the constant A_d could be varied considerably without any significant modification of $R_\pi(p_\pi)$. However, on the other hand, a change in A_c by a factor of $\frac{4}{3}$ would already produce too large a first peak of $R_\pi(p_\pi)$, due exclusively to the Σ-Λ–conversion amplitude.

By comparing our result of Fig. 2 with the curve (b) of Fig. 7 of I, we observe that, indeed, the assumption of $2P$ capture gives a better semiquantitative fit to the data[7] than does the assumption of the $1S$ capture. The now very pronounced first peak corresponds to the Λ-Σ threshold which occurs at $p_\pi \cong 164$ MeV/c (cf. the pygmy peak of Fig. 7 of I).

In summing up the discussion of both Figs. 1 and 2, we may say that while the rough agreement with the data[4] is about the same for the $2P$ capture as for the $1S$ capture of I, the $R_\pi(p_\pi)$ pion data appear to definitely favor the former case.[9]

[9] Actually, the smallness of the first peak of the curve $1S$ of Fig. 2 is a little exaggerated by the interference terms calculated in I with an inappropriate choice of one of the relative phases.

Obviously, one should regard our results with caution because of the crudeness of the treatment of the Σ-Λ–conversion amplitude by the two-component distorted-wave approximation as in I.

Our numerical results correspond to a simple but completely arbitrary choice of the phase of the conversion amplitude relative to the rest. In view of the inevitable smallness of the corresponding interference terms, only a very small improvement of our $R_3(p_3)$ could be achieved by any required shift of this phase, while $R_\pi(p_\pi)$ would suffer practically no change.

Many more data, particularly on angular distributions and on the absolute values of the reaction rates, should be avilable before a more detailed analysis becomes warranted.

The author wishes to acknowledge the invaluable help of Melle M. T. Commeault with the numerical computations and several useful conversations with P. Said.

PHYSICAL REVIEW VOLUME 152, NUMBER 4 23 DECEMBER 1966

Uncoupled-Phase Method with Many Perturbing Channels

R. E. KREPS AND P. NATH
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania
(Received 31 May 1966)

The uncoupled-phase method is a nonperturbative formalism which describes the influence of any given (nth) channel on the dynamics of an n-channel scattering reaction. The method relates scattering amplitudes for the uncoupled reaction, obtained by switching off interactions to the nth channel while the interactions among the rest remain unchanged, to the scattering amplitudes describing the full reaction. We extend the uncoupled-phase method further, under exactly the same assumptions used to derive the previous uncoupled phase relations. We remove the restriction that there is only one perturbing channel and allow for the possibility of an arbitrary number of perturbing channels. The more general set of uncoupled-phase relations reduces to the previous uncoupled-phase relations when the number of perturbing channels is equal to 1. Some elementary applications of these relations is made, and their possible application in elementary particle reactions is indicated.

I. INTRODUCTION

ONE of the crucial facts about high-energy scattering is the large number of channels that become available for scattering, and a correct description of any one of them involves all the channels that are significantly coupled at the relevant energy. In many important physical situations it may be sufficient to consider only the coupled two-body channels (some of which may be closed in the energy region of interest). For example, in the meson-baryon scattering, $(\pi\Lambda,\pi\Sigma,\bar{K}N,\eta\Sigma,K\Xi)$ may be coupled significantly near the energy region of the Y_1^* resonance, and $(\pi\Xi,\bar{K}\Lambda,\bar{K}\Sigma,\eta\Xi)$ near the region of the $\Xi_{1/2}^*$ resonance. However, even though such systems can be handled by matrix N/D dispersion relations, the calculations are generally involved and in practice most calculations ignore all except the nearest channel. On the other hand, some channels may not be significantly important and could be safely neglected, but one needs some semiquantitative criterion in ignoring them.

The uncoupled-phase method (UPM) developed by Ross and Shaw[1,2] relates the actual amplitudes describing n coupled channels to the "uncoupled" amplitudes which describe the scattering, when the couplings to the nth channel are switched off, the other interactions remaining unchanged. Although the UPM was originally developed in the framework of a potential model,[1–4] it has subsequently been extended to relativistic N/D matrix calculations.[5] The usual weak-coupling approxi-

[1] M. Ross and G. L. Shaw, Ann. Phys. (N. Y.) **9**, 391 (1960).
[2] G. L. Shaw and M. Ross, Phys. Rev. **126**, 806 (1962).
[3] P. Nath, G. L. Shaw, and C. K. Iddings, Phys. Rev. **133**, B1085 (1964).
[4] Interactions with hard cores were also investigated.
[5] P. Nath and G. Shaw, Phys. Rev. **137**, B711 (1965).

mations which include the neglected channel as a perturbation, appear as limiting cases of the UPM.[6] The UPM was tested numerically by comparison with the exact solutions and was found to have a large range of validity both in the nonrelativistic and the relativistic situations under significant modifications of the uncoupled phases.[2,3,5,7] Several applications of the UPM have been made with semiquantitative success.[8–11]

In the present paper, we develop the UPM further by removing the restriction that there is only one perturbing channel. Such an extension of the UPM will be of importance to handle situations where more than one perturbing channel influences the scattering reaction significantly. Such a situation cannot realistically be approximated by a single artificial channel. This is particularly true if the energy region of interest lies close to or between the thresholds of the perturbing channels, especially when there are resonances present. Another natural application of the extended UPM would be in context of peripheral scattering.

In Sec. II we review the salient points of the UPM. In Sec. III we derive the extended form of the UPM under exactly the same set of assumptions used to derive the restricted uncoupled phase relations. The extended UPM, therefore, is expected to have the same range of validity as the restricted UPM. In Sec. IV we discuss some elementary applications of the uncoupled phase relations. Section V is devoted to discussion and conclusions.

II. REVIEW OF THE UNCOUPLED-PHASE METHOD

Before we derive the uncoupled phase method for the case of many perturbing channels, it would perhaps be desirable to review in brief the uncoupled phase method for the case of one decoupled channel.[5] Let us consider n coupled two-body channels and define the invariant partial-wave amplitude A by

$$A\equiv ND^{-1}\equiv\rho^{-1/2}[(S-1)/2i]\rho^{-1/2}. \tag{2.1}$$

The diagonal matrix ρ depends on kinematics

$$\rho_{ij}=\delta_{ij}k_i^{2l+1}/\sqrt{s}, \tag{2.2}$$

where s is the square of the total energy and k_i is the relative momentum in the ith channel in the overall center-of-mass system, and N and D have their usual meaning. The reaction amplitude K is defined by

$$K=N(\mathrm{Re}D)^{-1}=A(1-i\rho A)^{-1}. \tag{2.3}$$

Under approximations of the uncoupled-phase method,[12] K satisfies the $n\times n$ matrix equation for a given partial wave

$$K(s)=B(s)+B(s)L^{-1}(s)K(s), \tag{2.4}$$

where $B(s)$ is the left-hand cut of A, i.e.,

$$B(s)=\frac{1}{\pi}\int_{\mathrm{Left}}\frac{\mathrm{Im}A(s')ds'}{(s'-s)}. \tag{2.5}$$

$L(s)$ is a diagonal matrix, the element L_k being related to the range of the forces in the kth channel:

$$L_k(s)=\left(\frac{\sqrt{s}}{\pi}\frac{1}{B_{kk}^2(s)}P\int_{s_k}^{\infty}\frac{B_{kk}^2(s')\rho_k(s')ds'}{(s'-s)}\right)^{-1}, \tag{2.6}$$

where s_k is the production threshold for the kth channel. The parameter L_k depends only on the interaction in the kth channel.[5]

When the couplings to the nth channel are switched off, the uncoupled $\tilde{K}$ matrix elements $\tilde{K}_{ij}$ are given by

$$\tilde{K}_{ij}=(B_{ij}+\sum_{k=1}^{n-1}B_{ik}L_k^{-1}\tilde{K}_{kj})(1-\delta_{in})(1-\delta_{jn}). \tag{2.7}$$

From (2.4) and (2.7) one gets

$$\sum_{k=1}^{n}(\delta_{ik}-B_{ik}L_k^{-1})\times[K_{kj}-\tilde{K}_{kj}(1-\delta_{jn})(1-\delta_{kn})+L_n\delta_{kn}\delta_{jn}]=0. \tag{2.8}$$

In order that solutions to (2.8) exist, the 2nd bracket in (2.8) must satisfy $(n^2-n)/2$ independent relations. There are $(n-1)$ relations for $i\neq n$,

$$\det\begin{pmatrix}K_{ii}-\tilde{K}_{ii} & K_{in}\\ K_{in} & K_{nn}+L_n\end{pmatrix}=0 \tag{2.9a}$$

and $(n-1)(n-2)/2$ relations for $i\neq n$, $j\neq n$,

$$\det\begin{pmatrix}K_{ii}-\tilde{K}_{ii} & K_{ij}-\tilde{K}_{ij}\\ K_{ij}-\tilde{K}_{ij} & K_{jj}-\tilde{K}_{jj}\end{pmatrix}=0. \tag{2.9b}$$

The number of uncoupled phase relations is equal to the number of the uncoupled amplitudes.

III. DERIVATION OF THE UNCOUPLED PHASE METHOD WITH MANY PERTURBING CHANNELS

We would now like to present a simple derivation of the uncoupled phase relations that result when the restriction that there is only one perturbing channel is removed. To be specific we shall consider that there are

[6] The weak coupling approximations usually employed are (1) $K\cong\tilde{K}$, where K and $\tilde{K}$ are perturbed and the unperturbed reaction amplitudes (2) $M\cong\tilde{M}$, where M and $\tilde{M}$ are essentially the inverse of the K and $\tilde{K}$ amplitudes. For details see Ref. 3.

[7] Even for cases where the influence of nth channel was large enough to cause bound states, the UPM reproduced the actual phases with an error of less than 15%.

[8] M. Ross and G. Shaw, Phys. Rev. **115**, 1773 (1959); Bull. Am. Phys. Soc. **5**, 504 (1960).

[9] G. Shaw and M. Ross, Phys. Rev. **126**, 814 (1962).

[10] P. Nath and Y. N. Srivastava, Phys. Rev. **138**, B404 (1965).

[11] M. DerSarkissian and P. Nath, Nuovo Cimento **38**, 1355 (1965); M. DerSarkissian (to be published).

[12] For a detailed discussion of the approximations and their numerical verification see Ref. 5.

n coupled channels of which m are finally decoupled. The $n\times n$ reaction matrix K satisfies the relation (2.4)

$$K=B+BL^{-1}K. \tag{3.1}$$

Let us now decouple m channels [by setting $B_{i\alpha}=0$, where $i=1\cdots(n-m)$ and $\alpha=(n-m+1)\cdots n$, while the rest of the interactions B_{ij} $(i, j=1\cdots(n-m))$ remain unchanged]. The reaction matrix $\tilde{K}$ which describes scattering for the remaining $(n-m)$ coupled channels satisfies the relation

$$\tilde{K}=\{PBP\}+\{PBP\}L^{-1}\tilde{K}. \tag{3.2}$$

P is the diagonal projection matrix onto the space of coupled channels whose first $(n-m)$ diagonal elements are equal to 1, the rest of the elements being zero. In order to facilitate further discussion, it is useful to define a matrix $\mathcal{K}$ by

$$\mathcal{K}=B+BPL^{-1}\rho\mathcal{K}. \tag{3.3}$$

We use the condition $P^2=P$ and Eqs. (3.2) and (3.3) to obtain

$$\tilde{K}=P\mathcal{K}P. \tag{3.4}$$

From (3.3) and (3.1) we can solve for $\mathcal{K}$ in terms of K:

$$\mathcal{K}=[I+KL^{-1}(1-P)]^{-1}K. \tag{3.5}$$

It is profitable to express K and L in the following fashion:

$$K=\begin{pmatrix}K_{CC} & K_{CD}\\ K_{DC} & K_{DD}\end{pmatrix},\quad L=\begin{pmatrix}L_C & 0\\ 0 & L_D\end{pmatrix} \tag{3.6}$$

where K_{CC} is a square matrix of dimension $(n-m)$, whose elements are those of PKP; K_{CD} has the elements of $PK(1-P)$, etc.; the subscripts C and D refer to coupled and decoupled channels, respectively.

$$[I+KL^{-1}(1-P)]^{-1}=\begin{pmatrix}I & -K_{CD}[L_D+K_{DD}]^{-1}\\ 0 & [I+K_{DD}L_D^{-1}]^{-1}\end{pmatrix}. \tag{3.7}$$

It is to be noted that the inverse in the elements of (3.7) is to be taken in the m-dimensional space of the decoupled channels. Using (3.5) and (3.7), $\mathcal{K}$ may be written in the form

$$\mathcal{K}=\begin{pmatrix}[K_{CC}-K_{CD}(L_D+K_{DD})^{-1}K_{DC}] & [K_{CD}(L_D+K_{DD})^{-1}L_D]\\ [L_D(L_D+K_{DD})^{-1}K_{DC}] & [L_D(L_D+K_{DD})^{-1}K_{DD}]\end{pmatrix}. \tag{3.8}$$

If we use (3.8) in (3.4) we find that

$$\tilde{K}=K_{CC}-K_{CD}[L_D+K_{DD}]^{-1}K_{DC}. \tag{3.9}$$

Though the uncoupled phase relations expressed by the matrix equation (3.9) have a compact form, it may be more convenient to work with an equivalent set of determinantal relations where matrix inversion as in (3.9) does not occur. Moreover, the equivalent set of relations appear as a natural generalization of the old uncoupled phase relations with one perturbing channel. To obtain the equivalent set of relations we define a matrix M by

$$M=K-\tilde{K}+(1-P)L^{-1}. \tag{3.10}$$

It is easily seen that

$$P(1-BL)M=0. \tag{3.11}$$

After a little manipulation, (3.11) can be recast in the form

$$P[1-t(1-P)]M=0, \tag{3.12}$$

where

$$t=BL(1-PBL)^{-1}. \tag{3.13}$$

Taking matrix elements of Eq. (3.12), we have explicitly

$$M_{ij}=\sum_{k=n-m+1}^{n}t_{ik}M_{kj},\quad i\leqslant n-m,\quad j\leqslant n-m, \tag{3.14}$$

and

$$M_{il}=\sum_{k=n-m+1}^{n}t_{ik}M_{kl},\quad l\geqslant n-m+1. \tag{3.15}$$

Since Eqs. (3.14) and (3.15) express the first column as a linear combination of the last m columns, the following set of determinants of dimension $(m+1)$ must vanish:

$$\det\begin{bmatrix}M_{ij} & M_{n-m+1,j} & \cdots & M_{nj}\\ M_{in-m+1} & \cdots & \cdots & M_{nn-m+1}\\ \vdots & & & \vdots\\ M_{in} & \cdots & \cdots & M_{nn}\end{bmatrix}=0 \quad \text{for} \quad i, j=1\cdots(n-m). \tag{3.16}$$

To display explicitly the relations (3.16) in terms of the coupled and the uncoupled reaction amplitudes, we write for $i, j<(n-m+1)$,[13]

$$\det\begin{bmatrix} (K_{ij}-\tilde{K}_{ij}) & K_{i(n-m+1)} & \cdots & K_{in} \\ K_{(n-m+1)j} & (K_{(n-m+1),(n-m+1)}+L_{(n-m+1)}) & \cdots & K_{(n-m+1),n} \\ \vdots & \vdots & & \vdots \\ K_{nj} & K_{n,(n-m+1)} & \cdots & (K_{nn}+L_n) \end{bmatrix}=0. \tag{3.17}$$

It is also possible to derive more determinantal relations involving several elements of $\tilde{K}$ similar to relations (2.9b). However, the total number of independent relations is exactly $(n-m)(n-m+1)/2$ equal to the number of independent elements of $\tilde{K}$. This is evident from Eq. (3.9) where the $\tilde{K}$ matrix has been explicitly expressed.

We shall consider now some special cases:

(I) $m=1$; This case corresponds to decoupling one channel, say the nth channel We have for $i, j\leqslant(n-1)$

$$\tilde{K}_{ij}=K_{ij}-\frac{K_{in}K_{nj}}{K_{nn}+L_n}. \tag{3.18}$$

These relations can be seen to be equivalent to the uncoupled phase relations (2.9a) and (2.9b).

(II) $m=n-1$; In this case we are decoupling $(n-1)$ channels which we shall regard as labeled 2 to n. We get only one uncoupled-phase relation

$$\det\begin{vmatrix} K_{11}-\tilde{K} & K_{12} & \cdots & K_{1n} \\ K_{21} & (K_{22}+L_2) & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & (K_{nn}+L_n) \end{vmatrix}=0. \tag{3.19}$$

(III) To illustrate the form (3.9) of the uncoupled phase relations we consider the case when there are 5 coupled channels where the channels 4 and 5 are regarded as the perturbing channels for the system of coupled channels 1, 2, and 3. Equation (3.9) becomes

$$\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix}=\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}-\frac{1}{\Delta}\begin{bmatrix} K_{14} & K_{15} \\ K_{24} & K_{25} \\ K_{34} & K_{35} \end{bmatrix}\begin{pmatrix} (K_{55}+L_5) & -K_{45} \\ -K_{54} & (K_{44}+L_4) \end{pmatrix}\begin{pmatrix} K_{41} & K_{42} & K_{43} \\ K_{51} & K_{52} & K_{53} \end{pmatrix}, \tag{3.20}$$

where

$$\Delta=(K_{44}+L_4)(K_{55}+L_5)-K_{45}K_{54}. \tag{3.21}$$

IV. SOME ELEMENTARY APPLICATIONS

Let us consider a situation where the reaction matrix has a pole (as is the case when the system has a bound state or a resonance). Let $\tilde{M}$ be the position of the pole in $\tilde{K}$. We may write

$$\tilde{K}_{ij}=-\frac{\tilde{\Gamma}_i\tilde{\Gamma}_j}{s-\tilde{M}}+\tilde{R}_{ij}. \tag{4.1}$$

Due to the influence of m perturbing channels the pole will shift to some value M so that (for $i, j=1\cdots n$)

$$K_{ij}=-\frac{\Gamma_i\Gamma_j}{s-M}+R_{ij}. \tag{4.2}$$

If we refer back to the uncoupled phase relations (3.9) it is clear that as we approach $s=\tilde{M}$, we have the condition

$$\det\left(\mathfrak{L}_D-\frac{\Gamma_D\otimes\Gamma_D}{\tilde{M}-M}\right)=0, \tag{4.3}$$

where

$$\mathfrak{L}_D=L_D+R_{DD}, \tag{4.4}$$

and $\Gamma_D\otimes\Gamma_D$ is the outer product of a column and a row vector whose elements run from $(n-m+1)$ to n. From (4.3) it is readily seen that the mass shift is given by

$$M=\tilde{M}-\Lambda(\tilde{M}), \tag{4.5}$$

where

$$\Lambda(s)=\sum_{i,j=n-m+1}^{n}\Gamma_i\mathfrak{L}_{ij}^{-1}(s)\Gamma_j. \tag{4.6}$$

If we make a further simplifying assumption and assume that the background term R may be neglected in comparison with L in (4.4), we have

$$\mathfrak{L}_{ij}=L_i\delta_{ij} \tag{4.7}$$

and

$$M=\tilde{M}-\sum_{k=n-m+1}^{n}\frac{\Gamma_k^2}{L_k}. \tag{4.8}$$

[13] These relations are exact if the left-hand cut is replaced by a pole. An exactly similar set of relations can be obtained in the potential model. The uncoupled phase parameters L_i's are related to the range of potentials. See Refs. 2 and 3.

For poles below the perturbing channels, since L_k is then positive, Eq. (4.8) tells us that the bound or resonant state is forced into a tighter and tighter configuration due to the presence of each additional perturbing channel. The result is in conformity with the recent detailed computer experiments involving several coupled channels.[10,14]

In order to investigate the modifications of the residues we examine Eq. (3.9) as we approach $s=\tilde{M}$. It is straightforward then to obtain the relation

$$\tilde{\Gamma}_i=\frac{\Gamma_i-r_i}{[1-\Lambda'(\tilde{M})]^{1/2}} \tag{4.9}$$

where

$$r_i=\sum_{k,l=n-m+1}^{n} R_{ik}\mathfrak{L}_{kl}^{-1}(\tilde{M})\Gamma_l.$$

In the same spirit as before if we neglect the background term R in comparison to L, we get the following interesting result[15]

$$\frac{\delta\Gamma_i}{\Gamma_i}=\frac{\delta\Gamma_j}{\Gamma_j}. \quad i,\ j\leqslant(n-m). \tag{4.10}$$

From Eq. (4.10) we can approximately compute the residue changes in all the channels from a knowledge of the change in residue in one channel due to the influence of a set of perturbing channels.

V. DISCUSSION AND CONCLUSIONS

The uncoupled phase relations obtained previously have been tested numerically both in the potential model as well as in the relativistic case. It was found that they can accommodate large modifications of the uncoupled phases due to the presence of one perturbing channel. Under exactly the same set of assumptions used to derive the previous uncoupled phase relations we obtain a natural generalization, which is expected to have the same range of validity, by removing the restriction that there is only one perturbing channel. In particular, we can study many situations where two or more perturbing channels significantly influence a scattering process. In an elementary application of the relations to the resonance situation, we find the change in pole position and residues, and observe the interesting result that the modifications in resonance widths are approximately proportional to the widths themselves.

As in the case of old uncoupled-phase method, we envision two main uses of the extended relations. First, in many physical situations, because of experimental difficulties, only a few (say m) channels may be available as incident channels. In this case, therefore, we can measure m complex-scattering lengths and $[mn-\frac{1}{2}m(m+3)]$ production ratios into the other channels. The total number of experimental quantities known in this case is $mn-\frac{1}{2}m(m-1)$. If we use these in conjunction with $\frac{1}{2}(n-m)(n-m+1)$ uncoupled phase relations we have $\frac{1}{2}n(n+1)$ quantities which allow us to construct the full K matrix. Thus some theoretical knowledge about $(n-m)$ channels and experimental knowledge about the m available channels can enable one to construct the complete $n\times n$ scattering matrix. For the case of one available channel $(m=1)$ this procedure has explicitly been employed to the s-wave $\bar{K}N$ scattering with two and three coupled channels.[9]

The second situation is one where all relevant scattering amplitudes are known experimentally from the measurement of cross sections. In this case the $n\times n$ reaction amplitude K is known. On the other hand a theoretical model (for example, from symmetry schemes) predicts a $\tilde{K}$ matrix which ignores some of the channels in the $n\times n$ reaction. In this case we can use the relations (3.9) to construct a $\tilde{K}$ matrix from the experimental K matrix, which may then be used to test the predicted $\tilde{K}$ matrix.

[14] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).

[15] Actually, a somewhat weaker condition, namely, $r_i\propto\Gamma_i$ is sufficient to obtain relation (4.10). Equation (4.10) is also obtained when $r_i/\Gamma_i\ll\Lambda'/\{1+[1-\Lambda'(\tilde{M})]^{1/2}\}$.