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Radius of the Nucleon in a Bound-State Model

KASHYAP V. VASAVADA*
Laboratory for Theoretical Studies, National Aeronautics and Space Administration, Goddard Space Flight Center, Greenbelt, Maryland

AND

Y. S. KIM†
Department of Physics and Astronomy, University of Maryland, College Park, Maryland

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We estimate the nucleon radius from the pion-nucleon phase shifts, assuming that the nucleon is a bound state of the pion and the nucleon. The calculation is based on the one-pole approximation of the Gelfand-Levitan formalism on the inverse scattering problem. The numerical result of 0.15–0.50 F is obtained for the radius. In this connection, we discuss the properties of the phase shift and also of the D function for the appropriate channel.

I. INTRODUCTION

In previous papers[1,2] we have studied bound-state problems in the S-matrix theory and have pointed out that it is essential to maintain the locality (normalizability) of the bound-state wave function. Furthermore, we have discussed a method (based on the Gelfand-Levitan formalism) of constructing an approximate bound-state wave function from the S-matrix parameters. It was indicated that crude one-pole approximation will give a reasonably accurate wave function.

In this paper we use this method to estimate the nucleon radius, assuming that the nucleon is a p-wave bound state of the pion and the nucleon in the sense of the Schrödinger theory. We first construct a localized (normalizable) bound-state wave function and then obtain the expectation value of the radius. We realize that this nonrelativistic picture is not adequate for the pion-nucleon problem. However,[3] being the only method available in this direction, the present treatment will certainly give an insight into the structure of the "bound-state" nucleon.

In order to handle the p-wave problem, we first generalize the s-wave formalism of Ref. 2 to the general partial-wave case. It is shown that, in this general case also, an approximate wave function can be constructed from the three bound-state parameters, namely, the binding energy, the residue of the bound-state pole, and the first derivative of the D function at the binding energy.

The residue and the binding energy in the pion-nucleon system are well-known parameters. However, in order to determine the derivative of the D function which is normalized to be unity at infinite energy, it is necessary to know the phase shift for all energies. In this paper, we use the known phase shifts up to the highest energy where experimental data are available. Beyond this energy we use various extrapolations. It is shown that the expectation value of the radius is not too sensitive to the type of extrapolations and the present method gives an acceptable numerical value for the nucleon radius.

II. FORMULATION OF THE PROBLEM

In one of the previous papers[2] we have discussed an approximation method of constructing a bound-state wave function for the s-wave case. Although the generalization to higher partial waves is straightforward, we discuss this problem here for completeness. For the higher partial-wave case, one has to be careful about the phase factors of wave functions and other quantities associated with the normalization condition

$$\lim_{|x|\to\infty} D_l(x)=1. \tag{1}$$

We then discuss the evaluation of the parameters in the approximate wave function from the experimental data.

Let us first write the lth partial-wave radial wave function as

$$\phi_l(r)=j_l(kr)+\frac{1}{k}\int_0^r K_l(r,r')j_l(kr')dr', \tag{2}$$

where the spherical Bessel function $j_l(kr)$ is defined in Ref. 1 and differs from that of the conventional definition by a factor of (kr). Then according to the generalization of the Gelfand-Levitan theorem, the $K_l(r,r')$ function is obtained from the integral equation

$$K_l(r,r')+g_l(r,r')+\int_0^r K_l(r,t)g_l(r't)dt=0,$$

* National Academy of Sciences—National Research Council Resident Research Associate. Present address: Department of Physics, University of Connecticut, Storrs, Connecticut.

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[1] Y. S. Kim, Phys. Rev. **142**, 1150 (1966).

[2] Y. S. Kim and K. V. Vasavada, Phys. Rev. **150**, 1236 (1966).

[3] In fact it has been recently shown in the framework of relativistic dispersion relations that because of the large amount of inelasticity in the $I=\frac{1}{2}$, $J=\frac{1}{2}$ p wave of the pion-nucleon system, it is rather unlikely that the pion-nucleon two-body channel *by itself* can support the nucleon bound state, and it seems that other channels should play an important role in binding the nucleon [P. Nath and K. V. Vasavada, Phys. Rev. **152**, 1254 (1966).] However, as a crude first approximation we have considered here a single-channel bound-state model with only the elastic contribution.

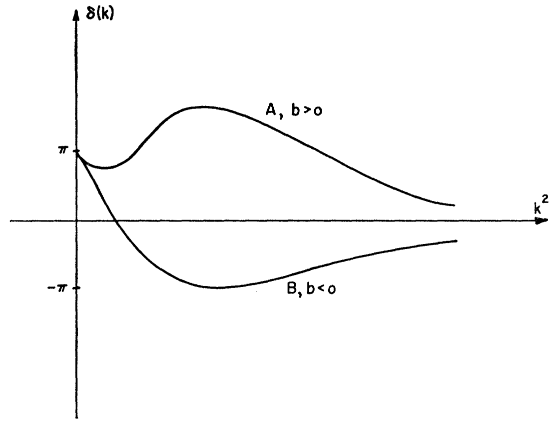

FIG. 1. Possible behaviors of the phase shift $\delta(k)$. For curve A, the parameter b is positive; for B, b can be negative. The $I=\frac{1}{2}$, $J=\frac{1}{2}$ ($l=1$) pion-nucleon phase shift behaves (qualitatively) like curve A.

where the $g_l(r,r')$ function is constructed from the spectral function

$$\begin{aligned} d\rho_l(x)/dx &= C_l\delta(x+x_0), \qquad x<0 \\ &= \frac{k}{\pi}[|D_l(x)|^{-2}-1], \quad x>0 \end{aligned} \tag{3}$$

in the following way:

$$g_l(s,t)=\int_{-\infty}^{\infty}\left(\frac{j_l(ks)}{k}\right)\left(\frac{j_l(kt)}{k}\right)d\rho_l(x).$$

We are using x and k for the energy and momentum variables, respectively. Here again it is assumed that there is only one bound state at $x=-x_0$. The constant C_l is related to the bound-state wave function by

$$\left(\int_0^{\infty}|\phi_l(r)|^2dr\right)\bigg|_{x=-x_0}=x_0/C_l. \tag{4}$$

Then, in the plane-wave approximation for the continuous spectrum we have the approximate solution

$$\phi_l(r)=j_l(i\alpha r)\bigg/\left[1+\frac{C_l}{x_0}\int_0^r|j_l(i\alpha t)|^2dt\right] \tag{5}$$

for the bound-state momentum $k=i\alpha$. This approximate solution satisfies the normalization condition of Eq. (4).

By taking the expectation value using the above wave function, we obtain the following expression for the radius:

$$\langle r\rangle=\int_0^{\infty}dr\left[1+\frac{C_l}{x_0}\int_0^r|j_l(i\alpha t)|^2dt\right]^{-1}. \tag{6}$$

We conjecture here that a more accurate expression would be one wherein the spherical Bessel function $j_l(i\alpha t)$ is replaced by a more complicated function having similar properties at both small and large t. The presence of this Bessel function is certainly a characteristic of the plane-wave approximation for the continuous spectrum.

The constant C_l can be determined by a trivial generalization of the s-wave case outlined in Ref. 2. But, since there is a complication in introducing phase factors associated with the normalization condition of Eq. (1), we have discussed this problem in considerable detail in the Appendix. According to Eq. (A8),

$$1/C_l=-(-1)^lR_l[D_l'(-x_0)]^2, \tag{7}$$

where R_l is the residue of the bound-state pole.

Let us now return to the expression for $\langle r\rangle$ in Eq. (6). The radius $\langle r\rangle$ is completely determined by the constant C_l. In the limit $C_l\to 0$, we obtain a large value for $\langle r\rangle$. In the other limit, $C_l\to\infty$, we obtain a very small value. According to Eq. (7), the constant C_l is determined from the derivative of the D_l function while the residue R_l and the binding energy x_0 are regarded as the well-known input parameters. In order to evaluate this derivative we use the following form for the D_l function:

$$D_l(k^2)=\frac{k^2+\alpha^2}{k^2}\exp\left[-\frac{1}{\pi}\int_0^{\infty}\frac{\delta_l(k')dk'^2}{k'^2-k^2-i\epsilon}\right], \tag{8}$$

where the phase shift $\delta_l(k)$ satisfies the boundary conditions

$$\delta_l(0)=\pi, \quad \delta_l(\infty)=0,$$

which are consistent with Levinson's theorem and the normalization condition

$$\lim_{|k^2|\to\infty}D_l(k^2)=1.$$

The above $D_l(k^2)$ vanishes at the binding energy $k^2=-\alpha^2$.

In actual calculations it is more convenient to regard the radius as a function of the parameter b_l defined as

$$\begin{aligned} b_l &\equiv \frac{1}{\pi}\int_0^{\infty}\frac{\delta_l(k)dk^2}{k^2+\alpha^2} \\ &= -\ln[-\alpha^2D'(-\alpha^2)]. \end{aligned} \tag{9}$$

This quantity of course depends on the behavior of the phase shift. If the phase shift behaves (qualitatively) like curve A of Fig. 1, then b_l will be positive. If, on the other hand, it behaves like curve B, then b_l will be negative. In either case, b_l is assumed to be finite. We shall discuss this and other numerical points for the pion-nucleon problem and estimate the nucleon radius in the following section.

III. APPLICATION TO THE PION-NUCLEON SYSTEM

In this section we use the formalism of Sec. II to estimate the nucleon radius, assuming that the nucleon is a bound state in the $I=\frac{1}{2}$, $J=\frac{1}{2}$ $(l=1)$ channel of pion-nucleon scattering. We first plot the curve for the radius $\langle r\rangle$ as a function of the parameter b and then estimate the numerical value of $\langle r\rangle$ by evaluating this parameter from the known phase shifts.

Since we are concerned here only with the p-wave amplitude, we drop the partial-wave index l. Let us first determine the residue. By retaining only the pole term in the p-wave amplitude, we have

$$A=(1/k)e^{i\delta}\sin\delta=R/(k^2+\alpha^2)+\cdots, \tag{10}$$

and by retaining only the amplitude corresponding to the direct one-nucleon diagram,

$$\begin{aligned} A&=(1/k)e^{i\delta}\sin\delta \\ &=(3f^2/m)1/(k^2+\alpha^2)+\cdots, \end{aligned} \tag{11}$$

where $f^2=0.08$. Here m is the nucleon mass, and k is the center-of-mass momentum for the π-N system. α^2 in this case is μ^2, where μ is the pion mass. By comparing Eqs. (10) and (11) we obtain

$$R=3f^2/m. \tag{12}$$

Using the numerical value of R, one can now calculate $\langle r\rangle$ for various values of the parameter b. We have plotted $\langle r\rangle$ against this parameter in Fig. 2. According to Eqs. (6), (7), and (9), and as is indicated in Fig. 2,

$$\begin{aligned} &\lim_{b\to\infty}\langle r\rangle=0\,, \\ &\lim_{b\to-\infty}\langle r\rangle=\infty\,. \end{aligned} \tag{13}$$

It is assumed that the parameter b is finite and therefore that the radius $\langle r\rangle$ is a nonzero finite number. In order to determine this parameter, one has to know the phase shifts for all energies. We use here the known phase shifts up to the pion lab energy 700 MeV, where experimental data are available.[4] Beyond this energy we use the following three different extrapolations:

$$\begin{aligned} \delta(k)&=A_1/k^2\,, \\ \delta(k)&=A_2/k\,, \\ \delta(k)&=A_3e^{-k2}\,, \end{aligned} \tag{14}$$

where the constants A_1, A_2, and A_3 are adjusted in such a way that $\delta(k)$ will join the experimental curve smoothly. We realize that the phase shift takes on complex values above the production threshold. For this region, only the real part is used. The phase shift then behaves qualitatively like curve A of Fig. 1.

For all three of the above extrapolations we obtain

$$\langle r\rangle=0.15\text{–}0.50\text{ F}.$$

The above numerical result is an acceptable value for the nucleon radius. Although the present analysis was based on a crude one-pole approximation, this result is quite encouraging and may perhaps be the key to a more satisfactory treatment.

We have noted earlier that the radius is determined by $D'(-\alpha^2)$ [or equivalently by b]. Conversely, the radius $\langle r\rangle$ determines the absolute value of $D'(-\mu^2)$. If the radius $\langle r\rangle$ is to be in the range of the above acceptable values, then $|\mu^2D'(-\mu^2)|$ must be in the range 0.003–0.05.

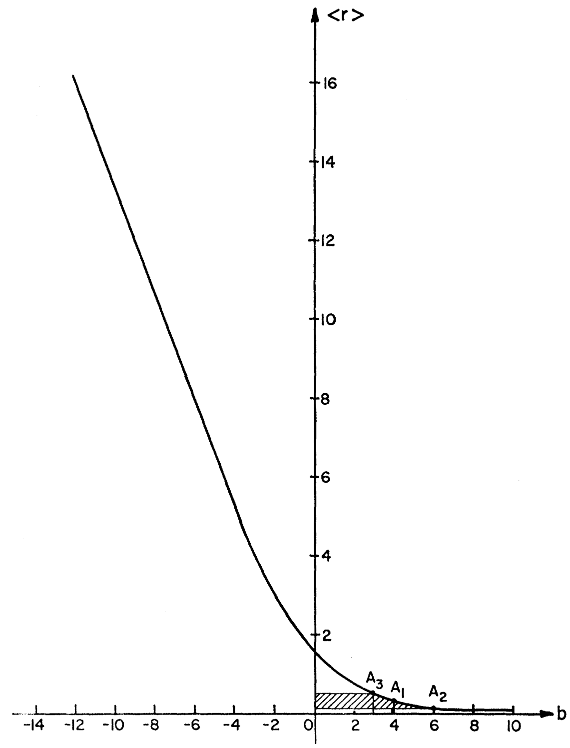

FIG. 2. The radius $\langle r\rangle$ (measured in fermis) as a function of the parameter b. According to the present phase-shift analysis, b is such that the radius $\langle r\rangle$ ranges over 0.15 to 0.50 F.

IV. CONCLUDING REMARKS

The bound-state model for the nucleon is not new. Probably the most fruitful subject in this school of thought has been the bootstrap. However, the bootstrap theory primarily deals with the location and the residue of the bound-state poles but not directly with the structure of the bound-state particle. In this paper we have attempted to discuss the spatial extension of this bound-state particle and estimated its size purely in the framework of the two-body scattering formalism. It should be noted that the present method does not involve any arbitrary parameters. Our numerical result is encouraging, and the present analysis gives a good insight into the bound-state nature of the nucleon.

For the historical development of the Gelfand-Levitan formalism, it is quite fortunate that the residue R is

[4] L. D. Roper and R. M. Wright, Phys. Rev. 138, B921 (1965); L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965); P. Auvil, A. Donnachie, A. Lea, and C. Lovelace, Phys. Letters 12, 76 (1964).

now being regarded as a known input parameter, whereas it was an undeterminable quantity in the early stage of application (thus generating phase-equivalent potentials).[5] It is also gratifying that the computational facilities are available today for practical applications, such as the one discussed here. These important factors may reshape the role of the Gelfand-Levitan formalism in scattering theory.

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APPENDIX

In order to determine the constant C_l consistent with the normalization condition of Eq. (1), we start with the following form of the regular solution:

$$\phi_l(r)=j_l(kr)+\frac{1}{k}\int_0^r[n_l(kr)j_l(kr')-j_l(kr)n_l(kr')]\times V(r')\phi_l(r')dr', \quad \text{(A1)}$$

where the spherical Bessel functions $j_l(kr)$ and $n_l(kr)$ are defined in Ref. 1. This solution can be written as a linear combination of the Jost solutions.

$$\phi_l(r)=\frac{1}{2i}e^{i\pi l/2}[f_l(k)f_l(-k,r)-f_l(-k)f_l(k,r)]. \quad \text{(A2)}$$

The phase factor $e^{i\pi l/2}$ is introduced to guarantee the desired normalization condition for the Jost functions $f_l(k)$ and $f_l(-k)$. The Jost solutions $f_l(\mp k, r)$ satisfy the following boundary conditions

$$\lim_{r\to\infty} f_l(\mp k, r)=\lim_{r\to\infty} e^{\pm i\pi(l+1)/2}h^{(1),(2)}(kr)=e^{\pm ikr}, \quad \text{(A3)}$$

where the spherical Hankel functions are defined as

$$h^{(1),(2)}(kr)=[j_l(kr)\pm in_l(kr)] \quad (+ \text{ for } h^{(1)}, - \text{ for } h^{(2)}).$$

The partial-wave S matrix is constructed from the Jost functions $f_l(\pm k)$ by

$$S_l(k)=[f_l(k)/f_l(-k)]e^{i\pi l}. \quad \text{(A4)}$$

Now by taking the Wronskians of $\phi_l(r)$ of Eq. (A1) with respect to $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$, we obtain

$$\begin{aligned} f_l(-k)&=1-\frac{1}{ik}\int_0^\infty h_l^{(1)}(kr)V(r)\phi_l(r)dr, \\ f_l(k)e^{i\pi l}&=1+\frac{1}{ik}\int_0^\infty h_l^{(2)}(kr)V(r)\phi_l(r)dr. \end{aligned} \quad \text{(A5)}$$

Then it is clear that the above Jost functions satisfy the normalization conditions

$$\begin{aligned} \lim_{k\to\infty} f_l(-k)&=1, \\ \lim_{k\to\infty} f_l(k)e^{i\pi l}&=1. \end{aligned} \quad \text{(A6)}$$

Next, by following the steps outlined by De Alfaro and Regge,[6] and by using the expression of Eq. (A2) for the regular solution, we derive

$$\int_0^\infty \phi_l^2(r)dr=\tfrac{1}{4}ie^{i\pi l}\left[f_l(k)\frac{df_l(-k)}{dk}\right]\Bigg|_{k=i\alpha}. \quad \text{(A7)}$$

From the behavior near the origin of $\phi_l(r)$,

$$\int_0^\infty |\phi_l(r)|^2dr=(-1)^{l+1}\int_0^\infty \phi_l^2(r)dr,$$

and from Eq. (A4) for the S matrix,

$$f_l(i\alpha)=-2\alpha R_l D_l'(-\alpha^2)e^{-i\pi l},$$

where

$$D_l'(-\alpha^2)=\left(\frac{1}{2i\alpha}\right)\frac{df_l(-k)}{dk}\Bigg|_{k=i\alpha}.$$

From the preceding three equations, we obtain

$$\begin{aligned} 1/C_l&=\frac{1}{\alpha^2}\left[\int_0^\infty |\phi_l(r)|^2dr\right]_{k=i\alpha} \\ &=-(-1)^l R_l[D_l'(-x_0)]^2. \end{aligned} \quad \text{(A8)}$$

[5] For historical development of the Gelfand-Levitan theory, see L. D. Faddeev (translated by B. Seckler), New York University Report No. EM-165, 1960 (unpublished).

[6] V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Company, Amsterdam, 1965).