},\Omega_{J_4M_4}{}^{\epsilon_4}).\end{aligned} \quad (3.43)$$

In other words, all NDT tests for reaction (3.4) can be obtained if we require that all relations of the form (3.43) be equal to zero. The coefficients of observable components in these relations are *products* of certain coefficients of relations ($\mathcal{L}$ or $\mathcal{L}'$) referring to the two constituent reactions (3.5). In this sense we can write schematically for the set of all ($\mathcal{L}^{\alpha\beta}$)

$$\{(\mathcal{L})_1\} = \{(\mathcal{L})_2\}\otimes\{(\mathcal{L})_3\} + \{(\mathcal{L}')_2\}\otimes\{(\mathcal{L}')_3\}. \quad (3.44)$$

Similarly, one can show that

$$\{(\mathcal{L})_1\}_{\text{complete}} = \{(\mathcal{L})_2\}_{\text{complete}}\otimes\{(\mathcal{L})_3\}_{\text{complete}} + \{(\mathcal{L}')_2\}_{\text{complete}}\otimes\{(\mathcal{L}')_3\}_{\text{complete}}. \quad (3.45)$$

Analogous statements hold for sufficient and necessary sets. Equations (3.44) and (3.45) are a simple consequence of the fact expressed in Eq. (2.16), that a symmetric $(M)_1$ matrix for reaction (3.4) consists of two parts, one is a product of two symmetric M matrices, referring to the constituent reactions, and the other part is a product of two antisymmetric M matrices.

IV. EXAMPLES

In this section we will discuss three reactions in detail as examples for the results of Secs. II and III. If we wish to emphasize that a certain observable (component) switches sign under parity transformation, then we will call it[3] a "pseudo-observable (component)". In other words, we use the phrases "observable (component)" and "pseudo-observable (component)" analogously to the phrases "scalar" and "pseudoscalar."

1. $\frac{1}{2}+0 \to \frac{1}{2}+0$, Rotation Invariance Only

For this reaction we have

$$M = b_0 + b_1\boldsymbol{\sigma}\cdot\hat{l} + b_2\boldsymbol{\sigma}\cdot\hat{m} + b_3\boldsymbol{\sigma}\cdot\hat{n}, \quad (4.1)$$

where σ is the Pauli spin matrix. If T invariance holds, we have

$$M_T = b_0 + b_2\boldsymbol{\sigma}\cdot\hat{m} + b_3\boldsymbol{\sigma}\cdot\hat{n}. \quad (4.2)$$

For a discussion of the subclass structure of this reaction, see Ref. 11, Table II. This table presents all observable components and pseudo-observable components for the case of rotation invariance and parity conservation, from which we can obtain the pure rotation invariant case by ignoring all headings which refer to parity quantum numbers.

The total number of observable components in this case is $4^2=16$. The number of bilinear combinations of form factors, if T invariance holds, is, on the other hand, only 9. Thus there must be seven relations among the observable components, imposed by the requirement of T invariance. Of these, six are mirror-type relations, which can be obtained immediately from the subclass tables by making $b_1=0$.

$$L(0,m;0,0) = L(0,0;0,m), \quad (4.3)$$

$$L(0,l;0,n) = -L(0,n;0,l), \quad (4.4)$$

$$L(0,l;0,0) = -L(0,0;0,l), \quad (4.5)$$

$$L(0,n;0,m) = L(0,m;0,n), \quad (4.6)$$

$$L(0,n;0,0) = -L(0,0;0,n), \quad (4.7)$$

$$L(0,l;0,m) = L(0,m;0,l). \quad (4.8)$$

These are the set of relations denoted in Sec. III by $\{\mathcal{L}\}_{\text{mirror}}$. The seventh relationship cannot be a mirror type, since Eqs. (4.3)–(4.8) exhaust all the possibilities in this respect. A more detailed inspection reveals that the seventh relationship comes from subclass $L(I\text{-}1)$ and is

$$L(0,0;0,0) - L(0,m;0,m) = L(0,n;0,n) - L(0,l;0,l). \quad (4.9)$$

Although in this rather trivial case the above relationship can be obtained virtually by inspection alone, we will now derive this relation again as an illustration of the general techniques developed in Sec. III. In this case, Eq. (4.9) represents the only relation in the set $\{\mathcal{L}(\gamma,\delta)\}$. To find this set, we have to find the set of all

[11] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Rev. Mod. Phys. (to be published).

$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4}(\gamma,\delta)$ which satisfy $\mu_1<\mu_2$, $\mu_3<\mu_4$, $\mu_3<0$. Since every μ can take only the value $+\frac{1}{2}$ and $-\frac{1}{2}$, only the choice $\mu_1=-\frac{1}{2}$, $\mu_2=\frac{1}{2}$, $\mu_3=-\frac{1}{2}$, $\mu_4=+\frac{1}{2}$ satisfies all these conditions. Furthermore, since in this case $\mu_1=\mu_3$ and $\mu_2=\mu_4$, according to Eq. (3.28c) this $\mathcal{L}$ vanishes identically for $\gamma=-$. At the same time, because $\mu_1=-\mu_2$, $\mu_3=-\mu_4$, we conclude from Eq. (3.29′) that $\mathcal{L}$ vanishes identically for $\delta=-$. Therefore we have to evaluate only $\mathcal{L}_{-1/2,+1/2,-1/2,+1/2}$.

For this purpose we use Eq. (3.31a), and find

$$\mathcal{L}_{-1/2,+1/2,-1/2,+1/2}=\sum_{J=0}^{1}\sum_{J'=0}^{1}\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},+\tfrac{1}{2}|J,0\rangle\langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|J',0\rangle[L(\Omega_{J,0}{}^{(+)},\Omega_{J',0}{}^{(+)})-L(\Omega_{J,0}{}^{(-)},\Omega_{J',0}{}^{(-)})]$$
$$+(-1)^{1+1/2+1/2}\sum_{J=1}^{1}\sum_{J'=1}^{1}\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}|J,-1\rangle\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}|J',-1\rangle$$
$$\times[L(\Omega_{J,-1}{}^{(+)},\Omega_{J',-1}{}^{(+)})-L(\Omega_{J,-1}{}^{(-)},\Omega_{J',-1}{}^{(-)})],\quad(4.10)$$

where $J+J'$ can take only even values, since $\delta=+$.

According to Eq. (3.12), we have $\Omega_{J,0}{}^{(-)}=0$. Therefore we can write

$$\mathcal{L}_{-1/2,+1/2,-1/2,+1/2}(+,+)=\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},+\tfrac{1}{2}|0,0\rangle\langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|0,0\rangle L(\Omega_{0,0}{}^{(+)},\Omega_{0,0}{}^{(+)})$$
$$+\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},+\tfrac{1}{2}|1,0\rangle\langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|1,0\rangle L(\Omega_{1,0}{}^{(+)},\Omega_{1,0}{}^{(+)})$$
$$+\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}|1,-1\rangle\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}|1,-1\rangle[L(\Omega_{1,-1}{}^{(+)},\Omega_{1,-1}{}^{(+)}-L(\Omega_{1,-1}{}^{(-)},\Omega_{1,-1}{}^{(-)})].\quad(4.11)$$

Looking up the Clebsch-Gordan coefficients we find

$$\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},+\tfrac{1}{2}|0,0\rangle=-\langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|0,0\rangle=-\frac{1}{\sqrt{2}},$$
$$\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},+\tfrac{1}{2}|1,0\rangle=+\langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|1,0\rangle=+\frac{1}{\sqrt{2}},$$
$$\langle\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}|1,-1\rangle=+1.$$

Therefore the only linearly independent nonmirror-type relation is

$$\mathcal{L}_{-1/2,+1/2,-1/2,+1/2}(+,+)=-\tfrac{1}{2}L(\Omega_{0,0}{}^{(+)},\Omega_{0,0}{}^{(+)})+\tfrac{1}{2}L(\Omega_{1,0}{}^{(+)},\Omega_{1,0}{}^{(+)})+L(\Omega_{1,-1}{}^{(+)},\Omega_{1,-1}{}^{(+)})-L(\Omega_{1,-1}{}^{(-)},\Omega_{1,-1}{}^{(-)})=0.\quad(4.12)$$

Finally we can express this result in terms of the usual σ matrices by using

$$\Omega_{0,0}=1,$$
$$\Omega_{1,0}=\boldsymbol{\sigma}\cdot\hat{m},$$
$$\Omega_{1,1}{}^{(+)}=\frac{i}{\sqrt{2}}\boldsymbol{\sigma}\cdot\hat{l},\quad(4.13)$$
$$\Omega_{1,1}{}^{(-)}=\frac{i}{\sqrt{2}}\boldsymbol{\sigma}\cdot\hat{n}.$$

Thus we finally obtain

$$\mathcal{L}_{-1/2,+1/2,-1/2,+1/2}(+,+)=\tfrac{1}{2}[L(0,0;0,0)-L(0,m;0,m)-L(0,n;0,n)+L(0,l;0,l)]=0\quad(4.14)$$

which is Eq. (4.9).

2. $\frac{1}{2}+0\rightarrow\frac{1}{2}+0$, Rotation and Reflection Invariance

In this case we have

$$M^{+}=b_0+b_2\boldsymbol{\sigma}\cdot\hat{m},\quad(4.15)$$

and hence T invariance (which is expressed by $b_1=0$) has no effect on the observable components at all. This result has already been proven by other methods in Ref. 3.

3. $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$, Rotation and Reflection Invariance

The subclass structure of this reaction has been given in Ref. 12. In terms of the notation used there, T invariance requires :

$$A_6=A_7=0.\quad(4.16)$$

The mirror relations have been indicated also in Ref. 12. In addition, however, there are nonmirror-type relations. They occur in those subclasses where, as Ref. 12 indicates, the number of independent observable components decreases when T invariance is imposed, but where mirror-type relations do not exist. These are subclasses I-1, I-4, and I-7. A brief inspection shows that these nonmirror-type relations are as follows:

For subclass I-1:

$$L^{++}(0,0;0,0)-L^{++}(0,m;0,m)=L^{++}(l,l;l,l)-L^{++}(l,n;l,n),\quad(4.17)$$

$$L^{++}(0,l;0,l)-L^{++}(0,n;0,n)=L^{++}(l,0;l,0)-L^{++}(n,0;n,0).\quad(4.18)$$

For subclass I-4:

$$L^{++}(m,0;0,m)=L^{++}(m,m;0,0)+L^{++}(n,n;l,l)+L^{++}(n,l;l,n).\quad(4.19)$$

[12] M. J. Moravcsik, in Proceedings of the International Conference on Polarization Phenomena of Nucleons, Karlsruhe, 1965 (to be published).

TABLE IV. Pseudo-observables L^{+-} for the reaction $0+1 \to 0+1$.

			Class I				
Subclass *R-I* 1	$C_{22}C_{23}^*$	$C_{13}C_{12}^*$	$C_{33}C_{32}^*$	Subclass *R-I* 8	$C_{22}C_{23}^*$	$C_{13}C_{12}^*$	$C_{33}C_{32}^*$
$L^{+-}(0,l)$	$+i$	$-i$	$-i$	$L^{+-}(0,nm)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$L^{+-}(ll,l)$	$+\frac{1}{3}i$	$+\frac{2}{3}i$	$-\frac{1}{3}i$	$L^{+-}(ll,nm)$	$-\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{1}{6}$
$L^{+-}(nn,l)$	$+\frac{1}{3}i$	$-\frac{1}{3}i$	$+\frac{2}{3}i$	$L^{+-}(nn,nm)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$+\frac{1}{3}$
Subclass *R-I* 4	$C_{22}C_{21}^*$	$C_{11}C_{12}^*$	$C_{31}C_{32}^*$	Subclass *R-I* 7	$C_{22}C_{21}^*$	$C_{11}C_{12}^*$	$C_{31}C_{32}^*$
$L^{+-}(0,lm)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$L^{+-}(0,n)$	$-i$	$+i$	$+i$
$L^{+-}(ll,lm)$	$-\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{1}{6}$	$L^{+-}(ll,n)$	$-\frac{1}{3}i$	$-\frac{2}{3}i$	$+\frac{1}{3}i$
$L^{+-}(nn,lm)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$+\frac{1}{3}$	$L^{+-}(nn,n)$	$-\frac{1}{3}i$	$+\frac{1}{3}i$	$-\frac{2}{3}i$
Subclass *R-I* 2	$C_{13}C_{32}^*$	$C_{33}C_{12}^*$		Subclass *R-I* 3	$C_{13}C_{32}^*$	$C_{33}C_{12}^*$	
$L^{+-}(ln,l)$	$+\frac{1}{4}i$	$+\frac{1}{4}i$		$L^{+-}(m,l)$	-1	$+1$	
$L^{+-}(m,nm)$	$+\frac{1}{2}i$	$-\frac{1}{2}i$		$L^{+-}(ln,nm)$	$+\frac{1}{8}$	$+\frac{1}{8}$	
Subclass *R-I* 5	$C_{11}C_{32}^*$	$C_{31}C_{12}^*$		Subclass *R-I* 6	$C_{11}C_{32}^*$	$C_{31}C_{12}^*$	
$L^{+-}(ln,lm)$	$+\frac{1}{8}$	$+\frac{1}{8}$		$L^{+-}(m,lm)$	$+\frac{1}{2}i$	$-\frac{1}{2}i$	
$L^{+-}(m,n)$	$+1$	-1		$L^{+-}(ln,n)$	$-\frac{1}{4}i$	$-\frac{1}{4}i$	
			Class II				
Subclass *R-II* 1	$C_{22}C_{32}^*$	$C_{31}C_{21}^*$	$C_{33}C_{23}^*$	Subclass *R-II* 8	$C_{22}C_{32}^*$	$C_{31}C_{21}^*$	$C_{33}C_{23}^*$
$L^{+-}(l,0)$	$-i$	$+i$	$+i$	$L^{+-}(nm,0)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$L^{+-}(l,ll)$	$-\frac{1}{3}i$	$-\frac{2}{3}i$	$+\frac{1}{3}i$	$L^{+-}(nm,ll)$	$-\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{1}{6}$
$L^{+-}(l,nn)$	$-\frac{1}{3}i$	$+\frac{1}{3}i$	$-\frac{2}{3}i$	$L^{+-}(nm,nn)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$+\frac{1}{3}$
Subclass *R-II* 4	$C_{22}C_{12}^*$	$C_{11}C_{21}^*$	$C_{13}C_{23}^*$	Subclass *R-II* 7	$C_{22}C_{12}^*$	$C_{11}C_{21}^*$	$C_{13}C_{23}^*$
$L^{+-}(lm,0)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$L^{+-}(n,0)$	$+i$	$-i$	$-i$
$L^{+-}(lm,ll)$	$-\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{1}{6}$	$L^{+-}(n,ll)$	$+\frac{1}{3}i$	$+\frac{2}{3}i$	$-\frac{1}{3}i$
$L^{+-}(lm,nn)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$+\frac{1}{3}$	$L^{+-}(n,nn)$	$+\frac{1}{3}i$	$-\frac{1}{3}i$	$+\frac{2}{3}i$
Subclass *R-II* 2	$C_{31}C_{23}^*$	$C_{33}C_{21}^*$		Subclass *R-II* 3	$C_{31}C_{23}^*$	$C_{33}C_{21}^*$	
$L^{+-}(l,ln)$	$-\frac{1}{4}i$	$-\frac{1}{4}i$		$L^{+-}(l,m)$	-1	$+1$	
$L^{+-}(nm,m)$	$-\frac{1}{2}i$	$+\frac{1}{2}i$		$L^{+-}(nm,ln)$	$+\frac{1}{8}$	$+\frac{1}{8}$	
Subclass *R-II* 5	$C_{11}C_{23}^*$	$C_{13}C_{21}^*$		Subclass *R-II* 6	$C_{11}C_{23}^*$	$C_{13}C_{21}^*$	
$L^{+-}(lm,ln)$	$+\frac{1}{8}$	$+\frac{1}{8}$		$L^{+-}(lm,m)$	$-\frac{1}{2}i$	$+\frac{1}{2}i$	
$L^{+-}(n,m)$	$+1$	-1		$L^{+-}(n,ln)$	$+\frac{1}{4}i$	$+\frac{1}{4}i$	

For subclass *I*-7:

$$L^{++}(0,n;m,l)+L^{++}(0,l;m,n)=L^{++}(l,0;n,m)+L^{++}(n,0;l,m). \quad (4.20)$$

Of course, had we not assumed that reflection invariance holds then the number of nonmirror relations would be higher.

4. $0+1 \to 0+1$, Rotation Invariance Only

This reaction was discussed in Ref. 13, and the subclass structure of the observable components was given in Table III of that reference. Since we are considering the case of rotation invariance only, we will have

$$M_1 = M_1^+ + M_1^-, \quad (4.21)$$

where M_1^+ and M_1^- are the M matrices in Ref. 13 referring to this reaction in the presence of reflection invariance. The table in this reference was prepared with rotation and reflection invariance in mind. To use it for the case of rotation invariance only, one should simply ignore the headings referring to the reflection properties $(++, --, +-, -+)$ and consider the whole line for each observable.

The pseudo-observable component subclasses were not discussed in Ref. 13, so they are given here in Table IV.

If T invariance holds, we have

$$C_{13}=-C_{31}, \quad C_{21}=-C_{12}, \quad C_{23}=C_{32}. \quad (4.22)$$

Thus the number of independent form factors is reduced by 3.

[13] M. J. Moravcsik, in Proceedings of the Williamsburg Conference on Intermediate Energy Physics, 1966 (to be published), p. 517.

Let us use for this example the general formulas developed in Sec. 2. In this case we have

$$\begin{aligned} x=9, \quad & N_0=9, \quad & N_T=6, \\ Q_0=Q_T=81, \quad & B_0=81, \quad & B_T=36, \\ A_T=45, \quad & & V_0=V_T=36. \end{aligned} \tag{4.23}$$

Thus we see that T invariance will impose in this case 45 relations among the observable components, of which 36 are of the mirror type, and 9 are not. Further use of the formulas of Sec. II also reveals that of the 36 mirror relations, 16 will be among the observable components and 20 among the pseudo-observable components, while of the 9 nonmirror-type relations 5 will involve observable components and 4 will involve pseudo-observable components.

The mirror relations are easy to write down and will not be reproduced here. Of the nonmirror relations among the observable components, three will involve observable components from subclass I-1, in conjunction with subclasses I-5, II-1, and II-6, respectively, because if Eq. (4.22) holds, the product sets of these subclasses will partially overlap. A straightforward calculation yields

$$4L(ln,ln)+\tfrac{1}{4}L(m,m)+\tfrac{1}{9}L(0,0)-\tfrac{1}{3}L(0,ll) -\tfrac{1}{3}L(0,nn)+L(ll,nn)=0, \tag{4.24}$$

$$L(nn,nn)+L(ll,nn)-\tfrac{1}{9}L(0,0)-\tfrac{1}{3}L(0,ll) +\tfrac{1}{2}L(l,l)+2L(nm,nm)=0, \tag{4.25}$$

$$\tfrac{1}{9}L(0,0)+\tfrac{1}{3}L(0,nn)-L(ll,ll)-L(ll,nn) +\tfrac{1}{2}L(n,n)+2L(lm,lm)=0. \tag{4.26}$$

Similar overlaps between the subclasses I-2 and II-2, and I-3 and II-3, respectively, yield the other two relations

$$2L(lm,nm)-\tfrac{1}{2}L(n,l)=\tfrac{2}{3}L(ln,0)+L(ln,ll) +L(ln,nn), \tag{4.27}$$

$$L(lm,l)-L(n,nm)=-\tfrac{1}{6}L(m,0)-\tfrac{1}{2}L(m,ll) -\tfrac{1}{2}L(m,nn). \tag{4.28}$$

Similarly, one can find four relations among pseudo-observable components, resulting from the partial overlap in the product sets of the following pairs of pseudo-observable component subclasses: I-1 and I-6; I-2 and I-7; I-3 and I-8, and finally I-4 and I-5. The relations are

$$L^{+-}(m,lm)+2L^{+-}(ln,n)-L^{+-}(ll,l) +\tfrac{1}{3}L^{+-}(0,l)=0, \tag{4.29}$$

$$L^{+-}(m,nm)+2L^{+-}(ln,l)-L^{+-}(nn,n) +\tfrac{1}{3}L^{+-}(0,n)=0, \tag{4.30}$$

$$\tfrac{2}{3}L^{+-}(0,nm)-2L^{+-}(ll,nm)+\tfrac{1}{2}L^{+-}(m,n) -4L^{+-}(ln,lm)=0, \tag{4.31}$$

$$\tfrac{2}{3}L^{+-}(0,lm)-2L^{+-}(nn,lm)+\tfrac{1}{2}L^{+-}(m,l) -4L^{+-}(ln,nm)=0. \tag{4.32}$$

This completes the list of nonmirror-type relations for this reaction.

V. INELASTIC REACTIONS

The case of inelastic processes turns out to be trivial. This makes their discussion very simple, but less interesting. There are only the "obvious" nondynamical relations, and no "hidden" ones.

Consider the reactions

$$s_1+s_2+\cdots+s_m \to s_{m+1}+s_{m+2}+\cdots+s_n, \tag{5.1}$$

and

$$s_{m+1}+s_{m+2}+\cdots+s_n \to s_1+s_2+\cdots+s_m, \tag{5.2}$$

where the $s_1, s_2, \cdots, s_n$ are particles with spin $s_1, s_2, \cdots, s_n$, respectively.

We show that a complete set of linearly independent NDT tests is given by all relations of the type

$$L(S_1,S_2,\cdots,S_m;S_{m+1},\cdots,S_n) =(-1)^{l_s}L'(S_{m+1},\cdots,S_n;S_1,\cdots,S_m), \tag{5.3}$$

where the observable components L and L' refer to reactions (5.1) and (5.2), respectively, $S_1, \cdots, S_n$ in their arguments refer to particles $s_1, \cdots, s_n$, respectively. The l_S is defined as

$$l_S \equiv l(S_1)+l(S_2)+\cdots+l(S_n),$$

and $l(S_1)$ is the number of l's appearing in S_1, etc.

First of all it is clear that if T invariance holds, then, Eq. (5.3) is valid for all $S_1, \cdots, S_n$. Thus any relation of the type (5.3) is an NDT test. Furthermore, no other nondynamical relations are implied by T invariance. This follows from the observation that all equations of the form (5.3) completely determine all observable components of the time-reversed reaction, once all observable components of the direct reaction are given. This proves the statement.