Exact Sum Rules as Consequences of Low-Energy Theorems and Unsubtracted Dispersion Relations*

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Working with the equal-time commutation relations of vector current densities including the Schwinger term, the low-energy theorems for the Compton scattering of isovector photons by nucleons have been obtained. Assuming unsubtracted dispersion relations, sum rules which have been previously obtained by several authors have been reproduced and the relations between the previous derivations and the present work are discussed.

I. INTRODUCTION

It has been suggested by Drell and Hearn[1] that an interesting sum rule for nucleon magnetic moments follows from the low-energy theorem and the assumption of an unsubtracted dispersion relation for the spin-dependent part of the forward Compton scattering amplitude. Recently, Kawarabayashi and Suzuki[2] have shown that the same sum rule can be derived from the commutation relations between vector charge densities, and thus have made it clear that as far as this sum rule is concerned, the two approaches are entirely equivalent.

The purpose of this paper is to extend these arguments and to show that not only the sum rule mentioned above, but also other exact sum rules, derived from commutation relations between vector current densities such as the Cabibbo-Radicati sum rule[3] and the one proposed in Ref. 2, are consequences of combinations of low-energy theorems and unsubtracted dispersion relations for the isovector photon Compton scattering amplitude.[4] Moreover, in the course of the proof of the theorems, we show how the theorems depend on the local commutation relations $[j_0{}^a(\mathbf{x},0),j_0{}^b(\mathbf{0},0)]$ and $[j_0{}^a(\mathbf{x},0),j_\lambda{}^b(\mathbf{0},0)]$, and thus how the two approaches—the current algebra at infinite momentum and the one presented here—are equivalent in deriving the sum rules.

II. LOW-ENERGY THEOREMS AND SUM RULES

Let us define the *time-ordered product part* of the forward scattering amplitude of isovector photons [represented, for instance, by the Yang-Mills fields[5] with $SU(2)$ indices] with nucleons in the laboratory system by

$$F_{\lambda\sigma}{}^{ab}(\nu)=\delta_{\lambda\sigma}\{\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b\}f_1(\nu)+\delta_{\lambda\sigma}[\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b]f_3(\nu)+i\epsilon_{\lambda\sigma\kappa}\sigma_\kappa\{\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b\}f_2(\nu)+i\epsilon_{\lambda\sigma\kappa}\sigma_\kappa[\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b]f_4(\nu)\,,\quad(1)$$

where[6]

$$\lim_{\substack{q'\to q,\\ p'\to p}} T_{\lambda\sigma}{}^{ab}=-\lim_{\substack{q'\to q,\\ p'\to p}}\int d^4x d^4y \times e^{-iq'x+iqy}\langle p'|T(j_\lambda{}^a(x),j_\sigma{}^b(y))|p\rangle =-i(2\pi)^4\delta^4(-q'-p'+q+p)F_{\lambda\sigma}{}^{ab}.\quad(2)$$

The isospin indices a and b run 1, 2, and 3, and λ and σ stand for the polarization indices, and ν the laboratory energy, of the photons.

The amplitudes $f_1(\nu)$ and $f_4(\nu)$ are even functions of ν, while $f_2(\nu)$ and $f_3(\nu)$ are odd, as consequences of the crossing symmetry.

We can then prove the following low-energy theorems[7]:

$$f_1(\nu)\underset{\nu\to0}{\longrightarrow}1/2M-S\,,\quad(3)$$

$$f_2(\nu)\underset{\nu\to0}{\longrightarrow}\nu(\mu_a{}^V/2M)^2\,,\quad(4)$$

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1 S. D. Drell and A. C. Hearn, Phys. Rev. Letters **16**, 908 (1966).

2 K. Kawarabayashi and M. Suzuki, Phys. Rev. **150**, 1181 (1966); M. Hosoda and K. Yamamoto (to be published).

3 N. Cabibbo and L. A. Radicati, Phys. Letters **19**, 697 (1966); S. L. Adler, Phys. Rev. **143**, B1144 (1966); J. D. Bjorken, *ibid.* 148, 1467 (1966); R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1966).

4 During the course of this study, one of us (KK) was informed through Dr. Y. Dothan that Dr. M. A. B. Bég derived the low-energy theorem for $f_3(\nu)$. After completing the essential part of this work, we had an opportunity to see his paper [Phys. Rev. **150**, 1276 (1966)] in which he derived the low-energy theorems not only for $f_3(\nu)$ but also for $f_4(\nu)$. However, some of our conclusions are different from those of his paper.

5 C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

6 For the connection to the true forward-scattering amplitude, see Sec. IV.

7 For the real Compton scattering, (3) and (4) are equivalent to the Thomson limit [see W. Thirring, Phil. Mag. **41**, 1193 (1950) and N. Kroll and M. R. Ruderman, Phys. Rev. **93**, 233 (1954)] and the Low-Gell-Mann-Goldberger limit [see Ref. 10 and M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954)], respectively. The constant S is absorbed in the amplitude $\tilde{f}_1(\nu)$ in this case (see Ref. 15 and Sec. IV of this paper).

$$f_3(\nu) \underset{\nu\to 0}{\longrightarrow} \nu[-\tfrac{1}{3}\langle r^2\rangle_E{}^V + (\mu_T{}^V/2M)^2], \tag{5}$$

$$f_4(\nu) \underset{\nu\to 0}{\longrightarrow} \mu_T{}^V/2M, \tag{6}$$

where the parameters are contained in the matrix element of the vector current

$$\langle p'|j_\lambda{}^a(0)|p\rangle = \left(\frac{M^2}{p_0'p_0}\right)^{1/2} \bar{u}(p')[i\gamma_\lambda F_1{}^V(q^2) - i\sigma_{\lambda\nu}q_\nu F_2{}^V(q^2)]\tfrac{1}{2}\tau^a u(p), \tag{7}$$

with the normalizations

$$F_1{}^V(0)=1, \quad F_2{}^V(0)=\mu_a{}^V/2M, \tag{8}$$

and

$$\mu_T{}^V = 1+\mu_a{}^V.$$

The quantity $\langle r^2\rangle_E{}^V$ is the mean square of the isovector electric charge radius of the nucleon. S is the so-called Schwinger constant, and is defined below [Eqs. (21) and (22)].

Now, with the aid of the optical theorem, we have ($\nu\geq 0$)

$$\mathrm{Im} f_1(\nu) = -\nu[\sigma_P{}^V(\nu)+\sigma_A{}^V(\nu)], \tag{9}$$

$$= -2\nu\sigma_T{}^V(\nu), \tag{10}$$

$$\mathrm{Im} f_2(\nu) = \nu[\sigma_P{}^V(\nu)-\sigma_A{}^V(\nu)], \tag{11}$$

$$\mathrm{Im} f_3(\nu) = \nu[\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)], \tag{12}$$

$$\mathrm{Im} f_4(\nu) = -\tfrac{1}{2}\nu\{[\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)]_P - [\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)]_A\}, \tag{13}$$

where $\sigma_P{}^V(\nu)$ ($\sigma_A{}^V(\nu)$) is the total cross section for the absorption of a circularly polarized isovector photon by a proton polarized with its spin parallel (antiparallel) to the photon spin, while $\sigma_{1/2}{}^V(\nu)$ ($\sigma_{3/2}{}^V(\nu)$) is the total cross section for the absorption of an unpolarized isovector photon in the $I=\frac{1}{2}$ ($I=\frac{3}{2}$) channel.

The sum rules are immediately obtained by introducing (3) through (6) into (9) through (13), under the assumption that the amplitudes $f_2(\nu)/\nu$, $f_3(\nu)/\nu$, and $f_4(\nu)$ satisfy unsubtracted dispersion relations:

$$\left(\frac{\mu_a{}^V}{2M}\right)^2 = \frac{1}{2\pi^2\alpha}\int_0^\infty \frac{\sigma_P{}^V(\nu)-\sigma_A{}^V(\nu)}{\nu}d\nu, \tag{14}$$

$$-\tfrac{1}{3}\langle r^2\rangle_E{}^V + \left(\frac{\mu_T{}^V}{2M}\right)^2 = \frac{1}{2\pi^2\alpha}\int_0^\infty \frac{\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)}{\nu}d\nu, \tag{15}$$

$$\frac{\mu_T{}^V}{2M} = \frac{-1}{4\pi^2\alpha}\int_0^\infty \{[\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)]_P - [\sigma_{3/2}{}^V(\nu)-2\sigma_{1/2}{}^V(\nu)]_A\}d\nu. \tag{16}$$

The relation (14) is the analog of the one suggested by Drell and Hearn[1] and can be derived also from the commutation relation of the vector charge densities.[8] We note in passing that the sum rule (14) is also satisfied when the isoscalar part of the anomalous magnetic moment $\mu_a{}^S$ and the total cross sections $\sigma_P{}^S(\nu)$ and $\sigma_A{}^S(\nu)$ in the absorption of an isoscalar photon are appropriately substituted. The relation (15) is the Cabibbo-Radicati sum rule,[3] while (16) has already been proposed in Ref. 2.[4]

If one further assumes that $f_1(\nu)$ satisfies the unsubtracted dispersion relation, one formally gets, from (3) and (10), a sum rule expressing the Schwinger constant in terms of the total cross section[9]:

$$S = \frac{1}{2M} + \frac{1}{\pi^2\alpha}\int_0^\infty \sigma_T{}^V(\nu)d\nu. \tag{17}$$

III. DERIVATION OF THE THEOREMS

For the proof of the low-energy theorems, we make use of the following identities:

$$\int d^4x d^4y\, e^{-iq'x+iqy}\langle p'|T(j_0{}^a(x),j_0{}^b(y))|p\rangle$$

$$= q_\lambda' q_\sigma \int d^4x d^4y\, e^{-iq'x+iqy} \times\langle p'|T(j_\lambda{}^a(x),j_\sigma{}^b(y))|p\rangle, \tag{18}$$

$$= q_0' q_0 \int d^4x d^4y\, e^{-iq'x+iqy}\langle p'|T(j_0{}^a(x),j_0{}^b(y))|p\rangle - iq_0'\int d^4x d^4y\, e^{-iq'x+iqy} \times\langle p'|\delta(x_0-y_0)[j_0{}^a(x),j_0{}^b(y)]|p\rangle - iq_\lambda\int d^4x d^4y\, e^{-iq'x+iqy} \times\langle p'|\delta(x_0-y_0)[j_0{}^a(x),j_\lambda{}^b(y)]|p\rangle, \tag{19}$$

where λ and σ run 1, 2, and 3. Equations (18) and (19) are easily derived by the current-conservation law and repeated integrations by parts. The relationship between (18) and (19) is a generalization of Low's formula for the scattering of photons by spin-$\frac{1}{2}$ systems,[10] and has been extensively used by Adler, for the special case of $q'=q$, in connection with the derivation of the sum rules for high-energy neutrino reactions.[11]

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[8] By choosing $a=b=3$ in Ref. 2, and repeating the same calculation, one can easily get (14).

[9] If the integral over the cross section is divergent, the sum rule (17) of course would not make any sense. On the other hand, obviously, we are not allowed, for the *real* Compton scattering, to assume the unsubtracted dispersion relation for the true amplitude $\tilde{f}_1(\nu)$, since this is inconsistent with the Thomson limit. It is amusing to note that if $S=1/M$ is set, one obtains a sum rule which is essentially the Thomas-Reiche-Kuhn sum rule.

[10] F. E. Low, Phys. Rev. **96**, 1428 (1954).

[11] S. L. Adler, Phys. Rev. **143**, B1144 (1966).

Let us now assume

$$[j_0^a(\mathbf{x},0),j_0^b(\mathbf{0},0)]=i\epsilon_{abc}j_0^c(\mathbf{x},0)\delta(\mathbf{x}), \quad (20)$$

$$[j_0^a(\mathbf{x},0),j_\lambda^b(\mathbf{0},0)]=i\epsilon_{abc}j_\lambda^c(\mathbf{x},0)\delta(\mathbf{x}) -i\partial_\lambda\delta(\mathbf{x})S^{ab}(\mathbf{x},0), \quad (21)$$

where $S^{ab}(\mathbf{x},0)$ is a scalar operator whose nature we do not specify here. Note that we have not assumed that S^{ab} is either symmetric or antisymmetric with respect to a and b. With the assumption (21), however, one can show from (18) and (19) that the matrix element $\langle p|S^{ab}(0)|p\rangle$ is in fact symmetric in a and b by virtue of the crossing symmetry. Thus, writing

$$\langle p|S^{ab}(0)|p\rangle=\delta_{ab}S\frac{P_0}{M}, \quad (22)$$

it will contribute only to the amplitude f_1.

From (18) and (19), we get

$$-q_\lambda' T_{\lambda\sigma}{}^{ab}q_\sigma=-q_0'q_0T_{00}{}^{ab}-i(2\pi)^4\delta^4(-q'-p'+q+p) \times[iq_0'\epsilon_{abc}\langle p'|j_0^c(0)|p\rangle +i\epsilon_{abc}\langle p'|j_\lambda^c(0)|p\rangle q_\lambda +q_\lambda q_\lambda\langle p'|S^{ab}(0)|p\rangle]. \quad (23)$$

By invoking Low's argument, we can evaluate $T_{00}{}^{ab}$ correctly to terms linear in the momenta of the photons by considering only the single-nucleon intermediate states.

Let us write

$$T_{\lambda\sigma}{}^{ab}=\{\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b\}T_{\lambda\sigma}{}^{(s)}+[\tfrac{1}{2}\tau^a,\tfrac{1}{2}\tau^b]T_{\lambda\sigma}{}^{(a)}, \quad (24)$$

and

$$T_{\lambda\sigma}{}^{(s,a)}=T_{\lambda\sigma}{}^{(0)(s,a)}+\delta_{\lambda\sigma}A^{(s,a)}+i\epsilon_{\lambda\sigma\kappa}\sigma_\kappa B^{(s,a)}, \quad (25)$$

where $T_{\lambda\sigma}{}^{(0)(s)}$ and $T_{\lambda\sigma}{}^{(0)(a)}$ are the single-particle T-product matrix element symmetric and antisymmetric in a and b, respectively. The terms $A^{(s,a)}$ and $B^{(s,a)}$ contain all contributions from the excited intermediate states, and are determined from (23) to order linear in q. The calculation of $T_{\lambda\sigma}{}^{(s)}$ proceeds exactly in the same way as that discussed by Low,[10] and the result, together with the Schwinger constant in (23), leads to the low-energy forward amplitudes (3) and (4).

Proceeding to the antisymmetric case, we first obtain[12]

$$T_{00}{}^{(a)}=-i(2\pi)^4\delta^4(-q'-p'+q+p)(M/p_0')^{1/2}\{1-\tfrac{1}{6}\langle r^2\rangle_E{}^V \times[(\mathbf{q}')^2+(\mathbf{q})^2]\}\nu^{-1}. \quad (26)$$

The target nucleon is taken to be standing still. The term proportional to ν^{-1} in (26) cancels in (23) with the term $q_0'\langle-\mathbf{q}'+\mathbf{q}|j_0^c(0)|0\rangle$. Unlike $T_{00}{}^{(s)}$, the spin-dependent part of $T_{00}{}^{(a)}$ is quadratic in q, and hence can be disregarded.

[12] It is very essential at this point to allow first the photons to be time-like, retain the lowest order terms in $|\mathbf{q}|^2$ in (26), and then take the limit $|\mathbf{q}|^2\to 0$, followed by $\nu=q_0\to 0$. If we keep $|\mathbf{q}|^2$ finite and take a limit $\nu\to 0$, from (18) and (19), we get a sum rule for arbitrary $\mathbf{q}^2>0$. The Cabibbo-Radicati sum rule is then obtained by the differentiation with respect to $\mathbf{q}^2$. See Ref. 11.

We next obtain

$$T_{\lambda\sigma}{}^{(0)(a)}=-i(2\pi)^4\delta^4(-q'-p'+q+p) \times(M/p_0')^{1/2}[-(4M^2)^{-1}(q_\lambda'q_\sigma-q_\lambda q_\sigma-q_\lambda'q_\sigma') +i\mu_T{}^V(2M)^{-1}\{(-\mathbf{q}'+\mathbf{q})_\lambda[\boldsymbol{\sigma}\times\mathbf{q}]_\sigma +[\boldsymbol{\sigma}\times\mathbf{q}']_\lambda(-\mathbf{q}+\mathbf{q}')_\sigma\}+(\mu_T{}^V)^2 \times\{\delta_{\lambda\sigma}(\mathbf{q}'\cdot\mathbf{q})-q_\lambda q_\sigma'+i\epsilon_{\lambda\sigma\kappa}q_\kappa'(\mathbf{q}\cdot\boldsymbol{\sigma}) -i[\mathbf{q}'\times\mathbf{q}]_\lambda(\sigma)_\sigma\}]\nu^{-1}. \quad (27)$$

Substituting (25), (26), and (27) into (23), we then get to linear in ν,

$$A^{(a)}=-\tfrac{1}{3}\langle r^2\rangle_E{}^V\nu,\quad B^{(a)}=\mu_T{}^V/2M. \quad (28)$$

Retaining only the terms proportional to $\delta_{\lambda\sigma}$ and $\epsilon_{\lambda\sigma\kappa}\sigma_\kappa$ in the forward amplitude $T_{\lambda\sigma}{}^{(a)}$, we obtain

$$T_{\lambda\sigma}{}^{(a)}=-i(2\pi)^4\delta^4(-q'-p'+q+p)(M/p_0')^{1/2} \times[\delta_{\lambda\sigma}\{\mu_T{}^2(\mathbf{q})^2\nu^{-1}-\tfrac{1}{3}\langle r^2\rangle_E{}^V\nu\} +i\epsilon_{\lambda\sigma\kappa}\sigma_\kappa\mu_T{}^V/2M]. \quad (29)$$

The low-energy forward-scattering amplitudes (5) and (6) follow immediately from (29).

IV. DISCUSSION AND SUMMARY

We wish to discuss briefly the relations between the forward-scattering amplitudes $f_1,\cdots,f_4$ obtained in the previous sections and the corresponding *true forward-scattering amplitudes* $\tilde{f}_1,\cdots,\tilde{f}_4$.

As is well known, the true forward-scattering amplitude is given by[13]

$$\tilde{F}_{\lambda\sigma}{}^{ab}(\nu)=F_{\lambda\sigma}{}^{ab}(\nu)+C_{\lambda\sigma}{}^{ab}(\nu), \quad (30)$$

where $F_{\lambda\sigma}{}^{ab}(\nu)$ is defined by (2), and $C_{\lambda\sigma}{}^{ab}(\nu)$, the so-called "seagull" term, is given by

$$C_{\lambda\sigma}{}^{ab}(\mathbf{q})=-i\int d^4x\, e^{-iqx}\delta(x_0) \times\langle p|[A_\lambda{}^a(x),j_\sigma{}^b(0)]|p\rangle. \quad (31)$$

In this expression, $A_\lambda{}^a(x)$ is the vector field satisfying the following equation of motion and the subsidiary condition:

$$-\Box^2A_\lambda{}^a(x)=j_\lambda{}^a(x), \quad (32)$$

and

$$\partial_\mu A_\mu{}^a(x)|\rangle=0. \quad (33)$$

First, it can easily be shown, under the assumption of invariance under time reversal and parity conjugation,

[13] We have assumed throughout the rest of this paper that the current $j_\lambda{}^a$ does not contain the time derivative of the vector field $A_\lambda{}^a$. Although in the Yang-Mills theory, the current $j_\lambda{}^a$ contains the vector strength $f_{\mu\nu}{}^a$ explicitly, the term proportional to $\dot{A}_\lambda{}^a$ can be eliminated by the subsidiary condition (33), since we are taking the matrix element like (31). Moreover, it can be shown that even if $j_\lambda{}^a$ contains the time derivative of the vector field $A_\lambda{}^a$, (36) still holds and therefore (37) and (39) are still valid. Consistency problems of the Lorentz subsidiary condition and the canonical quantization for the non-Abelian gauge field has been discussed by J. Schwinger [Phys. Rev. 130, 482 (1963)].

that

$$C_{\lambda\sigma}{}^{ab}(\mathbf{q})=C_{\sigma\lambda}{}^{ab}(-\mathbf{q}). \tag{34}$$

The crossing symmetry, on the other hand, states

$$C_{\lambda\sigma}{}^{ab}(\mathbf{q})=C_{\sigma\lambda}{}^{ba}(-\mathbf{q}). \tag{35}$$

Combining (34) and (35), we get

$$C_{\lambda\sigma}{}^{ab}(\mathbf{q})=C_{\lambda\sigma}{}^{ba}(\mathbf{q}). \tag{36}$$

Thus, the seagull terms vanish for the amplitudes antisymmetric in the isospins, and we obtain

$$\tilde{f}_3(\nu)=f_3(\nu), \tag{37}$$

$$\tilde{f}_4(\nu)=f_4(\nu). \tag{38}$$

Next, it can be shown from (31) that, by virtue of (32) and (33), the following identity exists[14]:

$$C_{\lambda\sigma}{}^{ab}+q_\rho(\partial/\partial q_\lambda)C_{\rho\sigma}{}^{ab}=S\delta_{ab}\delta_{\lambda\sigma}, \tag{39}$$

where S is the Schwinger constant defined by (22). In obtaining (39), the usual assumption that $j_\lambda{}^a$ does not contain the time derivative of the vector field has been made.[13]

If $C_{\lambda\sigma}{}^{ab}$ is independent of q, then we obtain immediately[15]

$$\tilde{f}_1(\nu)=f_1(\nu)+S, \tag{40}$$

$$\tilde{f}_2(\nu)=f_2(\nu). \tag{41}$$

Thus, it is evident that the low-energy theorem (3) corresponds exactly to the Thomson limit of the true forward-scattering amplitude.

In this manner, we find that the low-energy theorems for f_1, f_3, and f_4, i.e., (3), (5), and (6), can be replaced by those for the corresponding true scattering amplitudes $\tilde{f}_1$, $\tilde{f}_3$, and $\tilde{f}_4$. In addition, if $C_{\lambda\sigma}{}^{ab}$ is constant, the low-energy theorem for f_2, i.e. (4), can be replaced by that for $\tilde{f}_2$.[16] In this case, the low-energy theorems for the *real* Compton scattering amplitudes $\tilde{f}_1$ and $\tilde{f}_2$ are also reproduced.[17]

Summarizing our principal result, the low-energy theorems for the forward amplitudes in the Compton scattering of isovector photons by nucleons have been obtained. Assuming that each of these amplitudes satisfies unsubtracted dispersion relations, we have obtained sum rules that can possibly be tested experimentally.

Although it is evident that the low-energy theorems for these amplitudes can always be obtained, the precise forms of the theorems would naturally depend upon the commutation relations of vector current densities expressed in (20) and (21) through identities (18) and (19). For instance, if we modify the relation (20) by introducing nonminimal current, the sum rules of Cabibbo and Radicati, as well as the rest, get correspondingly modified.[18]

Thus, within the assumption of unsubtracted dispersion relations, these sum rules may provide a possible test of the current commutation relations (20) and (21).

Note added in proof. After our paper was submitted for publication, we came across the paper by M. A. B. Bég, [Phys. Rev. Letters **17**, 333 (1966)] in which he has derived the low-energy theorems for $f_1, \cdots f_4$ and the sum rules which correspond to (14), (15), and (16) of the present work. With regard to the main difference between his results and ours, we wish to make the following remarks:

(i) There are in general two different points of view with respect to the assumption of unsubtracted dispersion relations, namely: (a) assume unsubtracted dispersion relations for the amplitudes corresponding to the time-ordered product part $f_1(\nu), \cdots f_4(\nu)$ of the true scattering amplitudes; or (b) assume the same for the true scattering amplitudes $\tilde{f}_1(\nu), \cdots \tilde{f}_4(\nu)$. In the present paper, we have restricted ourselves to the point of view (a) and assumed unsubtracted dispersion relations for $f_1(\nu), \cdots f_4(\nu)$, while it would seem that the point of view (b) is taken by the above author. The difference between these two points of view is by no means trivial.

The crucial point is that from the point of view (a), the low-energy theorems for $f_1(\nu)$ and $f_4(\nu)$ and therefore the resulting sum rules (16) and (17) depend upon Schwinger terms (or seagull terms), while from the point of view (b), they do not. This situation is explicitly stated in the text in connection with the difference of the low-energy forms for $f_1(\nu)$ and $\tilde{f}_1(\nu)$. For instance, if we were allowed to assume that the $C_{\lambda\sigma}{}^{ab}$

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[14] Although we have used the equation of motion and the subsidiary condition for $A_\lambda{}^a(x)$ to prove (39), this identity can also be proved by the following argument. Multiply q_μ to the true forward amplitude defined by (30). Then, by integration by parts, one obtains

$$q_\mu F_{\mu\nu}{}^{ab}=-\int d^4x\, e^{-iqx}\delta(x_0)\langle p|[j_0{}^a(x),j_\nu{}^b(0)]|p\rangle+q_\mu C_{\mu\nu}{}^{ab}$$
$$=i\epsilon_{abc}\langle p|j_\nu{}^c(0)|p\rangle.$$

This relation was first derived by R. P. Feynman (California Institute of Technology seminar, November 1965; see also S. L. Adler and Y. Dothan, Phys. Rev. **150**, 000 (1966). The identity (39) is easily obtained by differentiating the above with respect to q_λ.

Since $C_{\mu\nu}{}^{ab}(\mathbf{q})=C_{\mu\nu}{}^{ba}(\mathbf{q})$, the higher order gradient terms of the δ-function that might possibly exist in (21) in addition to the one given should also be symmetric in a and b. This gives support to the conjecture by Adler and Callan (unpublished CERN report) with regard to the isospin symmetry of the Schwinger term.

[15] On the other hand, if $C_{\lambda\sigma}{}^{ab}$ depends on ν, since it arises from the equal-time commutator, it must be a polynomial in ν in general, in which case it would be necessary to assume higher order gradient terms of the δ-function in the commutation relation (21) because of the equation stated in Ref. 14. However, we still have

$$\tilde{f}_1(0)=f_1(0)+S,$$

and

$$\lim_{\nu\to 0}\tilde{f}_2(\nu)/\nu=\lim_{\nu\to 0}f_2(\nu)/\nu.$$

[16] In Low's derivation of the low-energy theorem, $\tilde{f}_2(\nu)=f_2(\nu)$ was shown explicitly on the basis of a model field theory.

[17] Since we are working with the isovector photons, there is a factor 2 difference between our formula (3) and (4) and the Thomson and Low–Gell-Mann–Goldberger limits. However, it is easy to apply our method to real Compton scattering. For the amplitudes symmetric in a and b we then reproduce the usual low-energy theorems.

[18] K. Kawarabayashi and M. Suzuki, this issue, Phys. Rev. **152**, 1383 (1966).

contains a term antisymmetric in both a and b, and λ and σ, the low-energy theorem for $f_4(\nu)$ and the corresponding sum rule (16) would have been modified. We have shown that such is not the case if one starts from the *physical S-matrix element* and applies the reduction formula to it, thereby having an explicit form (31) for the seagull term.

(ii) As is shown in the text, all exact sum rules that have been derived from the current commutation relations by making use of the method of the so-called "current algebra at $p\to\infty$," can be derived from low-energy limit theorems and the assumption of unsubtracted dispersion relations. However, the two approaches are equivalent only if we take the point of view (a).

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Shrinking of the p-p Scattering Diffraction Peak for Large Momentum Transfers

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The p-p elastic-scattering cross section can be explicitly calculated in quantum field theory endowed with a fundamental length in the limit of very high energy and large momentum transfer, assuming that a simple (vector-meson) interaction is dominant. Plots of $X\equiv\log_{10}[(d\sigma/d\Omega)_{\text{c.m.}}/(\sigma p/4\pi)^2]$ are given for various lab momenta $P\gtrsim 10$ GeV/c and momentum transfers $-t\gtrsim 10(\text{GeV}/c)^2$. In this range X involves only a *single, additive parameter*. A strong energy dependence, which gives the observed great shrinking, enters through the "kinematical form factors" attached to the external lines and is thus an immediate consequence of a fundamental length. The fit to the experimental points is excellent for a coupling constant of strong-interaction size if $\lambda\approx 0.5\times 10^{-14}$ cm.

THE curves shown in Fig. 1 are theoretical curves for p-p elastic scattering at very high energies and momentum transfers obtained from quantum field theory endowed with a universal fundamental length, or high-momentum cutoff. To illustrate the typical behavior in this range of s and t caused by a fundamental length, we chose the interaction to be mediated by a single neutral vector meson for simplicity. The features arising from a fundamental length are not very sensitive to the details of a more realistic interaction. In this range the cross section and thus X depend on only a *single multiplicative unknown parameter*. Thus in a logarithmic plot one has only the freedom of displacing all the curves rigidly up or down, without, of course, changing their shapes of relative orientation. In view of this, the fit obtained to the "shrinking" (energy dependence of the diffraction peak) seems good to us, better than any theoretical fit we have yet seen. The open circles show Serber's energy-independent theoretical curve,[1] obtained from an optical model, for comparison. Although the curves have been plotted back to zero momentum transfer, the small $-t$ $[-t\lesssim 10\ (\text{GeV}/c)^2]$ is not meant to be significant.

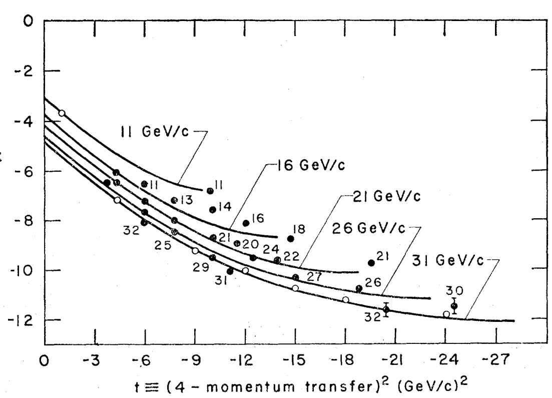

FIG. 1. The normalized p-p elastic scattering cross section in the very high energy and large-momentum-transfer range, computed from Eq. (2.6) with X defined by Eq. (2.7). ● ≡ experimental points from Ref. 12. ○ ≡ Serber's theoretical optical model (Ref. 1). The curves correspond to $\beta_p/\lambda^5\approx 1.46\times 10^4$ (GeV/c)^5. The behavior for $-t\lesssim 10$ (GeV/c)^2 is not significant.

[1] R. Serber, Rev. Mod. Phys. 36, 649 (1964).