Theory of Parametric Coupling in Plasmas

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Based on a simple fluid approach, the theory of the instability due to the nonlinear coupling of an electron plasma oscillation and an ion acoustic oscillation to a driving transverse field is worked out and extended to the case in which the driving field is a longitudinal field. It is shown that the effects of transverse and longitudinal driving fields on the instability are the same in the limit of long wavelength as one would expect physically. In addition, the effect of an electron drift velocity on the nonlinearity-induced instability mentioned above is also examined. It is found that the electron plasma oscillation can be greatly enhanced when the drift velocity approaches the ion wave velocity. Derivation of the above results by the Vlasov equation is indicated in an Appendix.

I. INTRODUCTION

RECENTLY, there has been considerable interest in the nonlinear interactions in plasmas.[1] In particular, DuBois and Goldman[2] suggested the possibility of an instability of electron plasma oscillations induced by an external transverse electromagnetic wave. There, they considered the nonlinear coupling of the external transverse wave to the longitudinal electron plasma oscillation and the ion acoustic oscillation, using the diagrammatic approach of quantum statistical mechanics.

Since this instability has received a great deal of attention, it seems of interest to investigate it by another approach. In Sec. II, on the basis of a simple fluid approach, the results of DuBois and Goldman[2] are rederived.

In a recent experiment,[3] in which high-frequency, large-amplitude electron-density fluctuations were generated in a plasma column by a driving electric field, a low-frequency ion-acoustic oscillation was observed as the electric field exceeded a definite threshold. It has been established[4] that when a transverse field is incident on a plasma column, a resonant longitudinal field may be induced inside the plasma, the strength of which is linearly proportional to the incident transverse field. Therefore, the experiment mentioned above is thought to be a manifestation of the nonlinear coupling of the electron plasma oscillation and the ion acoustic oscillation to a driving longitudinal field instead of to a driving transverse field. This motivates our analysis of such a coupling in Sec. III. In Sec. IV we examine the very interesting situation when the plasma has its electrons drifted relative to the ions in addition to being under the influence of a strong driving field. We find, in this case, that because of the usual linear ion-wave instability, the threshold for attaining the nonlinear instability of the electron plasma oscillation described in Sec. II or III can be greatly reduced when the electron drift velocity approaches the ion acoustic wave velocity. An experimental confirmation of this enhancement should provide a striking evidence of the theory of parametric coupling in a plasma. From here on, we shall call the electron plasma oscillation the l wave, and the ion acoustic oscillation the s wave for convenience.

II. COUPLING OF EXTERNAL TRANSVERSE WAVE TO ELECTRON PLASMA OSCILLATION AND ION ACOUSTIC OSCILLATION

For a two-component (electrons and ions) plasma under the influence of a transverse wave of the form

$$\mathbf{E}^{\mathrm{tr}}=\tfrac{1}{2}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}\exp[i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)]+\text{c.c.}$$

where $\mathbf{k}_0\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}=0$ and ω_0 is slightly above ω_p, the electrons obey the equation of motion:

$$mn_0\left[\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\cdot\mathbf{v}\right]+m\delta n\left[\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\cdot\mathbf{v}\right]+k_BT_e\nabla\delta n$$
$$=-en_0\mathbf{E}^{\mathrm{tr}}+en_0\nabla\Phi-e\delta n\mathbf{E}^{\mathrm{tr}}+e\delta n\nabla\Phi,\quad(1)$$

where n_0 is the average electron density in equilibrium; δn is the electron density perturbation induced by the external disturbance; Φ is the scalar potential satisfying the Poisson equation; $\mathbf{E}^{\mathrm{tr}}$ is the external transverse field after it enters into the plasma, obeying the dispersion relation $\omega_0^2=\omega_p^2+c^2k_0^2$. In Eq. (1), the quantities $\mathbf{v}$, δn, Φ are assumed to be small, being induced by the

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[2] D. F. DuBois and M. V. Goldman, Phys. Rev. Letters **14**, 544 (1965); M. V. Goldman, Research Report No. 342, Hugh Research Laboratories, 1965 (unpublished).

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[4] P. Weissglass, Phys. Rev. Letters **10**, 206 (1963); Plasma Phys. **6**, 251 (1964); J. C. Nickel, J. V. Parker, and R. W. Gould, Phys. Rev. Letters **11**, 183 (1963); Phys. Fluids **7**, 1489 (1964); F. C. Hoh, Phys. Rev. **133**, A1016 (1964); P. E. Vandenplas and R. W. Gould, Plasma Phys. **6**, 449 (1964); P. E. Vandenplas and A. M. Messiaen, *ibid.* **6**, 459 (1964).

external disturbance. For weak $\mathbf{E}^{tr}$, the linear theory applies and no mode coupling takes place. For strong $\mathbf{E}^{tr}$, the $\mathbf{k}_0,\omega_0$ mode will be coupled to other modes of different wave vectors and frequencies.

A. The $\mathbf{k}_0,\omega_0$ Component

When we take the $\mathbf{k}_0,\omega_0$ Fourier component of Eq. (1) the influence of the heavy ions can be neglected if ω_0 is assumed to be close to (slightly above) the electron plasma frequency. Also, we may linearize the equations and obtain

$$\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}=\frac{-ie}{2m\omega_0}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr},$$

and

$$n_{\mathbf{k}_0,\omega_0}=\Phi_{\mathbf{k}_0,\omega_0}=0. \tag{2}$$

B. The $\mathbf{k},\omega$ Component

Similarly we take the $\mathbf{k},\omega$ Fourier component of Eq. (1) where $\mathbf{k},\omega$ is now assumed to be the wave vector and frequency of the l wave. However, we must now retain the nonlinear terms which are responsible for the mode coupling. Following Refs. 2 and 5, we shall neglect all longitudinal fields propagating at frequencies other than ω and $\omega-\omega_0$, i.e., only the l wave and the s wave (at the low frequency $\omega-\omega_0$) are assumed to be important in the nonlinear mode-mode coupling. Thus we have

$$\begin{aligned}-i\omega mn_0\mathbf{v}_{\mathbf{k},\omega}{}^l&+imn_0[(-\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr})\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s\\&+(\mathbf{k}_0\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s)\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}]\\&-im\omega_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s+i\mathbf{k}k_BT_en_{\mathbf{k},\omega}{}^l\\&=ien_0\mathbf{k}\Phi_{\mathbf{k},\omega}{}^l-(e/2)n_{-\Delta\mathbf{k},-\Delta\omega}{}^s\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr},\\&\Delta\omega=\omega_0-\omega,\quad\Delta\mathbf{k}=\mathbf{k}_0-\mathbf{k},\end{aligned}\tag{3}$$

where the superscripts l and s are used to denote quantities for the l wave and the s wave, respectively. We can now estimate and compare the various nonlinear terms. On the left-hand side of Eq. (3), the nonlinear term $mn_0\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s$ is of the same order of magnitude as $mn_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}\Delta\mathbf{k}\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s$ which can in turn be approximated by $m\Delta\omega\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s$ from the linearized equation of continuity. The term $m\mathbf{k}_0\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}$ is also of the same order of magnitude as long as k_0 is smaller than or close to Δk. In comparison with the term $(e/2)n_{-\Delta\mathbf{k},-\Delta\omega}{}^s\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}$ on the right-hand side of (3), the two terms discussed above can be neglected by noting Eq. (2) and $|\Delta\omega|\ll\omega_0$. (The s-wave frequency is smaller by a factor of $\Delta k/k_D\sqrt{(m/M)}$ than the l-wave frequency, k_D being the Debye wave number, M being the ion mass.) Upon taking the scalar product of $\mathbf{k}$ with Eq. (3), one obtains

$$\omega n_0\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l-k^2v_{\text{th}}{}^2n_{\mathbf{k},\omega}{}^l=-\frac{en_0k^2}{m}\Phi_{\mathbf{k},\omega}{}^l, \tag{4}$$

where

$$v_{\text{th}}{}^2=\frac{k_BT_e}{m}.$$

The quantities $\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l$ and $n_{\mathbf{k},\omega}{}^l$ are related by the continuity equation

$$\begin{aligned}-\omega n_{\mathbf{k},\omega}{}^l+n_0\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l&=-\sum_{\mathbf{k}',\omega'}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}_{\mathbf{k}',\omega'}n_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\&=+\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s,\end{aligned}\tag{5}$$

where the last step is in the spirit of our chosen mode coupling scheme. When ω_0 is slightly above ω_p, $k_0\sim0$ from the dispersion relation $\omega_0{}^2=\omega_p{}^2+c^2k_0{}^2$. Substituting (5) into (4) and again neglecting $\Delta\omega$ compared with ω_0 we obtain

$$(\omega^2-k^2v_{\text{th}}{}^2)n_{\mathbf{k},\omega}{}^l=-\frac{en_0}{m}k^2\Phi_{\mathbf{k},\omega}{}^l-\frac{ie}{2m}\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s. \tag{6}$$

In the l waves, the heavy ions can hardly respond to the high-frequency oscillations, and the Poisson equation can be written as

$$n_{\mathbf{k},\omega}{}^l\simeq-\frac{k^2}{4\pi e}\Phi_{\mathbf{k},\omega}{}^l. \tag{7}$$

The electron density fluctuation $n_{-\Delta\mathbf{k},-\Delta\omega}{}^s$ in the nonlinear term in Eq. (6) can be obtained from the linearized electron equation of motion by neglecting the inertial term

$$n_{-\Delta\mathbf{k},-\Delta\omega}{}^s\simeq\frac{n_0e}{mv_{\text{th}}{}^2}\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s. \tag{8}$$

Substituting (7), (8) into (6) we finally obtain

$$(\omega^2-\omega_p{}^2-k^2v_{\text{th}}{}^2)\Phi_{\mathbf{k},\omega}{}^l=\frac{ie}{2m}\frac{k_D{}^2}{k^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr})\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s, \tag{9}$$

or, in another form

$$\epsilon(\mathbf{k},\omega)\Phi_{\mathbf{k},\omega}{}^l=\frac{ie}{2m}\frac{k_D{}^2}{k^2}\frac{1}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr})\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s, \tag{10}$$

where

$$\epsilon(\mathbf{k},\omega)=1-\frac{\omega_p{}^2}{\omega^2}-\frac{k^2v_{\text{th}}{}^2}{\omega^2}$$

is the usual dielectric constant for the plasma when ω is close to ω_p.

C. The $\Delta\mathbf{k},\Delta\omega$ Component

Now we consider the low-frequency component which represents the s wave. Here, the dynamics of the ions must be taken into account. The electron equation of

[5] V. N. Oraevskii and R. Z. Sagdeev, Zh. Tekhn. Fiz. **32**, 1291 (1962) [English transl.: Soviet Phys.—Tech. Phys. **7**, 955 (1963)].

motion (1) now takes the form

$$-i\Delta\omega m n_0 \mathbf{v}_{\Delta\mathbf{k},\Delta\omega}{}^s + imn_0[(-\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr})\mathbf{v}_{-\mathbf{k},-\omega}{}^l + (\mathbf{k}_0\cdot\mathbf{v}_{-\mathbf{k},-\omega}{}^l)\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}] - im\omega_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr} n_{-\mathbf{k},-\omega}{}^l + i\Delta\mathbf{k}k_BT_e n_{\Delta\mathbf{k},\Delta\omega}{}^s = ien_0\Delta\mathbf{k}\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s - (e/2)n_{-\mathbf{k},-\omega}{}^l E_{\mathbf{k}_0,\omega_0}{}^{tr}. \quad (11)$$

Similarly, we can write down the corresponding equation of motion for the ions including all the nonlinear terms. The nonlinear terms, typically like

$$Mn_0(-\mathbf{k}\cdot\mathbf{V}_{\mathbf{k}_0,\omega_0}{}^{tr})\mathbf{V}_{-\mathbf{k},-\omega}{}^l,$$

where we have used capital letters to denote quantities pertaining to ions, represent the coupling of a low-frequency ion motion to two high-frequency ion motions.[6] However, the heavy mass of the ion makes it difficult for the ions to participate in high frequency motions. Therefore, quantities such as V^l and N^l are negligibly small. Although one could be more precise, the above argument should suffice to convince ourselves that both the ion equation of motion and the ion continuity equation can be linearized. These linearized equations yield the result that

$$N_{\Delta\mathbf{k},\Delta\omega}((\Delta\omega)^2-\Omega_p{}^2-(\Delta k)^2V_{th}{}^2) = -\Omega_p{}^2 n_{\Delta\mathbf{k},\Delta\omega}, \quad (12)$$

where

$$V_{th}{}^2 = \frac{k_BT_i}{M}, \quad \Omega_p{}^2 = \frac{4\pi n_0 e^2}{M}.$$

From Eq. (12) we see that the ions do not feel the external field directly. They respond linearly to the influence of the electrons which react to external disturbances. Together with the Poisson equation

$$(\Delta k)^2\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s = 4\pi e(N_{\Delta\mathbf{k},\Delta\omega}{}^s - n_{\Delta\mathbf{k},\Delta\omega}{}^s), \quad (13)$$

Eq. (12) gives, upon neglecting $(\Delta k)^2V_{th}{}^2$,

$$n_{\Delta\mathbf{k},\Delta\omega}{}^s = \left(\frac{\Omega_p{}^2}{(\Delta\omega)^2}-1\right)\frac{(\Delta k)^2}{4\pi e}\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s \simeq \frac{\Omega_p{}^2(\Delta k)^2}{4\pi e(\Delta\omega)^2}\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s. \quad (14)$$

In Eq. (14), the last step is justified by recalling that the frequency of the s wave $\omega_s \simeq \Delta k v_{th}\sqrt{(m/M)}$ is considerably less than Ω_p for $\Delta k \ll k_D$.

Substituting the continuity equation

$$-\Delta\omega n_{\Delta\mathbf{k},\Delta\omega}{}^s + n_0\Delta\mathbf{k}\cdot\mathbf{v}_{\Delta\mathbf{k},\Delta\omega}{}^s = (\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr})n_{-\mathbf{k},-\omega}{}^l \quad (15)$$

into the scalar product of $\Delta\mathbf{k}$ with Eq. (11), one obtains

$$((\Delta\omega)^2-(\Delta k)^2v_{th}{}^2)n_{\Delta\mathbf{k},\Delta\omega}{}^s + n_0(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr})(\Delta\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k},-\omega}{}^l) - n_0(\mathbf{k}_0\cdot\mathbf{v}_{-\mathbf{k},-\omega}{}^l)(\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr}) + \omega(\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{tr})n_{-\mathbf{k},-\omega}{}^l = -\frac{en_0}{m}(\Delta k)^2\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s - \frac{ie}{2m}(\Delta\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr})n_{-\mathbf{k},-\omega}{}^l. \quad (16)$$

[6] We note that decays within the ion acoustic branch are prohibited (see Ref. 5).

Also, as $k_0 \to 0$, $\Delta\mathbf{k} \to -\mathbf{k}$, $n_0\Delta\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k},-\omega}{}^l \to -\omega n_{-\mathbf{k},-\omega}{}^l$. Therefore, in the limit of $k_0 \to 0$, Eqs. (14), (7), (2) and (16) combine to give

$$[(\Delta\omega)^2-\omega_s{}^2]\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s = +\frac{ie}{2m}\frac{(\Delta\omega)^2}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr})\Phi_{-\mathbf{k},-\omega}{}^l, \quad (17)$$

where

$$\omega_s{}^2 = (\Delta k)^2 k_BT_e/M.$$

After some algebra, Eq. (17) can be written in another form,

$$\epsilon(\mathbf{k},\omega)\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s = +\frac{ie}{2m}\frac{k_D{}^2}{k^2}\frac{1}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr})\Phi_{-\mathbf{k},-\omega}{}^l, \quad (18)$$

where

$$\epsilon(\mathbf{k},\Delta\omega) = 1 + \frac{k_D{}^2}{k^2} - \frac{\Omega_p{}^2}{(\Delta\omega)^2} \quad (19)$$

is the usual longitudinal dielectric constant for $\Delta\omega$ close to ω_s.

Following DuBois and Goldman,[2] we may now identify the nonlinear susceptibility by rewriting Eqs. (10) and (18) as (when $k_0 \to 0$)

$$\epsilon(\mathbf{k},\omega)\Phi_{\mathbf{k},\omega}{}^l = -\chi^{NL}(\mathbf{k}_0,\omega_0;-\Delta\mathbf{k},-\Delta\omega)\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s, \quad (20)$$

and

$$\epsilon(-\Delta\mathbf{k},-\Delta\omega)\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s = -\chi^{NL}(-\mathbf{k}_0,-\omega_0;\mathbf{k},\omega)\Phi_{\mathbf{k},\omega}{}^l, \quad (21)$$

where the nonlinear susceptibilities are given by

$$\chi^{NL}(\mathbf{k}_0,\omega_0;-\Delta\mathbf{k},-\Delta\omega) = -\frac{ie}{2m}\frac{k_D{}^2}{k^2}\frac{1}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}), \quad (22)$$

and

$$\chi^{NL}(-\mathbf{k}_0,-\omega_0;\mathbf{k},\omega) = +\frac{ie}{2m}\frac{k_D{}^2}{k^2}\frac{1}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{-\mathbf{k}_0,-\omega_0}{}^{tr}). \quad (23)$$

These results are in complete agreement with those in Ref. 2 provided we identify our $\epsilon(\mathbf{k},\omega)$ and $\epsilon(-\Delta\mathbf{k},-\Delta\omega)$ as the full dielectric constants, including their imaginary parts. It is well known that in the self-consistent-field approximation, the imaginary part of $\epsilon(\mathbf{k},\omega)$ for real $\mathbf{k},\omega$ gives rise to Landau damping which the present fluid approach cannot adequately handle. We could have gotten some damping by introducing a phenomenological collision frequency in the beginning. However, it seems intuitively clear that to take the particle-wave interaction, or Landau damping into account, all the modification that is required is just this identification of ϵ as the full dielectric constant. This step is indeed justified in the Appendix where the Vlasov equations are used instead of the fluid equations.

In the next section, we shall continue to use the fluid equations with the above understanding about the dielectric constant ϵ.

From Eqs. (22) and (23) we observe that

$$\chi^{NL}(\mathbf{k}_0,\omega_0; -\Delta\mathbf{k}, -\Delta\omega)=\chi^{NL}(-\mathbf{k}_0, -\omega_0; \mathbf{k},\omega)^*. \quad (24)$$

This relation is actually quite general and has been proved in nonlinear optics.[7]

A dispersion relation may now be derived from Eqs. (20) and (21), and the nonlinear damping rate may be obtained. However, we shall defer a discussion of these until the next section.

III. COUPLING OF EXTERNAL LONGITUDINAL FIELD TO ELECTRON PLASMA OSCILLATION AND ION ACOUSTIC OSCILLATION

We now assume that there exists a longitudinal field $\mathbf{E}^{\text{long}}=\frac{1}{2}\mathbf{E}_{\mathbf{k}_0,\omega_0}\exp[i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)]+\text{c.c.}$ inside the plasma due to some external driving source, where $\mathbf{k}_0\times\mathbf{E}_{\mathbf{k}_0,\omega_0}=0$ and ω_0 is close to ω_p. When the field strength of $\mathbf{E}^{\text{long}}$ is sufficiently weak, the linear theory holds and the only disturbances in the plasma are $\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}$ and $\mathbf{n}_{\mathbf{k}_0,\omega_0}$, given by

$$n_0\mathbf{k}_0\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}=\omega_0 n_{\mathbf{k}_0,\omega_0}=-\frac{i\omega_0}{4\pi e}\mathbf{k}_0\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}. \quad (25)$$

We see that, in addition to a longitudinal velocity $\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}$, there is also a $\mathbf{k}_0,\omega_0$ Fourier component of the perturbed electron density $n_{\mathbf{k}_0,\omega_0}$. This nonvanishing $n_{\mathbf{k}_0,\omega_0}$ is the only essential difference between a transverse driving field and a longitudinal driving field inside the plasma.

As $\mathbf{E}^{\text{long}}$ becomes strong, oscillation modes other than that of $\pm\mathbf{k}_0$, $\pm\omega_0$ will be excited via mode coupling. Again, as in the preceding section, we shall consider the l wave and the s wave as the most prominant modes which participate in the mode-mode coupling.

Replacing $\mathbf{E}^{\text{tr}}$ by $\mathbf{E}^{\text{long}}$ in Eq. (1) and taking its $\mathbf{k},\omega$ Fourier component, which represents the wave vector and frequency of the l wave, we obtain

$$-i\omega m n_0\mathbf{v}_{\mathbf{k},\omega}{}^l+i\mathbf{k}k_BT_e n_{\mathbf{k},\omega}{}^l-ien_0\mathbf{k}\Phi_{\mathbf{k},\omega}{}^l=[im\omega_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s+imn_0(\Delta\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s]$$
$$-[imn_0(\mathbf{k}_0\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s)\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}-im\Delta\omega\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s n_{\mathbf{k}_0,\omega_0}]-\left[\frac{e}{2}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}+ien_{\mathbf{k}_0,\omega_0}\Delta\mathbf{k}\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s\right], \quad (26)$$

where

$$\Delta\mathbf{k}=\mathbf{k}_0-\mathbf{k},\quad \Delta\omega=\omega_0-\omega.$$

The nonlinear terms on the right-hand side of (26) can be examined by using the linearized relations as in the preceding section. It is not difficult to show that in each square bracket on the right-hand side of (26), the second term is negligibly small compared to the first term if we assume $k_0^2\sim k^2\ll k_D^2$, and $|\Delta\omega|\ll\omega_0$. If we take the scalar product of the vector $(i/m)\mathbf{k}$ and Eq. (26) and neglect those small nonlinear terms, we obtain

$$\omega n_0\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l-k^2v_{\text{th}}{}^2n_{\mathbf{k},\omega}{}^l+\frac{en_0}{m}k^2\Phi_{\mathbf{k},\omega}{}^l$$
$$=-\omega_0(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})n_{-\Delta\mathbf{k},-\Delta\omega}{}^s$$
$$+n_0(\mathbf{k}_0\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})$$
$$-\frac{ie}{2m}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})n_{-\Delta\mathbf{k},-\Delta\omega}{}^s. \quad (27)$$

The equation of continuity $\partial n/\partial t+\nabla\cdot(n\mathbf{v})=0$ gives

$$n_0\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l=\omega n_{\mathbf{k},\omega}{}^l-\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s$$
$$-\mathbf{k}\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s n_{\mathbf{k}_0,\omega_0}. \quad (28)$$

Substituting (28) into (27) one obtains

$$(\omega^2-k^2v_{\text{th}}{}^2)n_{\mathbf{k},\omega}{}^l+\frac{en_0}{m}k^2\Phi_{\mathbf{k},\omega}{}^l$$
$$=-\frac{ie}{2m}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})n_{-\Delta\mathbf{k},-\Delta\omega}{}^s$$
$$+n_0(\mathbf{k}_0\cdot\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})$$
$$-\Delta\omega(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})n_{-\Delta\mathbf{k},-\Delta\omega}{}^s. \quad (29)$$

By observing that on the right-hand side of (29), the last two terms can be neglected compared with the first term, Eq. (29) yields

$$(\omega^2-k^2v_{\text{th}}{}^2)n_{\mathbf{k},\omega}{}^l+\frac{en_0}{m}k^2\Phi_{\mathbf{k},\omega}{}^l$$
$$=-\frac{ie}{2m}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})n_{-\Delta\mathbf{k},-\Delta\omega}{}^s. \quad (30)$$

We note that Eq. (30) is exactly the same as Eq. (6) except that $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{tr}}$ is now replaced by $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}$. Without repeating the subsequent steps and by the use of (24) it is clear that for the present case, Eqs. (20)–(23) are also valid, except for the replacement of $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{tr}}$ by $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}}$, i.e.,

$$\chi^{NL}(\mathbf{k}_0,\omega_0; -\Delta\mathbf{k},-\Delta\omega)=-\frac{ie}{2m}\frac{k_D{}^2}{k^2}\frac{1}{\omega_p{}^2}(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\text{long}})$$
$$=\chi^{NL}(-\mathbf{k}_0, -\omega_0; \mathbf{k},\omega)^*. \quad (31)$$

[7] See for example, N. Bloembergen, *Nonlinear Optics* (W. A. Benjamin, Inc., New York, 1965), Chap. 1.

We can therefore conclude as one would expect physically that the effect of a longitudinal driving field is just the same as that of a transverse driving field as long as both their wavelengths are long compared with the Debye length and the wavelengths of the plasma oscillations which take part in the mode coupling. However, we should emphasize here that in Eqs. (22) and (23) as well as in Eq. (31), $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}$ or $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{long}}$ represents the driving field when it gets *inside* the plasma. Now we come to the discussion of the threshold. From Eqs. (20), (21), one can obtain the dispersion relation governing the nonlinear coupling. In terms of the nonlinear dielectric constant, this dispersion relation takes the form[2] (when $k_0 \to 0$)

$$\epsilon^{NL}(\mathbf{k},\omega)=\epsilon(\mathbf{k},\omega) - |\chi^{NL}(\mathbf{k}_0,\omega_0;\,\mathbf{k},\,\omega-\omega_0)|^2[\epsilon(\mathbf{k},\,\omega-\omega_0)]^{-1}=0. \quad (32)$$

If we consider real frequencies, ω, only the real part of $\epsilon^{NL}(\mathbf{k},\omega)$ must vanish, thereby determining the resonant frequency. The imaginary part gives the effective damping rate

$$\frac{\gamma^{NL}}{\omega_p}=\mathrm{Im}\,\epsilon^{NL}(\mathbf{k},\omega) =\mathrm{Im}\,\epsilon(\mathbf{k},\omega) - |\chi^{NL}(\mathbf{k}_0,\omega_0;\,\mathbf{k},\,\omega-\omega_0)|^2\,\mathrm{Im}\,\epsilon^{-1}(\mathbf{k},\,\omega-\omega_0) \quad (33)$$

for $|\gamma^{NL}|\ll\omega_p$. When $(\gamma^{NL}/\omega_p)<0$, the oscillations become unstable. Thus, the threshold condition for the onset of instability of both the l wave (electron plasma oscillation) and the s wave (ion acoustic oscillation) is given by

$$\mathrm{Im}\,\epsilon^{NL}(\mathbf{k},\omega)=0. \quad (34)$$

It has been shown in Ref. 2 that for a plasma of density $n_0=10^{13}$ electrons/cm^3 and $k_BT=1$ eV, $\mathrm{Im}\,\epsilon(\mathbf{k},\omega)\approx 10^{-3}$, the condition (34) can be met by a driving *transverse* field of 600 V/cm. With our present understanding of the effect of a longitudinal driving field, we can reinterpret the 600 V/cm as the threshold value for the driving field inside the plasma, may it be transverse, longitudinal or mixed. It is known that[4] when an external transverse field is incident on a bounded plasma, a longitudinal field which may become resonant is induced inside the plasma. The ratio of this field inside the plasma to the incident transverse field may become considerably greater than unity. Although this ratio is not precisely known in the experiment of Stern and Tzoar[3] described in Sec. I, a value of 10–20 for this ratio would be sufficient to bring the theoretical threshold into agreement with the expermiental threshold.

IV. PARAMETRIC COUPLING WHEN ELECTRONS ARE DRIFTED

It is well known that in a two-component plasma, when the electrons are drifted relative to the ions, a linear two-stream instability occurs as the drift velocity approaches or exceeds the ion acoustic wave velocity. It is interesting to investigate the effect of this two-stream instability on the nonlinear instability considered in Sec. II or III. In other words, we want to analyze the following situation: What will happen to a two-component plasma in which the electrons are drifted with respect to the ions with a net velocity $\mathbf{v}_D$ when the plasma is at the same time driven by a high-frequency (ω_0 slightly above ω_p) intense field (transverse, longitudinal, or mixed). For definiteness, we consider the case of a transverse driving field.

Similar to Sec. II, when the transverse driving field is sufficiently weak, the linearized theory gives

$$\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}=\frac{-ie}{2m(\omega_0-\mathbf{k}_0\cdot\mathbf{v}_D)}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}, \quad (35)$$

$$n_{\mathbf{k}_0,\omega_0}=\Phi_{\mathbf{k}_0,\omega_0}=0. \quad (36)$$

As the driving field $\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}$ gets stronger, it can couple to the l-wave and the s-wave. Analogous to Eqs. (3) and (11), the electron equations of motion are now given by (as $\mathbf{k}_0\to 0$)

$$-i(\omega-\mathbf{k}\cdot\mathbf{v}_D)mn_0\mathbf{v}_{\mathbf{k},\omega}{}^l-imn_0(\mathbf{\Delta k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}})\mathbf{v}_{-\Delta\mathbf{k},-\Delta\omega}{}^s -im\omega_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s+i\mathbf{k}k_BT_en_{\mathbf{k},\omega}{}^l =ien_0\mathbf{k}\Phi_{\mathbf{k},\omega}{}^l-\frac{e}{2}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}, \quad (37)$$

and

$$-i(\Delta\omega-\mathbf{\Delta k}\cdot\mathbf{v}_D)mn_0\mathbf{v}_{\Delta\mathbf{k},\Delta\omega}{}^s-imn_0(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}})\mathbf{v}_{-\mathbf{k},-\omega}{}^l -im\omega_0\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}n_{-\mathbf{k},-\omega}{}^l+i\mathbf{\Delta k}k_BT_en_{\Delta\mathbf{k},\Delta\omega}{}^s =ien_0\mathbf{\Delta k}\cdot\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s-\frac{e}{2}n_{-\mathbf{k},-\omega}{}^l\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}. \quad (38)$$

The continuity equations are given by

$$n_0\mathbf{k}\cdot\mathbf{v}_{\mathbf{k},\omega}{}^l=(\omega-\mathbf{k}\cdot\mathbf{v}_D)n_{\mathbf{k},\omega}{}^l +\mathbf{\Delta k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}}n_{-\Delta\mathbf{k},-\Delta\omega}{}^s, \quad (39)$$

and

$$n_0(\mathbf{\Delta k}\cdot\mathbf{v}_{\Delta\mathbf{k},\Delta\omega}{}^s)=(\Delta\omega-\mathbf{\Delta k}\cdot\mathbf{v}_D)n_{\Delta\mathbf{k},\Delta\omega}{}^s +(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}_0,\omega_0}{}^{\mathrm{tr}})n_{-\mathbf{k},-\omega}{}^l. \quad (40)$$

Following the same procedures as in Sec. II, Eqs. (37)–(40), together with Eqs. (7) and (14) yield, for $\mathbf{k}\cdot\mathbf{v}_D\ll\omega_p$,

$$\epsilon_D(\mathbf{k},\omega)\Phi_{\mathbf{k},\omega}{}^l=-\chi^{NL}(\mathbf{k}_0,\omega_0;\,-\mathbf{\Delta k},\,-\Delta\omega)\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s \quad (41)$$

and

$$\epsilon_D(-\Delta\mathbf{k}, -\Delta\omega)\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s = -\chi^{NL}(-\mathbf{k}_0, -\omega_0; \mathbf{k},\omega)\Phi_{\mathbf{k},\omega}{}^l, \quad (42)$$

where ϵ_D is the modified dielectric constant given by

$$\epsilon_D(\mathbf{k},\omega) = 1 - \frac{\omega_p{}^2}{(\omega-\mathbf{k}\cdot\mathbf{v}_D)^2 - k^2 v_{\text{th}}{}^2} - \frac{\Omega_p{}^2}{\omega^2}. \quad (43)$$

However, as discussed in Sec. II, we shall identify the ϵ_D in Eqs. (41), (42) as the full dielectric constant, including the imaginary part (see Appendix).

Comparing Eqs. (41), (42) with Eqs. (20), (21), we observe that the only modification due to the drift velocity[8] is to change ϵ into ϵ_D. Analogous to Eq. (32), (33), the resonant frequency ω is determined by

$$\text{Re}\epsilon_D{}^{NL}(\mathbf{k},\omega) = 0, \quad (44)$$

and the effective damping rate given by

$$\frac{\gamma^{NL}}{\omega_p} = \text{Im}\epsilon_D{}^{NL}(\mathbf{k},\omega), \quad (45)$$

where

$$\epsilon_D{}^{NL} = \epsilon_D(\mathbf{k},\omega) - |\chi^{NL}(\mathbf{k}_0, \omega_0; \mathbf{k}, \omega-\omega_0)|^2[\epsilon_D(\mathbf{k}, \omega-\omega_0)]^{-1}. \quad (46)$$

We do not expect the resonant frequency ω to shift much from ω_p for the high-frequency plasma oscillation. More interesting is the effect of the drift velocity on the damping rate of the plasma oscillation given by Eq. (45). We see from Eq. (A16) in the Appendix that as $\omega_0 \approx \omega + kv_{\text{th}}(m/M)^{1/2}$, the real part of the modified dielectric constant $\text{Re}\epsilon_D(k, \omega-\omega_0) \approx 0$, which is insensitive to $\mathbf{v}_D$ as long as $v_D \ll v_{\text{th}}$. The imaginary part,

$$\text{Im}\epsilon_D(\mathbf{k}, \omega-\omega_0) = \text{Im}\epsilon(\mathbf{k}, \omega-\omega_0-\mathbf{k}\cdot\mathbf{v}_D)$$

being proportional to the variable $(\omega-\omega_0-\mathbf{k}\cdot\mathbf{v}_D)$ [see Eq. (A14)], remains negative (or positive) as long as $\omega-\omega_0-\mathbf{k}\cdot\mathbf{v}_D < 0$ (or >0), provided the ion temperature can be neglected compared to the electron temperature. Therefore, the quantity

$$\text{Im}1/\epsilon_D(\mathbf{k}, \omega-\omega_0) = \frac{-\text{Im}\epsilon_D(\mathbf{k}, \omega-\omega_0)}{(\text{Re}\epsilon_D(\mathbf{k}, \omega-\omega_0))^2 + (\text{Im}\epsilon_D(\mathbf{k}, \omega-\omega_0))^2}, \quad (47)$$

remains positive and approaches infinity as $(\omega-\omega_0-\mathbf{k}\cdot v_D)$ approaches zero *from below*, provided ω_0 is so chosen that $\omega_0 \approx \omega + kv_{\text{th}}(m/M)^{1/2}$. This is of course the origin of the ion wave instability. From Eqs. (45) and (46), we see that the nonlinear damping rate has a negative damping contribution from the nonlinear coupling

$$-|\chi^{NL}(\mathbf{k}_0, \omega_0; \mathbf{k}, \omega-\omega_0)|^2 \,\text{Im}1/\epsilon_D(\mathbf{k}, \omega-\omega_0),$$

which becomes very large and negative when the ion wave instability is approached. We should remark that as $|\gamma^{NL}|$ becomes comparable to ω_p the result based on Eq. (45) is not valid. Although a more careful analysis of the dispersion relation, Eq. (32), is needed then to obtain the correct nonlinear growth rate, the indication that the drift velocity would enhance the electron plasma oscillation is clear. Experimentally, this should result in a great reduction of the threshold in inducing the nonlinear instability of the electron plasma oscillation.

[8] For a discussion of this, see Y. C. Lee and N. Tzoar, Phys. Rev. **140**, A396 (1965). Also see M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).

V. CONCLUSION

We have shown that the nonlinear instability of electron plasma oscillation and the ion acoustic oscillation arising from their nonlinear coupling to a driving transverse or longitudinal field can be derived by a simple fluid approach. As long as the wavelength of the driving field inside the plasma is long compared to the Debye length and the wavelengths of the participating coupled oscillations in the plasma, it is demonstrated explicitly that it does not make any difference whether the driving field *inside* the plasma is transverse or longitudinal, or mixed. However, since a longitudinal field inside the plasma may become large at resonance, less power at the external source is needed to drive the instability, provided the external source can be coupled linearly and resonantly to the longitudinal field inside the plasma. In the case when the electrons are imparted a drift velocity relative to the ions, the usual low-frequency ion wave instability can have a large effect on the nonlinear instability via the nonlinear coupling, resulting in an enhancement of the high-frequency plasma oscillation. This is physically reasonable since as the ion wave instability is approached, the large ion wave amplitude increases its coupling with the external field, which, in turn, induces a stronger electron plasma oscillation.

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APPENDIX

In this Appendix we shall indicate the steps in deriving Eqs. (41), (42) by using the nonlinear Vlasov equation. When the drift velocity is set to zero, Eqs. (41) and (42) reduce to Eqs. (20) and (21).

For a two-component collisionless plasma under the transverse field $\mathbf{E}^{tr}$, the Vlasov equations can be written as

$$\frac{\partial f}{\partial t}(\mathbf{x},\mathbf{v},t)+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}-\frac{e}{m}[\mathbf{E}^{tr}(\mathbf{x},t)-\nabla\Phi(\mathbf{x},t)]\cdot\frac{\partial f}{\partial \mathbf{v}}=0, \quad \text{(A1)}$$

$$\frac{\partial F}{\partial t}(\mathbf{x},\mathbf{v},t)+\mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}}+\frac{e}{M}[\mathbf{E}^{tr}(\mathbf{x},t)-\nabla\Phi(\mathbf{x},t)]\cdot\frac{\partial F}{\partial \mathbf{v}}=0, \quad \text{(A2)}$$

and the Poisson equation takes the form

$$\nabla^2\Phi(\mathbf{x},t)=4\pi e n_0\left[\int f(\mathbf{x},\mathbf{v},t)d\mathbf{v}-\int F(\mathbf{x},\mathbf{v},t)d\mathbf{v}\right], \quad \text{(A3)}$$

where $f(\mathbf{x},\mathbf{v},t)$ and $F(\mathbf{x},\mathbf{v},t)$ denote the electron distribution function and the ion distribution function, respectively. Analogous to Eqs. (35), (36), the $\mathbf{k}_0,\omega_0$ component of the above equations yields

$$f_{\mathbf{k}_0,\omega_0}(\mathbf{v})=\frac{ie}{2m}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\Big/\omega_0-\mathbf{k}_0\cdot\mathbf{v}+i\delta$$

$$=\frac{ie}{2m}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}\cdot\frac{\partial f_M(\mathbf{v}-\mathbf{v}_D)}{\partial \mathbf{v}}\Big/\omega_0-\mathbf{k}_0\cdot\mathbf{v}+i\delta \quad \text{(A4)}$$

and

$$\Phi_{\mathbf{k}_0,\omega_0}=0, \quad \text{(A5)}$$

where $f_M(\mathbf{v})$ is the equilibrium Maxwell distribution function for the electrons. The $\mathbf{k},\omega$ component of Eqs. (A1), (A3) gives

$$-i\omega f_{\mathbf{k},\omega}{}^l(\mathbf{v})+i\mathbf{k}\cdot\mathbf{v}f_{\mathbf{k},\omega}{}^l+\frac{ie}{m}\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}\Phi_{\mathbf{k},\omega}$$

$$+\frac{ie}{m}\Phi_{-\Delta\mathbf{k},-\Delta\omega}{}^s(-\Delta\mathbf{k})\cdot\frac{\partial f_{\mathbf{k}_0,\omega_0}(\mathbf{v})}{\partial \mathbf{v}}$$

$$=\frac{e}{2m}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}\cdot\frac{\partial f_{-\Delta\mathbf{k},-\Delta\omega}{}^s(\mathbf{v})}{\partial \mathbf{v}}, \quad \text{(A6)}$$

and

$$-k^2\Phi_{\mathbf{k},\omega}{}^l=4\pi e n_0\int f_{\mathbf{k},\omega}{}^l(\mathbf{v})d\mathbf{v}$$

(ions neglected here). (A7)

The $\Delta\mathbf{k},\Delta\omega$ components of Eqs. (A1), (A2), (A3) yield

$$-i\Delta\omega F_{\Delta\mathbf{k},\Delta\omega}{}^s(\mathbf{v})+i\Delta\mathbf{k}\cdot\mathbf{v}F_{\Delta\mathbf{k},\Delta\omega}{}^s$$

$$-\frac{ie}{m}\Delta\mathbf{k}\cdot\frac{\partial F_M(\mathbf{v})}{\partial \mathbf{v}}\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s=0, \quad \text{(A8)}$$

$$-\Delta\omega f_{\Delta\mathbf{k},\Delta\omega}{}^s(\mathbf{v})+i\Delta\mathbf{k}\cdot\mathbf{v}f_{\Delta\mathbf{k},\Delta\omega}{}^s$$

$$+\frac{ie}{m}\Delta\mathbf{k}\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\Phi_{\Delta\mathbf{k},\Delta\omega}$$

$$+\frac{ie}{m}\Phi_{-\mathbf{k},-\omega}{}^l(-\mathbf{k})\cdot\frac{\partial f_{\mathbf{k}_0,\omega_0}(\mathbf{v})}{\partial \mathbf{v}}$$

$$=\frac{e}{2m}\mathbf{E}_{\mathbf{k}_0,\omega_0}{}^{tr}\cdot\frac{\partial f_{-\mathbf{k},-\omega}{}^l(v)}{\partial \mathbf{v}}, \quad \text{(A9)}$$

and

$$-(\Delta k)^2\Phi_{\Delta\mathbf{k},\Delta\omega}{}^s$$

$$=4\pi e n_0\left[\int f_{\Delta\mathbf{k},\Delta\omega}{}^s(\mathbf{v})d\mathbf{v}-\int F_{\Delta\mathbf{k},\Delta\omega}{}^s(\mathbf{v})d\mathbf{v}\right]. \quad \text{(A10)}$$

One may then solve Eqs. (A6) and (A7), approximating the nonlinear terms by the use of linearized relations and taking the limit of $\Delta\omega\ll\omega$, $k_0\to 0$ to obtain Eq. (41). Similarly, Eqs. (A8), (A9), and (A10) lead to Eq. (42). However, the modified dielectric constant is given by

$$\epsilon_D(\mathbf{k},\omega)\equiv 1-\varphi_k Q_D{}^e(\mathbf{k},\omega)-\varphi_k Q^i(\mathbf{k},\omega) \quad \text{(A11)}$$

where

$$\varphi_k Q_D{}^e(\mathbf{k},\omega)\equiv-\frac{\omega_p{}^2}{k^2}\int\frac{\mathbf{k}\cdot\partial f_0(\mathbf{v})/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\delta}d\mathbf{v}$$

$$=-\frac{\omega_p{}^2}{k^2}\int\frac{\mathbf{k}\cdot\partial f_M(\mathbf{v})/\partial\mathbf{v}}{(\omega-\mathbf{k}\cdot\mathbf{v}_D)-\mathbf{k}\cdot\mathbf{v}+i\delta}d\mathbf{v}$$

$$\equiv\varphi_k Q^e(\mathbf{k},\,\omega-\mathbf{k}\cdot\mathbf{v}_D), \quad \text{(A12)}$$

and

$$\varphi_k Q^i(\mathbf{k},\omega)\equiv-\frac{\Omega_p{}^2}{k^2}\int\frac{\mathbf{k}\cdot\partial F_M(\mathbf{v})/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\delta}d\mathbf{v}. \quad \text{(A13)}$$

It is well known that

$$\operatorname{Im}\varphi_k Q^e(\mathbf{k},\omega)=-\sqrt{\frac{\pi}{2}}\frac{k_D{}^2}{k^2}\frac{\omega}{kv_{th}}\exp\left(-\frac{\omega^2}{2k^2v_{th}{}^2}\right), \quad \text{(A14)}$$

and

$\operatorname{Im}\varphi_k Q^i(\mathbf{k},\omega)\to 0$ as the ion temperature $T_i\to 0$.

When $\mathbf{k}\cdot\mathbf{v}_D\ll kv_{th}\ll\omega$,

$$\operatorname{Re}\epsilon_D(\mathbf{k},\omega)\simeq 1-\frac{\omega_p{}^2}{\omega^2}-\frac{3\omega_p{}^2k^2v_{th}{}^2}{\omega^4}-\frac{\Omega_p{}^2}{\omega^2}; \quad \text{(A15)}$$

when $\omega\ll kv_{th}$, $\mathbf{k}\cdot\mathbf{v}_D\ll kv_{th}$,

$$\operatorname{Re}\epsilon_D(\mathbf{k},\omega)\simeq 1+\frac{k_D{}^2}{k^2}-\frac{\Omega_p{}^2}{\omega^2}. \quad \text{(A16)}$$