Vertex Symmetry and the Reciprocal Bootstrap*

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(Received 6 July 1966)

We formulate a vertex-symmetric, time-reversal invariant bootstrap in a multichannel version of the static model. The results are applied to the coupled πN and πN^* channels, and to the infinite baryon tower of states with $I=J$. The coupled-channel problem is shown to be equivalent to the uncoupled problem, and the implications for approximate calculations are discussed briefly.

I. INTRODUCTION

In recent years there has been a revival of interest in the static model,[1] which we may hope is qualitatively correct in describing the low-energy behavior of mesons and baryons. In the context of the N/D method, the term "static model" has come to be used for the scheme[2] in which only the baryon-exchange pseudopoles are used for the left-hand cut, the D function is approximated by a straight line, and everything is taken to lowest order in the inverse baryon mass. This model was used by Chew[3] to introduce the idea of a reciprocal bootstrap for the N and the N^* and has since[4] been extended to $SU(3)$. A somewhat different approach to the model has been adopted by Cook, Goebel, and Sakita[5] and by Singh and Udgaonkar,[6] who have studied the group-theoretic aspects of the strong-coupling limit. The present work overlaps the last reference to a certain extent, though our approach is rather different.

Our interest in the static model is mainly in the reciprocal bootstrap, and in particular in the problem of vertex symmetry. The bootstrap hypothesis treats all particles on the same footing as bound states of each other, and hence it is highly desirable to make approximations in such a way as to maintain this symmetry. Specifically, we should like the πNN^* coupling constant derived from the residue of the N^* pole in πN scattering to be the same as the coupling constant derived from the residue of the N pole in πN^* scattering.[7] This attitude has been particularly emphasized by Cutkosky.[8]

Such symmetry is obviously difficult to maintain in a theory that treats the N and the N^* differently. Our purpose is therefore to consider the two-channel problem of coupled πN and πN^* channels and to formulate the reciprocal bootstrap for this case. A related problem has been considered by Abers, Balázs, and Hara (ABH),[9] but these authors treat each channel separately, and include only πN intermediate states in the unitarity equation for πN scattering and in $\pi N \to \pi N^*$, and only πN^* intermediate states in πN^* elastic scattering. Such a procedure obviously needs further justification; our aim is to provide it. We therefore consider the πN-πN^* system as a coupled two-channel problem and include both πN and πN^* intermediate states in all processes. Rather surprisingly, the extra intermediate states make no difference to the final answer, and the ABH procedure turns out to be equivalent to solving the coupled problem.

The advantage (or the disadvantage, depending on one's attitude) of the static model is that it reduces dynamics to group theory. That is, all dependence on masses, cutoffs, and density of states cancels out of the final equations, and we are left with relations between crossing matrix elements and coupling constants only. In this sense our conclusions are mainly about properties of Clebsch-Gordan coefficients, and it is a valid question whether they have any physical relevance. While we do not believe that a more realistic calculation will necessarily agree with this one in detail, it is probably true, however, that a fair part of our "intuition" about how a bootstrap calculation works is based on the results of the static model, and from this point of view our conclusions are of some relevance, even if their interest is mainly negative. The fact that the coupled-channel and the uncoupled-channel calculations agree is encouraging, though it comes about in a rather artificial way. Had they disagreed, however, it would have been a strong argument for believing neither.

A further reason for continuing to study the static model is just that any more realistic model is so difficult to handle. As emphasized by Ernst, Warnock, and Wali,[10] almost any approximate calculation that maintains vertex symmetry is likely to lead to a violation of time-reversal invariance.

A coupled-channel problem has vertex symmetry "built in," and if the N/D equations are solved exactly, time-reversal invariance of the solution will follow from

* Supported by the U. S. Atomic Energy Commission.
[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[2] D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. **12**, 359 (1962).
[3] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[4] I. Gerstein and K. T. Mahanthappa, Nuovo Cimento **32**, 239 (1964); R. Dashen, Phys. Letters, **11**, 89 (1964).
[5] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).
[6] V. Singh and B. Udgaonkar, Phys. Rev. **149**, 1164 (1966).
[7] We should, ideally, also get the coupling constant from the pion pole in $\bar{N}N^*$ scattering, but such an approach is beyond present techniques.
[8] K. Y. Lin and R. E. Cutkosky, Phys. Rev. **140**, B205 (1965).
[9] E. S. Abers, L. A. P. Balázs, and Y. Hara, Phys. Rev. **136**, B1382 (1964). We shall refer to this paper as ABH.
[10] F. J. Ernst, R. L. Warnock, and K. C. Wali, Phys. Rev. **141**, 1354 (1966).

time-reversal invariance of the input.[11] In any approximate calculation, however, it is difficult to maintain symmetry of the T matrix (which reflects the time-reversal invariance). We suggest a new approximation below [Eq. (7)] which guarantees the symmetry of the T matrix in the static model.[12]

We start by formulating a general theory of multichannel scattering in the static model. It turns out that the equivalence of the coupled and uncoupled versions of the model is most transparent for the infinite "baryon tower" of ABH, and we discuss this first. We then turn to a restriction of the tower to the N-N^* system, and finally discuss the results.

II. FORMULATION OF THE MULTICHANNEL BOOTSTRAP

In this section we would like to give a rather general discussion of a multichannel bootstrap in the static model. As is well known, spin and orbital angular momentum are completely decoupled in the model, and spin may be treated as an internal symmetry. We therefore take a model of n heavy spinless baryons N_i and consider the coupled πN_i channels.

We shall neglect differences in the masses of the N_i in external lines, though not in internal lines or in density-of-states factors. This assumption, though slightly inconsistent, avoids a lot of purely kinematic complications; and as pointed out by Ernst, Warnock, and Wali,[10] the most sensitive dependence on mass differences is in the density of states. As usual, we take the meson laboratory energy ω as our variable. The P-wave scattering amplitude may then be written[13]

$$\mathbf{T}=\mathbf{N}\mathbf{D}^{-1}, \tag{1}$$

where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{D}$ are $n\times n$ matrices with rows and columns labelled by the channel πN_i. Then

$$\mathbf{N}=\int_L \frac{\boldsymbol{\alpha}(\omega')\mathbf{D}(\omega')}{\omega'-\omega}d\omega', \tag{2}$$

$$\mathbf{D}=\mathbf{I}-\frac{\omega}{\pi}\int_R \frac{\boldsymbol{\varrho}(\omega')\mathbf{N}(\omega')}{\omega'(\omega'-\omega)}d\omega', \tag{3}$$

where $\boldsymbol{\varrho}$ is the density of states and $\boldsymbol{\alpha}$ is the left-hand discontinuity of $\mathbf{T}$. We replace the left-hand cut by a series of poles at $\omega=-\omega_r$ with residues $\boldsymbol{\Gamma}_r$, so that

$$\mathbf{N}=\sum_r \boldsymbol{\Gamma}_r\mathbf{D}(-\omega_r)/(\omega+\omega_r), \tag{4}$$

$$\mathbf{D}=\mathbf{I}-\boldsymbol{\Lambda}\omega, \tag{5}$$

with

$$\boldsymbol{\Lambda}=\sum_r\left[\frac{1}{\pi}\int\frac{\boldsymbol{\varrho}(\omega')}{(\omega'-\omega)\omega'(\omega'+\omega_r)}\right]\boldsymbol{\Gamma}_r\mathbf{D}(-\omega_r). \tag{6}$$

In the one-channel approach, the cutoff necessary is usually[2] introduced by assuming D is a linear function of ω; that is, by taking $\boldsymbol{\Lambda}$ to be constant. A similar approach in the multichannel case would be to regard the n^2 elements of $\boldsymbol{\Lambda}$ as constant parameters, and in fact as arbitrary apart from the constraints of time-reversal invariance. We should like to suggest that it is reasonable to constrain $\boldsymbol{\Lambda}$ further. We ignore the dependence of $\boldsymbol{\Lambda}$ on ω; the form of the factor in curly brackets in Eq. (6) then suggests that it is consistent to assume that this factor is also independent of ω_r and to write

$$\begin{aligned}\boldsymbol{\Lambda}&=\sum_r \mathbf{P}\boldsymbol{\Gamma}_r\mathbf{D}(-\omega_r)\\&=(\mathbf{I}-\sum\omega_r\mathbf{P}\boldsymbol{\Gamma}_r)^{-1}\mathbf{P}\boldsymbol{\Gamma},\end{aligned} \tag{7}$$

where $\mathbf{P}$ is a constant diagonal matrix, and

$$\boldsymbol{\Gamma}=\sum_r \boldsymbol{\Gamma}_r. \tag{8}$$

We have used Eq. (5) in deriving the second line of (7). The n diagonal elements of $\mathbf{P}$ are now the free parameters in the model, and represent the cutoffs. As we shall see, the form (7) for $\boldsymbol{\Lambda}$ maintains the symmetry of the T matrix.

If now $\mathbf{T}$ has a direct-channel pole at $\omega=\omega_0$ with residue $\mathbf{R}$, we have

$$\det(\mathbf{I}-\omega_0\boldsymbol{\Lambda})=0, \tag{9}$$

and

$$\begin{aligned}\mathbf{R}&=-\mathbf{N}(\omega_0)[\mathbf{D}'(\omega_0)]^{-1}\\&=-\boldsymbol{\Gamma}\boldsymbol{\Lambda}\,\mathrm{adj}(\mathbf{I}-\omega_0\boldsymbol{\Lambda})/(d/d\omega)[\det(\mathbf{I}-\omega\boldsymbol{\Lambda})]_{\omega=\omega_0},\end{aligned} \tag{10}$$

where "adj" denotes the adjugate, defined for any matrix $\mathbf{A}$ by

$$\mathbf{A}\,\mathrm{adj}\mathbf{A}=\det\mathbf{A}\cdot\mathbf{I}. \tag{11}$$

By choosing the representation in which $\boldsymbol{\Lambda}$ is triangular, it is easy to prove that

$$\mathrm{Tr}\boldsymbol{\Lambda}\,\mathrm{adj}(\mathbf{I}-\omega_0\boldsymbol{\Lambda})=(d/d\omega)[\det(I-\omega\boldsymbol{\Lambda})]_{\omega=\omega_0}, \tag{12}$$

and hence that[14]

$$\mathrm{Tr}[\mathbf{R}\,\mathrm{adj}\boldsymbol{\Gamma}]=\det\boldsymbol{\Gamma}, \tag{13}$$

where we write the equation this way since there is no *a priori* reason for $\boldsymbol{\Gamma}$ to be nonsingular. This equation involves only coupling constants and is completely independent of any masses or cutoffs. It is also independent of the specific form (7) we have taken for $\boldsymbol{\Lambda}$ and is the basic equation of our model. It is, however, only a necessary condition, and we must still verify that it is possible to choose masses to satisfy the full

[11] J. D. Bjorken and M. Nauenberg, Phys. Rev. 121, 1250 (1961).

[12] A more realistic model has been studied by R. C. Brunet and R. W. Childers, Phys. Rev. (to be published). Their calculation is not a complete bootstrap, however, and does not preserve symmetry of the T matrix.

[13] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960). See also J. B. Hartle and J. R. Taylor, Princeton University (unpublished report), 1966.

[14] Compare the paper by Dashen quoted in Ref. 4.

set of equations (10) with the values of the coupling constants given by (13).

In order to make contact with the work of other authors,[10] we consider the special case of these equations that occurs when all the ω_r are taken equal to some ω^*. Then using (7), after some manipulation we find

$$\mathbf{R} \propto \mathbf{\Gamma P \Gamma}\,\mathrm{adj}[\mathbf{I}-(\omega^*+\omega)\mathbf{P\Gamma}], \qquad (14)$$

where the omitted factors are all scalars (i.e., not matrices) and where Eq. (9) implies

$$\det[I-(\omega^*+\omega_0)\mathbf{P\Gamma}]=0. \qquad (15)$$

Since the elements of $\mathbf{R}$ are residues of one-particle poles, they factorize into the product of two coupling constants, one for each vertex. In fact, $\mathbf{R}$ is a rank-one matrix[15] and can be written

$$\mathbf{R}=|r\rangle\langle r| \qquad (16)$$

for some vector $|r\rangle$.

The last three equations then imply

$$\mathbf{\Gamma P}|r\rangle=(\omega^*+\omega_0)|r\rangle. \qquad (17)$$

A similar equation has been obtained by Ernst, Warnock, and Wali.[10] Though Eq. (17) is very simple, it holds only because we have taken all crossed poles at the same point. As we shall discuss later, this equation has certain disadvantages, and we shall not use it in the following.

Returning to the case of general masses, an important question is the symmetry of the right-hand side of Eq. (10) since this is necessary if our approximations do not violate time-reversal invariance. In the Appendix we show that

$$\mathbf{R}=\mathbf{\Gamma S \Gamma}, \qquad (18)$$

where $\mathbf{S}$ is a complicated matrix involving the ω_r, ω_0, and $\mathbf{P}$. $\mathbf{S}$ is symmetric, and the identify (12) ensures that

$$\mathrm{Tr}\mathbf{S\Gamma}=1. \qquad (19)$$

The symmetry of $\mathbf{R}$ is then manifest.

Up till now our discussion has been perfectly general. The bootstrap hypothesis now relates the coupling constants that appear in $\mathbf{R}$ to these that appear in $\mathbf{\Gamma}$ via crossing matrices and identifies ω_0 with one of the ω_r. Equation (13) now becomes a consistency condition on the coupling constant; whether or not it can be satisfied cannot be answered in general and depends on the particular crossing matrices and on the number of channels. Assuming that it can, it is an interesting question whether the full Eq. (10) implies any further conditions. This is impossible to answer in general; depending on the specific values of the coupling constants, Eq. (10) may be satisfied identically if (13) is, it may give an equation constraining the various ω_r, or it may merely determine the cutoff in terms of the ω_r. We shall give examples of these possibilities below.

To proceed further, it is necessary to choose specific forms $\mathbf{\Gamma}$ and $\mathbf{R}$. We therefore turn to a study of various examples.

III. EQUIVALENCE OF THE UNCOUPLED AND COUPLED CHANNEL BOOTSTRAPS

As already mentioned, the mechanism at work can be understood more clearly by a study of the infinite baryon "tower" of states with equal spin and isospin[16] introduced by ABH than by looking at the coupled N-N^* case. We therefore introduce a series of baryons N_t with $t=2I=2J=1, 3, 5, \cdots$, and define the $\pi N_t N_{t'}$ coupling constant[17] $g_{tt'}$ by requiring that the residue of the N_t pole in $\pi N_{t'} \to \pi N_{t''}$ scattering be $g_{tt'}g_{tt''}/(t+1)^2$. Suppose we now follow ABH in looking at πN_t scattering and keeping only πN_t intermediate states in the unitarity equation. There are then three equations of the type of Eq. (13) giving the residues of the N_{t-2}, N_t, and N_{t+2} direct poles in terms of the corresponding crossed residues. As is well known, these can be written as an eigenvalue equation.

$$\begin{bmatrix} g_{t,t-2}^2/(t-1)^2 \\ g_{t,t}^2/(t+1)^2 \\ g_{t,t+2}^2/(t+3)^2 \end{bmatrix} = \begin{bmatrix} \dfrac{4}{t^2(t+1)^2} & \dfrac{4}{t^2} & \left(\dfrac{t+3}{t+1}\right)^2 \\ \dfrac{4(t-1)^2}{t^2(t+1)^2} & \dfrac{(t^2+2t-4)^2}{t^2(t+2)^2} & \dfrac{4(t+3)^2}{(t+1)^2(t+2)^2} \\ \left(\dfrac{t-1}{t+1}\right)^2 & \dfrac{4}{(t+2)^2} & \dfrac{4}{(t+1)^2(t+2)^2} \end{bmatrix} \begin{bmatrix} g_{t,t-2}^2/(t-1)^2 \\ g_{t,t}^2/(t+1)^2 \\ g_{t,t+2}^2/(t+3)^2 \end{bmatrix}, \qquad (20)$$

where the matrix is the $\pi N_t \to \pi N_t$ crossing matrix.[18] Equation (20) has a solution for all t, namely,

$$g_{tt'}=[(t+1)(t'+1)]^{1/2}g, \qquad (21)$$

where g is arbitrary. The fact that a solution exists at all is a reflection of the very high symmetry of the

[15] J. M. Charap and E. Squires, Phys. Rev. **127**, 1387 (1962).

[16] We do not consider the problems of which states can occur in the model, but merely assume a consistent set. We also do not investigate whether the solution we find is unique, but merely show that a solution with the desired properties exists. Compare the discussion in Ref. 9.

[17] There is some problem in notation due to the fact that the residue of, say, the N^* pole in $\pi N \to \pi N$ and the N pole in $\pi N^* \to \pi N^*$ are not the same, but differ by kinematic factors. Unlike other authors, we prefer to follow common practice in introducing only one coupling constant at each vertex, so that $g_{tt'}=g_{t't}$. This has the unaesthetic feature of introducing various numerical factors into the residues, but avoids defining different coupling constants with extra relations between them.

[18] For the "internal" symmetry $SU(2)\otimes SU(2)$.

model[5]; and it is this fact that makes the uncoupled- and coupled-channel problems equivalent.

Equation (20) was the condition for a reciprocal bootstrap in the πN_t channel. It has another interpretation, however. For suppose we now look at the full coupled-channel problem using the formalism of Sec. II. Everything now becomes a 3×3 matrix with rows and columns labelled by πN_{t-2}, πN_t, and πN_{t+2}. The rows of the vector on the left side of (20) are residues of direct poles, and hence are elements of $\mathbf{R}$; those of the vector on the right are sums of residues at crossed poles, and are elements of $\mathbf{\Gamma}$. Equation (20) then states the equality of diagonal elements of $\mathbf{R}$ and $\mathbf{\Gamma}$. For the off-diagonal elements we need the $\pi N_t \to \pi N_{t+2}$ and the $\pi N_{t-2} \to \pi N_{t+2}$ crossing matrices; they are

$$\begin{bmatrix} \dfrac{4}{(t+2)^2} & \dfrac{t(t+3)(t+4)}{(t+1)(t+2)^2} \\ \dfrac{t(t+1)(t+4)}{(t+3)(t+2)^2} & \dfrac{4}{(t+2)^2} \end{bmatrix},$$

and

$$[1],$$

respectively. It is then straightforward to show that the solution (21) implies

$$\mathbf{R}=\mathbf{\Gamma}. \tag{22}$$

This equation was derived using values of the coupling constants which were a solution to the uncoupled-channel problem. We would now like to verify that these coupling constants also provide a solution to the coupled-channel problem; that is, that it is possible to choose the masses and cutoffs so that Eq. (22) furnishes a solution of the bootstrap Eq. (10), or equivalently, of Eq. (18). In fact, an even stronger result is true, for if we use the explicit form (16) of $\mathbf{R}$, and Eq. (22), then the right-hand side of Eq. (18) becomes

$$\mathbf{\Gamma S \Gamma}=|r\rangle\langle r|\mathbf{S}|r\rangle\langle r|,$$

while Eq. (19) implies the identity

$$\langle r|\mathbf{S}|r\rangle=1.$$

Thus for any values of the masses and cutoffs, Eq. (22) provides a solution of the coupled-channel Eq. (18) and *a fortiori* of Eqs. (13) and (17); which shows the equivalence of the coupled and uncoupled problems.[19] It is perhaps worth emphasizing the difference between the uncoupled and coupled cases. In the former, Eq. (22) is a necessary condition, and can always be satisfied for any one value of t, though not necessarily for all. In the latter, (22) is certainly sufficient, but it is by no means necessary, and whether it can even be satisfied for one value of t depends on the particular crossing matrices. That it can, in fact, be satisfied for all t is a consequence of all the relevant crossing matrices having an eigenvalue one.

The infinite baryon tower, though elegant mathematically, may not be very relevant physically in this context. Even if the higher resonances do exist, we would not expect them in a range of energies where the static model was reasonable. We therefore turn to the more direct problem of the two lowest members of the tower—the N and the N^*. Treating the πN-πN^* system as a coupled-channel problem, the two equations (13) (one for each value of t) provide two nonlinear conditions on the two unknown ratios of the three relevant coupling constants g_{11}, g_{13}, and g_{33}. These may be solved, and rather surprisingly yield

$$g_{11}^2:g_{13}^2:g_{33}^2=1:2:4, \tag{23}$$

which is exactly the value given by (21). As already mentioned, it is still necessary to verify the equations of Eq. (10); with these values for the coupling constants, each of equations (10) yields a value for the ratio of ω_1 and ω_3 independent of any cutoffs, but the two values obtained from $t=1$ and $t=3$ are inconsistent. Hence, although the coupled-channel problem reproduces the coupling-constant ratios of the uncoupled problem, it fails to be reciprocal.

The difficulty here is just that it is impossible to treat the N and N^* symmetrically. While the πN system can only have $t=1$ or 3, the πN^* system can also have $t=5$, and in fact it is only by introducing the infinite tower that all baryons are treated on the same footing. The effect of the higher resonances is small, however,—only the $\pi N^* \to \pi N^*$ channel gets any contribution from N_5, and N_7, N_9, $\cdots$, do not contribute to the $t=1$ or 3 states at all. Hence, we may regard the contribution of the N_5 exchange as a simple way of parametrizing the effect of higher states, and take g_{35} as a cutoff whose value is fixed not by a bootstrap in the $t=5$ channel, but by requiring that the N-N^* bootstrap be reciprocal. Notice that this procedure is quite reasonable—the values (23) of the coupling constants are independent of g_{35}, and only the ratio of ω_1 to ω_3 (which we do not expect the static model to predict reliably anyway) is affected. It is interesting that the coupling-constant ratios do not depend on g_{35}; this fact is a reflection of the high degree of redundancy in eigenvalue equations. (That is, Eq. (20) provides three equations for any value of t, while the off-diagonal elements of (22) provide another two. Only two of these five equations are necessary to fix the g's, and hence it is not surprising that a subset of Eq. (22) should yield the same value as the full set). From this point of view the equivalence is rather a fluke; it does, however, suggest that Eq. (13) is a useful bootstrap condition,

[19] It is amusing to note that Eq. (22) is exactly the condition that if the T matrix is approximated by the sum of all direct- and crossed-channel poles, then unitarity is satisfied at high energies. (See Ref. 2). The solution of a one-channel bootstrap always has this property; for a multichannel bootstrap, Eq. (22) overconstrains the problem in general and is impossible to satisfy.

because it is insensitive to the exact nature of the higher states.

IV. DISCUSSION

We have succeeded in formulating a time-reversal-invariant, vertex-symmetric bootstrap in the static model for the infinite baryon tower, though not for the N-N^* case. We pointed out, however, that the lack of reciprocity in the latter case was in an equation for the masses and could easily be removed by assuming some contribution of higher states to $\pi N^* \to \pi N^*$ scattering. The moral of the calculation is the perhaps not unsurprising one that in the static model, equations involving only coupling constants are more believable (or less sensitive to the effects ignored) than those involving masses. This remark is not completely idle; although Eq. (17) predicts the same coupling constants as Eq. (13) when the full bootstrap solution (22) exists, the solutions of the former are changed radically when the contribution of $t=5$ states is ignored, while those of the latter are not changed at all. This may explain the large deviations from vertex symmetry found by Ernst, Warnock, and Wali,[10] who use an equation very similar to Eq. (17). To what extent these features persist in a more realistic model is, of course, an open question. As already remarked, the problem of maintaining both vertex and time-reversal symmetry makes the answer difficult to find.

ACKNOWLEDGMENTS

I should like to thank Dr. R. Amado for provoking my interest in this problem, Dr. G. Kane for several stimulating discussions, and Dr. I. Gerstein for reading the manuscript and for several helpful suggestions.

APPENDIX

Let

$$\mathbf{K}=\mathbf{\Lambda}\,\mathrm{adj}(\mathbf{I}-\omega_0\mathbf{\Lambda})/(d/d\omega)[\det(\mathbf{I}-\omega_0\mathbf{\Lambda})]_{\omega=\omega_0}. \quad \text{(A1)}$$

Equation (12) gives

$$\mathrm{Tr}\mathbf{K}=1, \quad \text{(A2)}$$

so that in the following calculation we may drop all scalar factors, and use (A2) to normalize the final answer. Then

$$\begin{aligned}\mathbf{K} &\propto \mathbf{\Lambda}\,\mathrm{adj}(\mathbf{I}-\omega_0\mathbf{\Lambda})\\ &\propto \mathrm{adj}\mathbf{\Lambda}^{-1}\,\mathrm{adj}(\mathbf{I}-\omega_0\mathbf{\Lambda})\\ &=\mathrm{adj}(\mathbf{\Lambda}^{-1}-\omega_0)\\ &=\mathrm{adj}[(\mathbf{P\Gamma})^{-1}\{1-\sum_r(\omega_r+\omega_0)\mathbf{P\Gamma}_r\}]\end{aligned}$$

by Eq. (7). Thus

$$\begin{aligned}\mathbf{K} &\propto \mathrm{adj}[\mathbf{\Gamma}^{-1}\{\mathbf{P}^{-1}-\textstyle\sum(\omega_r+\omega_0)\mathbf{\Gamma}_r]\\ &\propto \mathrm{adj}[\mathbf{P}^{-1}-\textstyle\sum(\omega_r+\omega_0)\mathbf{\Gamma}_r]\cdot\mathbf{\Gamma}.\end{aligned}$$

Hence,

$$\mathbf{K}=\mathbf{S\Gamma}, \quad \text{(A3)}$$

with

$$\mathrm{Tr}[\mathbf{S\Gamma}]=1.$$

Since $\mathbf{P}$ and $\mathbf{\Gamma}_r$ are symmetric, so is $\mathbf{S}$. The above proof assumed that $\mathbf{\Gamma}$ was nonsingular. Since the final answer (A3) does not involve $\mathbf{\Gamma}^{-1}$, and holds as an algebraic identity, the result is true even when $\mathbf{\Gamma}$ is singular.