mass differences, this degeneracy implies $SU(3)$-independent quark forces in the idealized limit of equal-mass quarks. Suppose that in this idealized limit the quark forces were not completely $SU(3)$-independent so that the singlet had a mass M_0 slightly different from the octet mass M with $M/M_0=I$. From Ref. 1 we see that Eqs. (15) become

$$\begin{aligned} G_{\phi AB}{}^0 &= G_{\phi_0 AB}\cos\beta - IG_{\omega_0 AB}\sin\beta, \\ G_{\omega AB} &= G_{\phi_0 AB}\sin\beta + IG_{\omega_0 AB}\cos\beta. \end{aligned} \tag{37}$$

For the sake of argument, let us suppose that $G_{\omega_0 AB}$ and $G_{\phi_0 AB}$ are still related by the nonet ansatz. Then, for example,

$$G_{\phi\rho\pi}{}^0/G_{\omega\rho\pi}{}^0 = \tan(\theta-\beta'), \tag{38}$$

where the effective mixing angle β' is given by

$$\beta' \approx \beta + (\sqrt{2}/3)(I-1). \tag{39}$$

We see that we only need $I\approx 1.02$ to halve the $\phi\to 3\pi$ partial width and $I\approx 1.08$ would completely suppress $\phi\to 3\pi$ in the model considered. The predictions of the renormalized theory that are least sensitive to slight deviations from nonet degeneracy will be decays involving real or virtual photons and thus unfortunately are susceptible to the merits of the models used to describe these decays.

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High-Energy Behavior of the Scattering Amplitude in the Unphysical Region $0<t<4m^2$*

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It is proved within the framework of axiomatic field theory that the scattering amplitude must have a Regge behavior in the unphysical region $0<t<4m^2$ in the sense that the logarithmic derivative of the absorptive part is bounded by $C\ln s$ for very large s.

IT can be shown in axiomatic field theory that the absorptive part $A(s,t)$ of the elastic scattering amplitude is positive and monotonically increasing in the interval $0\leq t<4m^2$ and that the rate of increase is restricted by the upper bound[1,2]

$$A(s,t)<Cs^{1+(t/4m^2)^{1/2}+\epsilon}, \quad \epsilon>0. \tag{1}$$

This suggests strongly that $A(s,t)$ has a Regge-type behavior in this interval. However, since the inequality (1) does not tell much about the actual t dependence of $A(s,t)$, it is necessary to examine the behavior of $A(s,t)$ more closely in order to settle this question. The purpose of this paper is to show that $A(s,t)$ has in fact a Regge behavior in the interval $0<t<4m^2$ in the sense that t dependence stronger than that of the Regge type is ruled out. At present, it is not known whether or not this result can be extended to the physical region $t\leq 0$.

We start from the result of axiomatic field theory[3] that the absorptive part $A(s,t)$ for a fixed physical value of s is analytic in an ellipse in the t plane with foci $t=0$, $-4k^2$ and semimajor axis $4m^2+2k^2$. Thus $A(s,t)$ can be expanded in this ellipse into partial waves:

$$A(s,t)=(\sqrt{s}/2k)\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(1+t/2k^2), \tag{2}$$

where $a_l(s)$ is the absorptive part of the lth partial-wave amplitude satisfying the unitarity restriction

$$0\leq a_l(s)\leq 1, \quad l=0, 1, 2, \cdots, \tag{3}$$

as well as the analyticity requirement[4]

$$a_l(s)\leq Cs^2\exp[-2l(4m^2/s)^{1/2}]. \tag{4}$$

As is well known, for $s>4m^2$, $A(s,t)$ and all derivatives of $A(s,t)$ with respect to t are positive in the inter-

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[1] For simplicity, we treat only the elastic scattering of spinless particles with equal mass m. As usual, s and t are the square of the total energy and the momentum transfer in the center-of-mass system. We also use the center-of-mass momentum k, related to s by $s=4(m^2+k^2)$.
[2] The inequality (1) was first derived by K. Bardakci [Phys. Rev. **127**, 1832 (1962)] starting from the Mandelstam representation. A more general derivation was given by A. Martin (Ref. 4). We have put $N=2$ in their formulas, taking account of the recent result of Martin (Ref. 3).
[3] A. Martin, Nuovo Cimento **42A**, 930 (1966); **44A**, 1219 (1966).
[4] A. Martin, in *Strong Interactions and High Energy Physics* (Oliver and Boyd, London, 1964), p. 105.

val $0 \leq t < 4m^2$.[5] This follows from (3) and the fact that $P_l(z)$ and its derivatives are non-negative at $z=1$. At $t=0$ the contribution to the infinite series (2) from the terms with l greater than $C\sqrt{s}\ln s$ is less than s^{-N}, where N can be made as large as we wish by choosing a large enough C.[6] It is easy to see that the same remark applies to any t in the interval $0 < t < 4m^2 - \delta$, where δ is some fixed positive number. Next, we note that $A(s,t)$ is bounded from below by s^{-5} in the interval $0 \leq t < 4m^2$.[7] Thus, if we choose a fixed value of N larger than 5 and determine the corresponding value of C, we can approximate (2) by the finite sum

$$A(s,t) = (\sqrt{s}/2k) \sum_{l=0}^{L} (2l+1) a_l(s) \times P_l(1+t/2k^2) + o(s^{-5}) \quad (5)$$

for any t in $0 \leq t < 4m^2 - \delta$, where $L = C\sqrt{s}\,(\ln s)$.

In order to extend this to a complex region, we note that for any $P_l(1+t/2k^2)$ there is a certain neighborhood of the positive real axis $t \geq 0$ defined by the condition that the real part of $P_l(1+t/2k^2)$ is positive-definite. As is shown in the Appendix, the boundary curve of such a domain is given approximately by

$$v \simeq \pm \frac{\pi(\sqrt{s})(\sqrt{u})}{2l} \quad \text{for} \quad l > \sqrt{s}, \quad (6)$$

where $t = u + iv$ and $u > 0$. Thus, if we define the domain D as the intersection of the ellipse (foci $t=0$ and $-4k^2$, semimajor axis $4m^2 + 2k^2 - \delta$) and the parabolic domain determined by

$$|v| \leq \frac{\pi(\sqrt{s})(\sqrt{u})}{2C_1(\sqrt{s})\ln s} = \frac{\pi\sqrt{u}}{2C_1 \ln s}, \quad (7)$$

where C_1 can be chosen as $16(m/\delta)$, the dominance of the first term in (5) holds also in the extended domain D since the real parts of the P_l in the sum (5) are all positive in D. Consequently, Re $A(s,t)$ is also positive in D. It is important to note that the "height" of the domain D is of order $1/\ln s$ for any finite u in the interval $0 < u < 4m^2$.

We have thus seen that $\mathrm{Re}A(s,t)$ is harmonic and positive in the domain D. This leads us to some useful information about the high-energy behavior of $A(s,t)$. Let us choose t_0 satisfying $0 < t_0 < 4m^2 - \delta$ and consider the disk defined by $|t - t_0| < \pi(\sqrt{t_0})/2C_1 \ln s$, which is contained in the domain D. Then, according to Harnack's theorem,[8] for any t in the smaller disk

$$|t - t_0| < \frac{\pi r \sqrt{t_0}}{2C_1 \ln s}, \quad 0 < r < 1, \quad (8)$$

$\mathrm{Re}A(s,t)$ is bounded from above and below as

$$\frac{1-r}{1+r} A(s,t_0) < \mathrm{Re}A(s,t) < \frac{1+r}{1-r} A(s,t_0). \quad (9)$$

Since the radius of the circle is of order $1/\ln s$, this inequality means that $\mathrm{Re}A(s,t)$ cannot increase by more than a finite factor when t increases by $1/\ln s$. This is just what we expect if $A(s,t)$ is of the Regge form

$$\sum_n \beta_n(t) s^{\alpha_n(t)}. \quad (10)$$

Of course we have not proved that $A(s,t)$ can in fact be written in the form (10). But our result shows that $A(s,t)$ cannot behave more wildly than is expected of the Regge-pole terms at least in the interval $0 < t < 4m^2$. Applying Cauchy's inequality in the disk (8), we can also show that

$$\frac{d}{dt} \ln A(s,t) \leq C' \ln s \quad (11)$$

for $0 < t < 4m^2 - \delta$, which is another way of confirming the Regge behavior.

Unfortunately, the above argument does not give us any exciting result at $t=0$. The only thing we can prove is that $\mathrm{Re}A(s,t)$ is positive in a much smaller disk

$$|t| < C/(\ln s)^2, \quad (12)$$

which means that

$$\frac{d}{dt} \ln A(s,t)\,|_{t=0} < C''(\ln s)^2, \quad (13)$$

as is easily seen by making use of (A8).[9] This result is closely related to that of Bessis,[10] in which he shows that $A(s,t)$ has no zero at $t=0$ inside the circle of radius $(\ln s)^{-2}$. Obviously, if one could prove that $\mathrm{Re}A(s,t)$ is positive in a larger circle, for instance $|t| < C/\ln s$, one would immediately improve the Froissart bound, bounds on the slope of $A(s,t)$ at $t=0$, etc.

Finally, we note that the domain D constructed above is not the largest connected domain with the property that $\mathrm{Re}A(s,t)$ is positive. It is quite possible that there exists a domain D' which is larger than D. However, if the "height" of D' is larger than $(\ln s)^{-1+\epsilon}$, $\epsilon > 0$, in $0 \leq t < 4m^2$, then $A(s,t)$ has to grow more slowly than $s^{\alpha(t)}$ according to Harnack's theorem. This would be embarrassing if $A(s,t)$ should have a genuine Regge behavior.

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[5] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1375 (1964). See also T. Kinoshita, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1965), Vol. VIIB, p. 144.

[6] A. Martin, Nuovo Cimento **29**, 993 (1963).

[7] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[8] C. Carathéodory, *Theory of Functions* (Chelsea Publishing Company, New York, 1954), p. 153.

[9] T. Kinoshita (Ref. 5).

[10] J. D. Bessis, Nuovo Cimento (to be published).

APPENDIX. DERIVATION OF THE FORMULA (6)

In order to derive (6), we will express the Legendre polynomial $P_l(z)$ in the product form

$$P_l(z)=\prod_{\nu=1}^{l}((z-z_\nu)/(1-z_\nu)), \qquad \text{(A1)}$$

where $z_\nu=\cos\theta_\nu$ are the zeros of $P_l(z)$. As is well known, all zeros of $P_l(z)$ lie between -1 and 1. More precisely, they satisfy the inequality[11]

$$(\nu-\tfrac{1}{2})\frac{\pi}{l}<\theta_\nu<\nu\pi/(l+1), \quad \nu=1,2,\cdots,[\tfrac{1}{2}l], \qquad \text{(A2)}$$

where $[\frac{1}{2}l]=\frac{1}{2}l$ if l is even and $=\frac{1}{2}(l-1)$ if l is odd. θ_ν for $\nu>[\frac{1}{2}l]$ is determined by $\theta_\nu=\pi-\theta_{l+1-\nu}$. Let φ be the phase of $P_l(z)$ for $z=1+a+ib$. Then we have

$$\begin{aligned}\varphi&\equiv\arg P_l(z)\\&=\sum_{\nu=1}^{l}\arg(z-z_\nu) \qquad \text{(A3)}\\&=\sum_{\nu=1}^{l}\tan^{-1}\left(\frac{b}{1+a-\cos\theta_\nu}\right).\end{aligned}$$

We are interested in the boundary curve of the connected neighborhood of the semiaxis $z\geq 1$ in which $\mathrm{Re}P_l(z)$ is positive. As z moves away from the semiaxis $z\geq 1$, $\mathrm{Re}P_l(z)$ vanishes for the first time when $\varphi=\frac{1}{2}\pi$. Thus this curve can be defined by

$$\sum_{\nu=1}^{l}\tan^{-1}\left(\frac{b}{1+a-\cos\theta_\nu}\right)=\tfrac{1}{2}\pi. \qquad \text{(A4)}$$

This curve crosses the real axis vertically at $a=-1+\cos\theta_1=O(l^{-2})$. It is easy to see from (A4) that it crosses the vertical line $z=1+ib$ at the height $b\approx l^{-2}$.

Next we consider the region of positive a. For very large a the curve determined by (A4) behaves as $\tan^{-1}(b/a)\simeq\pi/2l$, as is seen from the fact that $P_l(z)\propto z^l$ for large $|z|$. Since we are primarily interested in large l, it will be sufficient for our purpose to assume that $b\ll a$. Then φ can be estimated accurately as follows:

$$\begin{aligned}\varphi&\simeq\frac{l}{\pi}\int_{\theta_1}^{\pi-\theta_1}d\theta\tan^{-1}\left(\frac{b}{1+a-\cos\theta}\right)\\&\simeq\frac{lb}{\pi}\int_{\theta_1}^{\pi-\theta_1}\frac{d\theta}{1+a-\cos\theta}\\&\simeq\frac{2lb}{\pi(2a+a^2)^{1/2}}\left[\tan^{-1}\left(\frac{2}{\theta_1}\sqrt{\left(\frac{2+a}{a}\right)}\right)-\tan^{-1}\left(\frac{\theta_1}{2}\sqrt{\left(\frac{2+a}{a}\right)}\right)\right]. \qquad \text{(A5)}\end{aligned}$$

From (A4) and (A5), the boundary curve of the domain where $\mathrm{Re}P_l(z)>0$ is given by

$$\frac{2lb}{\pi(2a+a^2)^{1/2}}\left[\tan^{-1}\left(\frac{2}{\theta_1}\sqrt{\left(\frac{2+a}{a}\right)}\right)-\tan^{-1}\left(\frac{\theta_1}{2}\sqrt{\left(\frac{2+a}{a}\right)}\right)\right]\simeq\tfrac{1}{2}\pi, \qquad \text{(A6)}$$

where θ_1 satisfies the inequality $\pi/2l<\theta_1<\pi/(l+1)$. This formula can be simplified in the region of interest where a and b are of order $1/k^2$. Putting $a=u/2k^2$ and $b=v/2k^2$ and considering the case of very large k, we find that

$$v\simeq\frac{1}{l}\left(\frac{\pi}{2}\right)^2(\surd s)(\surd u)/\cot^{-1}(\tfrac{1}{2}\theta_1\surd(s/u)).$$

This reduces to

$$v\simeq\pi(\surd s)(\surd u)/2l \qquad \text{(A7)}$$

when $l\gg\surd(s/u)$. For smaller values of l, the right-hand side of (A7) is multiplied by a factor greater than one. The formula (A7) becomes inaccurate in the neighborhood of $u=0$. This can be corrected approximately by shifting u by the amount $2k^2(1-\cos\theta_1)$. Thus we have

$$v\simeq(\pi/2l)(su+\xi s^2/l^2)^{1/2} \quad \text{for} \quad l\gg\surd(s/u), \qquad \text{(A8)}$$

where ξ is of order 1. The formula (A8) gives the boundary curve located in the upper half plane. The part of the boundary curve in the lower half plane is the mirror image of (A8) with respect to the real axis.

[11] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 118.