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Particles with Identical Quantum Numbers in Dispersion Theory and Field Theory*

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The general structure of the scattering amplitude is expressed in terms of the one-particle reducible and irreducible parts, when there are several particles present having the same quantum numbers in the channel in addition to the physical and unphysical cuts. A comparison with field theory is made to obtain the propagator, the vertex functions, and the matrix of the wave-function renormalization constant **Z** in terms of the N and D functions of the N/D method by making use of the Lehmann representation of the propagator. We then prove the equivalence between composite particles defined in the N/D method and elementary particles with a singular matrix of the wave-function renormalization constant, i.e., $\det\mathbf{Z}=0$ in a full field theory under the approximation of keeping only up to the two-particle intermediate states. We show also that **Z** becomes singular when the one-particle reducible part $A(s)$ of the scattering amplitude does not decrease as fast as s^{-1} at high energies, i.e., $B\equiv -\lim_{s\to\infty}s^{-1}A^{-1}(s)=0$. In particular, the condition $B=0$ so that $\det\mathbf{Z}=0$ makes all particles to become composites of other particles, while $B\neq 0$ but with $\det\mathbf{Z}=0$ allows mixture of the elementary and composite states. The one- and two-particle cases are discussed in detail to illustrate the compositeness condition $\det\mathbf{Z}=0$.

I. INTRODUCTION

It has been suggested in some field-theoretic models[1,2] that the composite state can be regarded as an elementary state with vanishing wave-function renormalization constant. It has further been proposed[3] that the wave-function renormalization constant will vanish precisely when a bound state is generated dynamically within the framework of the N/D method of the S-matrix theory.

In particular, for the one-particle case, the equivalence between the bound state so generated and the elementary particle with its wave-function renormalization constant set equal to zero has been discussed by many authors both in field-theoretic models[2,4] and in a full field theory.[5] This equivalence has been proved up to the approximation of neglecting all but two-particle intermediate states. In an earlier paper,[6] we have shown for an exactly soluble model that a vanishing wave-function renormalization constant is also equivalent to imposing a certain high-energy lower bound to the scattering amplitude, while the relation expressing this condition contains, in general, parameters other than the coupling constant and mass of the particle, unless one neglects all the Castillejo-Dalitz-Dyson (CDD) zeros of the amplitude.

Recently a good deal of attention[7–12] has been given to the problem of describing two different particles with identical quantum numbers. However, a number of authors[7,9] claimed that a peculiar situation, not known in the one-particle case, arises in the two-particle problem; fixing one of the elements in the matrix of the wave-function renormalization constant makes all the remaining elements zero simultaneously, so that the composite state is indistinguishable from the elementary state. Others[10] asserted that the vanishing of the particle's renormalization constants has nothing to do with the particle being composite. In view of these diverse remarks, it is of some interest to study the compositeness conditions in the case of many particles with the same quantum numbers.

In this paper, we shall prove the equivalence between (i) bound states as generated dynamically, and (ii) elementary particles with a singular matrix of the wave-function renormalization constant in a full field theory. We assume that the s-wave amplitude for scattering of two spin-zero particles has two cuts, the phsyical as well as the unphysical one, as well as that the N particles have the quantum numbers of this channel. We shall make the approximation of keeping intermediate states only up to the two-particle ones.

We shall also show that *the matrix* **Z** *of the wave-function renormalization constant becomes singular when the single-particle reducible part* $A(s)$ *of the scattering amplitude does not decrease as fast as* s^{-1}, *i.e., when* $B\equiv -\lim_{s\to\infty} s^{-1}A^{-1}(s)=0$. However, *the converse of this*

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[2] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961); J. S. Dowker, Nuovo Cimento **25**, 1135 (1962); M. L. Whippman and I. S. Gerstein, Phys. Rev. **134**, B1123.
[3] A. Salam, Nuovo Cimento **25**, 224 (1963); Phys. Rev. **130**, 1287 (1963); S. Weinberg, *ibid.* **130**, 776 (1963); **131**, 440 (1963).
[4] R. M. Rockmore, Phys. Rev. **132**, 878 (1963); J. S. Dowker and J. E. Paton, Nuovo Cimento **30**, 450 (1963).
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[7] P. K. Srivastava and S. Rai Choudhury, Nuovo Cimento **39**, 650 (1965).
[8] K. Kinoshita and H. Yabuki, Progr. Theoret. Phys. (Kyoto) 34, 825 (1965); 34, 981 (1965).
[9] H. Yabuki, University of Tokyo report (unpublished).
[10] M. Alexanian and R. L. Zimmerman, Lawrence Radiation Laboratory Report UCRL-14808-T, 1966 (unpublished).
[11] J. C. Houard and J. C. LeGuillou, Nuovo Cimento **44**, 484 (1966).
[12] M. M. Broido and J. G. Taylor, Phys. Rev. **147**, 993 (1966).

statement is not true in general; even if $\det\mathbf{Z}=0$, there are cases in which the quantity B can take a nonzero and positive-definite value in the presence of more than one particle with the same quantum numbers.

In particular, it will be seen that if $\det\mathbf{Z}=0$ with $B=0$, then the N particles can be exhibited as composites of other particles, and that if $\det\mathbf{Z}=0$ but with $B\neq 0$, then we obtain some composite and some elementary particles instead of all composite particles. In any case, we shall see that if $\det\mathbf{Z}=0$, then at least one of the particles becomes composite. Thus $\det\mathbf{Z}=0$ is a natural generalization of the compositeness condition in the many-particle case. This is in agreement with what was found by Houard and Le Guillou.[11] However, these authors considered the compositeness condition for only one of the two particles by using a special field-theoretic model. Our analysis will show that fixing one of the elements in $\mathbf{Z}$ does not automatically make all the remaining elements zero, contrary to what was claimed in Refs. 7 and 9.

We shall apply our condition to the one- and two-particle cases in detail. In the one-particle case, $Z=0$ is completely equivalent to $B=0$, thus confirming our previous result obtained for an exactly soluble model. Again it will be pointed out that the relation $Z=B=0$ contains in general parameters other than the coupling constant and mass of the particle. This means that, when the right- as well as the left-hand cuts are present, the vanishing of the wave-function renormalization constant is equivalent to imposing a certain high-energy lower bound only on the single-particle reducible part $A(s)$ of the scattering amplitude. The amplitude $A(s)$ will be shown to have only the right-hand cut in addition to the particle poles. In the two-particle case, we will consider $\det\mathbf{Z}=0$ both with $B=0$ and $B\neq 0$, and show that the former corresponds to having two composite particles and the latter to one elementary and one composite particle.

In Sec. II, the general structure of the scattering amplitude is analyzed within the framework of the N/D method.[13] Special emphasis is put on the decomposition of the scattering amplitude into the single-particle reducible part and the irreducible part. In Sec. III, the field-theoretic quantities like the propagator, the vertex functions, and the wave-function renormalization constants are obtained from the scattering amplitude by using the Lehmann representation[14] of the propagator in the two-particle approximation and the various forms of the unitarity relation. Section IV contains the discussion on the compositeness conditions in the form of $\det\mathbf{Z}=0$. Here the equivalence between the elementary particles with $\det\mathbf{Z}=0$ and the bound states generated dynamically is proved. Finally, some concluding remarks are given in Sec. V.

[13] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[14] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

II. GENERAL FORM OF THE SCATTERING AMPLITUDE

Let us consider the s-wave amplitude $T(s)$ for two-particle scattering. If there are particles of masses m_1, m_2, $\cdots$, m_N which have the quantum members of this channel, then the scattering amplitude can be expressed by a dispersion relation[13]

$$T(s)=\frac{1}{\pi}\int_{s_t}^{\infty} ds'\frac{\mathrm{Im}T(s')}{s'-s}+\frac{1}{\pi}\int_{c_L} ds'\frac{f(s')}{s'-s}+\sum_i^N \frac{g_i^2}{m_i^2-s}, \quad (1)$$

where $f(s)$ is the given discontinuity across the left-hand cut c_L, and s_t is the threshold value. In writing (1), we have overlooked the possible subtractions since we will not need the detailed solution of the scattering amplitude. Our amplitude is normalized as

$$T(s)=(1/\rho(s))e^{i\delta(s)}\sin\delta(s)\,, \quad (2)$$

where $\rho(s)$ is the phase-space factor and approaches a constant at infinity. The amplitude will have, in general, a zero between every two successive poles and at least one zero between the nearest pole and the branch point s_t if $T(s)$ obeys an unsubtracted dispersion relation *and* $T(s_t)>0$. If the unsubtracted $T(s)$ of (1) has a negative value at the left-hand branch point, then the amplitude will also have at least one zero between the left-hand branch point and the pole of the lowest mass. In what follows, we shall assume an unsubtracted dispersion relation for $T(s)$, since introducing one subtraction[15] will not make any essential difference, but will only complicate the algebra.

Following the usual N/D method,[13] we write the amplitude as

$$T(s)=N(s)/D(s)\,, \quad (3)$$

where

$$D(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty} ds'\frac{\rho(s')N(s')}{(s'-s)(s'-s_0)}, \quad (4a)$$

$$N(s)=\frac{1}{\pi}\int_{c_L} ds'\frac{f(s')D(s')}{s'-s}+\sum_i^N \frac{g_i^2 D(m_i^2)}{m_i^2-s}. \quad (4b)$$

Here we have normalized $D(s)$ at $s=s_0$. While the zeros of $T(s)$ between the two succsesive poles are automatically taken care of by our numerator function (4b), a number of pole terms should be added to the denominator function (4a) corresponding to other CDD zeros,[16] if there are any, of the amplitude in the regions between the threshold and the pole of the highest mass, and between the left-hand branch point and the pole of the

[15] We have recently given a proof that the partial-wave scattering amplitude needs at most only one subtraction, by making use of rather general assumptions of analyticity, unitarity, temperedness, the normal threshold behavior, and a finite number of sign changes of the left-hand-cut discontinuity. See Y. S. Jin and Kyungsik Kang (this issue), Phys. Rev. **152**, 1227 (1966). See also T. Kinoshita, Phys. Rev. Letters **16**, 869 (1966).
[16] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1955).

lowest mass. We shall assume throughout our paper that there are no other such zeros of the amplitude, and that $N(s)$ and $D(s)$ have no coinciding zeros.

If it were not for the particle poles at $m_1^2, m_2^2, \cdots, m_N^2$, the above procedure would have resulted in a unitary amplitude

$$t(s)=\frac{n(s)}{d(s)}=\frac{e^{i\delta_0(s)}}{\rho(s)}\sin\delta_0(s)\,, \tag{5}$$

where

$$d(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')n(s')}{(s'-s)(s'-s_0)}\,, \tag{5'}$$

$$n(s)=\frac{1}{\pi}\int_{c_L}\frac{f(s')d(s')}{s'-s}ds'\,. \tag{5''}$$

Then one can easily show that (4) and (5) are related by

$$D(s)=d(s)+E(s)\,, \tag{6a}$$

$$N(s)=n(s)+F(s)\,, \tag{6b}$$

where

$$E(s)=-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')F(s')}{(s'-s)(s'-s_0)} \tag{7}$$

and

$$F(s)=\sum_{i=1}^{N}\frac{g_i^2D(m_i^2)}{m_i^2-s}+\frac{1}{\pi}\int_{s_t}^{\infty}\frac{ds'}{s'-s}\times\left[B(s')-\frac{s-s_0}{s'-s_0}B(s)\right]\rho(s')F(s')\,. \tag{8}$$

In (8), the function $B(s)$ is given by the second term in (1):

$$B(s)=\frac{1}{\pi}\int_{c_L}\frac{f(s')}{s'-s}ds'\,. \tag{9}$$

Our amplitude $T(s)$ can now be decomposed into two parts as

$$T(s)=\frac{n(s)}{d(s)}+\frac{1}{D(s)}G(s)\frac{1}{d(s)}\,, \tag{10}$$

with

$$G(s)=d(s)F(s)-n(s)E(s)\,. \tag{11}$$

This function $G(s)$ has neither the right-hand nor the left-hand cuts, but has the poles at $m_1^2, m_2^2, \cdots, m_N^2$, and tends to zero at infinity. Thus from the Cauchy theorem, it follows that

$$G(s)=\sum_{i=1}^{N}\frac{D(m_i^2)g_i^2d(m_i^2)}{m_i^2-s}\,, \tag{12}$$

so that (10) becomes

$$T(s)=\frac{n(s)}{d(s)}+\sum_{i=1}^{N}\frac{D(m_i^2)}{D(s)}\frac{g_i^2}{m_i^2-s}\frac{d(m_i^2)}{d(s)}\,. \tag{13}$$

The meaning of this decomposition is obvious: The first term is the one-particle irreducible part of $T(s)$, while the second term is the contribution of the one-particle intermediate states. In order to see the connection between the two denominator functions $D(s)$ and $d(s)$, let us denote the one-particle reducible part of $T(s)$ in (13) by $a(s)$. Since this function has only the right-hand cut plus the pole terms it admits a representation

$$a(s)=\sum_{i=1}^{N}\frac{g_i^2}{m_i^2-s}+\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\mathrm{Im}\,a(s')}{s'-s}\,. \tag{14}$$

From unitarity of $T(s)$ and $t(s)$, we have[17]

$$\mathrm{Im}\,a_\pm(s)=\pm\rho(s)|a_\pm(s)|^2\pm 2\rho(s)\mathrm{Re}[a_+^*(s)t(s)]\,, \tag{15}$$

where

$$a_+(s)=a(s+i\epsilon)=a_-^*(s)\,.$$

After some manipulations, one can get

$$\left(\frac{1}{a(s)}\right)_\pm=\frac{\mp i\rho(s)}{\cos 2\delta_0(s)}+[1\mp i\tan\delta_0(s)]\mathrm{Re}\left(\frac{1}{a(s)}\right)_\pm\,. \tag{16}$$

This is a standard inhomogeneous Hilbert arc problem,[18] and the solution has the form

$$\left(\frac{1}{a(s)}\right)_\pm=X_+(s)\int_{s_t}^{\infty}\frac{ds'}{\pi}\frac{g(s')}{s'-s-i\epsilon}\,, \tag{17}$$

where $X_\pm(s)$ is the solution of the homogeneous problem

$$X_\pm(s)=[1\mp i\tan 2\delta_0(s)]\mathrm{Re}X_\pm(s)\,, \tag{18}$$

and the function $g(s)$ is given by

$$\left(\frac{1}{a(s)}\right)_+\Big/X_+(s)-\left(\frac{1}{a(s)}\right)_-\Big/X_-(s)=2ig(s)\,; \tag{19}$$

therefore,

$$g(s)=-[\rho(s)/p(s)]|d(s)|^2 \tag{20}$$

and

$$X_\pm(s)=p(s)\{d_\pm(s)\}^2\,. \tag{21}$$

Here $p(s)$ is an arbitrary polynomial which is assumed to have no zeros on the right-hand cut, and the function $d(s)$ is given by (5′) and has the phase representation

$$d(s)=\exp\left(-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}\frac{\delta_0(s')}{(s'-s)(s'-s_0)}ds'\right)\,. \tag{22}$$

Notice that

$$e^{2i\delta_0(s)}\left(\frac{1}{a(s)}\right)_+=R(s)-i\rho(s)\,, \tag{23}$$

[17] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

[18] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953). See also M. M. Islam and Kyungsik Kang, Phys. Rev. **139**, B973 (1965), Appendix.

with a real function $R(s)$. Thus, if we define a new function

$$A(s)\equiv a(s)\{d(s)\}^2, \tag{24}$$

then we get

$$A(s)=\sum_{i=1}^{N}\frac{g_i^2d^2(m_i^2)}{m_i^2-s}+\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')|A(s')|^2}{(s'-s)|d(s')|^2}. \tag{25}$$

Again we have assumed in writing (25) that the amplitude (24) needs no subtraction. Then (25) will have a zero between every two successive poles. These zeros will automatically be present in the solution by the N/D formalism. If $A(s_t)>0$, the amplitude (25) can have a zero in the region between s_t and the pole of the highest mass. However, if one subtraction was needed for (24), then $A(s)$ will have a zero in the region below the pole of the lowest mass. Those other zeros of $A(s)$ will give rise to the CDD pole terms in the denominator function of $A(s)$. We shall consistently assume no such zeros of $A(s)$. One can easily solve for $A(s)$ by putting

$$A(s)=\tilde{n}(s)/\tilde{d}(s), \tag{26}$$

where

$$\tilde{d}(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}\frac{\rho(s')\tilde{n}(s')}{(s'-s)(s'-s_0)|d(s')|^2}ds', \tag{27a}$$

$$\tilde{n}(s)=\sum_{i=1}^{N}\frac{\tilde{d}(m_i^2)g_i^2d^2(m_i^2)}{m_i^2-s}. \tag{27b}$$

It is interesting to notice that $A(s)$ of (25) is a Herglotz function[19] of s, i.e., $\mathrm{Im}A(s)>0$ for $\mathrm{Im}s>0$. Thus $-A^{-1}(s)$ is also a Herglotz function and, in general,

$$B\equiv\lim_{s\to\infty}\frac{-1}{A(s)s}\geq 0.$$

This means that

$$\tilde{d}(\infty)\bigg/\left(\sum_{i=1}^{N}\tilde{d}(m_i^2)g_i^2d^2(m_i^2)\right)\geq 0,$$

so that $\tilde{d}(\infty)\geq 0$, in general. Because $A(s)$ is a Herglotz function of s, it is in general bounded by $C'|s|^{-1}\leq|A(s)|\leq C|s|$, which implies $C_2|s|^{-2}\leq|\tilde{d}(s)|\leq C_1$. Thus, even if $\tilde{d}(\infty)\neq 0$, one subtraction is sufficient in (27a). If $\tilde{d}(\infty)=0$, then we may undo a subtraction in (27a). We shall see that $\tilde{d}(\infty)$ plays an important role in the discussion of the compositeness condition.

Finally, the amplitude $T(s)$ can be written as

$$T(s)=\frac{n(s)}{d(s)}+\sum_{i=1}^{N}\frac{d(m_i^2)\tilde{d}(m_i^2)}{d(s)\tilde{d}(s)}\frac{g_i^2}{m_i^2-s}\frac{d(m_i^2)}{d(s)}, \tag{28}$$

so that by comparing (28) with (13) one obtains

$$D(s)=d(s)\tilde{d}(s) \tag{29a}$$

and

$$N(s)=\frac{1}{d(s)}\sum_{i=1}^{N}\frac{d^2(m_i^2)g_i^2\tilde{d}(m_i^2)}{m_i^2-s}+n(s)\tilde{d}(s). \tag{29b}$$

It is clear from (1) that the solution (28) should satisfy the conditions

$$T^{-1}(m_i^2)=0 \tag{30a}$$

and

$$\left(\frac{dT^{-1}(s)}{ds}\right)_{s=m_i^2}=-\frac{1}{g_i^2}. \tag{30b}$$

III. FIELD-THEORETIC QUANTITIES

As we have started from the dispersion relation (1) for $T(s)$, we do not *a priori* have the field-theoretic quantities like the propagator and the vertex functions. In this section, we shall recover these quantities in terms of the scattering amplitude $T(s)$ given by (28).

The Lehmann representation[14] for the propagator has the expression

$$\Delta'_{ij}(s)=\frac{\delta_{ij}}{m_i^2-s}+\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\Lambda_i^*(s')\rho(s')\Lambda_j(s')}{s'-s}, \tag{31a}$$

where the quantity $\Lambda_i(s)$ is related to the form factor $F_i(s)$ by

$$F_i(s)=(m_i^2-s)\Lambda_i(s), \tag{32a}$$

and the form factor has the dispersion relation

$$F_i(s)=g_i+\frac{s-m_i^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\mathrm{Im}F_i(s')}{(s'-s)(s'-m_i^2)}. \tag{33}$$

Let us use the matrix notations

$$\mathbf{p}(s)\equiv\begin{bmatrix}1/(m_1^2-s) & 0 & \cdots & 0\\ 0 & 1/(m_2^2-s) & \cdots & 0\\ \vdots & & & \vdots\\ 0 & \cdots & \cdots & 1/(m_N^2-s)\end{bmatrix} \tag{34}$$

and

$$\mathbf{\Lambda}(s)=(\Lambda_1(s),\Lambda_2(s),\cdots,\Lambda_N(s)),\quad\text{etc.} \tag{35}$$

Then (31) and (32) become, respectively,

$$\mathbf{\Delta}'(s)=\mathbf{p}(s)+\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{s'-s}\mathbf{\Lambda}^\dagger(s')\mathbf{\Lambda}(s') \tag{31b}$$

and

$$\mathbf{F}(s)=\mathbf{\Lambda}(s)\mathbf{p}^{-1}(s). \tag{32b}$$

The quantity $\mathbf{\Lambda}(s)$ is related to the vertex function by

$$\mathbf{\Lambda}(s)=\mathbf{\Gamma}(s)\mathbf{\Delta}'(s), \tag{36}$$

so that

$$\mathrm{Im}\mathbf{\Delta}'(s)=\rho(s)\mathbf{\Lambda}^\dagger(s)\mathbf{\Lambda}(s)=\rho(s)\mathbf{\Delta}'^\dagger\mathbf{\Gamma}^\dagger\mathbf{\Gamma}\mathbf{\Delta}' \tag{37}$$

and

$$\mathrm{Im}\mathbf{\Delta}'^{-1}(s)=\rho(s)\mathbf{\Gamma}^\dagger(s)\mathbf{\Gamma}(s). \tag{38}$$

[19] J. A. Shohat and J. D. Tamarkin, *The Problems of Moments* (American Mathematical Society, New York, 1943).

From (31b) and (38), one gets[20]

$$\mathbf{Z}(s)\equiv\mathbf{p}(s)\mathbf{\Delta}'^{-1}(s)$$

$$=1-\left(\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{s'-s}\mathbf{p}(s')\mathbf{\Gamma}^{\dagger}(s')\mathbf{\Gamma}(s')\mathbf{p}(s')\right)\mathbf{p}^{-1}(s) \quad (39a)$$

or

$$Z_{ij}(s)=\delta_{ij}+\frac{s-m_j^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)}\frac{\Gamma_i^*(s')\Gamma_j(s')}{(m_i^2-s')(m_j^2-s')}. \quad (39b)$$

In writing (39), we have assumed no propagator zeros, since we are assuming no CDD zeros in $A(s)$ and $T(s)$. Notice, however, that $\Delta'_{ii}(s)$ can have a zero between m_i^2 and s_t if no subtraction is needed for (31a) and $\Delta'_{ii}(s_t)>0$, while it can have another zero below m_i^2 if one subtraction is needed.

From the definition of $\mathbf{Z}(s)$, it follows that

$$\mathbf{Z}^{-1}(s)=1+\left(\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{s'-s}\mathbf{\Lambda}^{\dagger}(s')\mathbf{\Lambda}(s')\right)\mathbf{p}^{-1}(s) \quad (40a)$$

or

$$(Z^{-1}(s))_{ij}=\delta_{ij}+\frac{m_j^2-s}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{s'-s}\frac{F_i^*(s')F_j(s')}{(m_i^2-s')(m_j^2-s')}, \quad (40b)$$

and

$$\Delta'_{ij}(s)=(m_j^2-s)^{-1}(Z^{-1}(s))_{ij}, \quad (41a)$$

$$(\Delta'^{-1}(s))_{ij}=(m_i^2-s)Z_{ij}(s). \quad (41b)$$

Let us make the following decomposition[21] of $T(s)$:

$$T(s)=\mathbf{\Gamma}(s)\mathbf{\Delta}'(s)\mathbf{\Gamma}^T(s)+H(s), \quad (42)$$

where $\mathbf{\Gamma}^T(s)$ is the transpose of $\mathbf{\Gamma}(s)$.

From the unitarity conditions, we know that

$$\mathrm{Im}T(s)=\rho(s)T^{\dagger}(s)T(s) \quad (43)$$

and

$$\mathrm{Im}\mathbf{F}(s)=\rho(s)T^{\dagger}(s)\mathbf{F}(s). \quad (44a)$$

By inserting (32b) into (44a), one obtains

$$\mathrm{Im}\mathbf{\Lambda}(s)=\rho(s)T^{\dagger}(s)\mathbf{\Lambda}(s). \quad (44b)$$

We notice here that $\mathbf{F}(s)$ and $\mathbf{\Lambda}(s)$ have the same phase as the scattering amplitude $T(s)$. By making use of (38), (43), and (44), one can easily verify that

$$\mathrm{Im}\mathbf{\Gamma}(s)=\rho(s)H^{\dagger}(s)\mathbf{\Gamma}(s) \quad (45a)$$

and

$$\mathrm{Im}\mathbf{\Gamma}^T(s)=\rho(s)H(s)\mathbf{\Gamma}^{\dagger}(s). \quad (45b)$$

To find $\mathrm{Im}H(s)$, we notice from (42) that

$$\mathrm{Im}T(s)=\mathrm{Im}[\mathbf{\Lambda}(s)\mathbf{\Gamma}^T(s)]+\mathrm{Im}H(s), \quad (46a)$$

while from (43),

$$\mathrm{Im}T(s)=\rho T^{\dagger}(s)\mathbf{\Gamma}(s)\mathbf{\Lambda}^T(s)+\rho\mathbf{\Lambda}^*(s)\mathbf{\Gamma}^{\dagger}(s)H(s)+\rho H^{\dagger}(s)H(s). \quad (46b)$$

On the other hand, from (44) and (45) it follows that

$$\mathrm{Im}[\Lambda(s)\mathbf{\Gamma}^T(s)]=\rho(s)T^{\dagger}(s)\mathbf{\Lambda}(s)\mathbf{\Gamma}^T(s)+\rho(s)\mathbf{\Lambda}^*(s)\mathbf{\Gamma}^{\dagger}(s)H(s), \quad (47)$$

so that

$$\mathrm{Im}H(s)=\rho(s)H^{\dagger}(s)H(s). \quad (48)$$

Thus $H(s)$ by itself satisfies unitarity, and furthermore it is nothing but the single-particle irreducible part $t(s)$ of the scattering amplitude $T(s)$ given in the preceding section. From (45), one can notice that the vertex function $\mathbf{\Gamma}(s)$ has the phase of $t(s)$. Since the form factor $\mathbf{F}(s)$ has the phase of the scattering amplitude $T(s)$, it is appropriate to define

$$F_i(s)=g_iD(m_i^2)/D(s), \quad (49a)$$

so that

$$\Lambda_i(s)=(g_i/(m_i^2-s))D(m_i^2)/D(s). \quad (49b)$$

Thus $\mathbf{F}(s)$ so defined has zeros only at CDD zeros of $T(s)$, which we are assuming not to exist in our problem. Then from (36), we see that

$$\mathbf{\Gamma}(s)=\mathbf{F}(s)\mathbf{Z}(s) \quad (50a)$$

or

$$\Gamma_i(s)=\sum_{j=1}^{N}\frac{g_jD(m_j^2)}{D(s)}Z_{ji}(s), \quad (50b)$$

and

$$\mathbf{\Lambda}(s)=\mathbf{\Gamma}(s)\mathbf{Z}^{-1}(s)\mathbf{p}(s) \quad (51a)$$

or

$$\Lambda_i(s)=\frac{1}{m_i^2-s}\sum_{j=1}^{N}\Gamma_j(s)(Z^{-1}(s))_{ji}. \quad (51b)$$

Since we know that the vertex function has the phase of $t(s)$, let us put

$$\Gamma_i(s)=g_id(m_i^2)/d(s). \quad (52)$$

Then (39b) becomes

$$Z_{ij}(s)=\delta_{ij}+\frac{s-m_j^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)}\times\frac{g_ig_jd(m_i^2)d(m_j^2)}{(m_i^2-s')(m_j^2-s')|d(s)|^2}, \quad (39c)$$

whereas (40b) can be rewritten as

$$(Z^{-1}(s))_{ij}=\delta_{ij}+\frac{m_j^2-s}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{s'-s}\times\frac{g_iD(m_i^2)g_jD(m_j^2)}{(m_i^2-s')(m_j^2-s')|D(s')|^2}. \quad (40c)$$

[20] Our definition of the function $\mathbf{Z}(s)$ is an extension of the one defined by M. Ida [Phys. Rev. **135**, B499 (1964); Progr. Theoret. Phys. (Kyoto) **34**, 92 (1965)] to the many particles with the identical quantum numbers.

[21] S. D. Drell and F. Zachariasen, Phys. Rev. **105**, 1407 (1957); M. Ida, *ibid.* **136**, B1767 (1964); Y. S. Jin and S. W. MacDowell, *ibid.* **137**, B688 (1965).

From (50b) and (52), we get

$$\frac{g_i d(m_i^2)}{d(s)}=\sum_{j=1}^{N}\left[\frac{g_j D(m_j^2)}{D(s)}\right]Z_{ji}(s); \tag{53}$$

and from (49b) and (51b),

$$\frac{g_i D(m_i^2)}{D(s)}=\sum_{j=1}^{N}\frac{g_j d(m_j^2)}{d(s)}(Z^{-1}(s))_{ji}. \tag{54}$$

Now one can easily check that

$$\left(\sum_{j=1}^{N} g_j D(m_j^2)Z_{ji}(s)\right)\Big/g_i d(m_i^2)=\tilde{d}(s)=\tilde{d}(m_i^2)+\frac{s-m_i^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)(m_i^2-s')|d(s')|^2}\sum_{j=1}^{N}\frac{\tilde{d}(m_j^2)g_j^2d^2(m_j^2)}{m_j^2-s'}$$
$$(i=1,2,\cdots,N) \tag{55}$$

and

$$\left(\sum_{j=1}^{N} g_j d(m_j^2)(Z^{-1}(s))_{ji}\right)\Big/g_i D(m_i^2)=1/\tilde{d}(s)=\frac{1}{\tilde{d}(m_i^2)}+\frac{s-m_i^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)(s'-m_i^2)|D(s')|^2}\sum_{j=1}^{N}\frac{D(m_j^2)g_j^2d(m_j^2)}{m_j^2-s'}$$
$$(i=1,2,\cdots,N). \tag{56}$$

By inserting (55) and (56) into (53) and (54), respectively, we see that (29) is reproduced. Thus the forms of the form factors and vertex functions given by (49) and (52) are indeed appropriately written. Finally, the propagator is obtained by inserting (39c) and (40c) into (41), and the scattering amplitude given by (42) becomes (28).

Before closing this section, a few words on the wave-function renormalization constants are in order. As Z_{ij} is given by $\lim_{s\to\infty}(m_i^2-s)^{-1}(\Delta'^{-1}(s))_{ij}=Z_{ij}(\infty)$ [which is obvious from (41)], by making use of either of (39) and (40), we obtain

$$Z_{ij}=\delta_{ij}-\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')g_ig_jd(m_i^2)d(m_j^2)}{(s'-m_i^2)(s'-m_j^2)|d(s')|^2} \tag{57}$$

or

$$(Z^{-1})_{ij}=\delta_{ij}+\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')g_ig_jD(m_i^2)D(m_j^2)}{(s'-m_i^2)(s'-m_j^2)|D(s')|^2}. \tag{58}$$

It is seen that the matrix of the wave-function renormalization constant $\mathbf{Z}$ is symmetric, as it should be. From (55), we observe that Z_{ij} is related to $\tilde{d}(\infty)$. This is a fact which was found for the one-particle case in an earlier paper.[6]

As we mentioned earlier, $\Delta'_{ii}(s)$ can have at most one zero (two zeros) if (31a) needs no (one) subtraction. These zeros would give rise to the CDD pole terms in (39) and thus in $\tilde{d}(s)$ from (55). On the other hand, the single-particle irreducible part $t(s)$ and the proper vertex function $\mathbf{\Gamma}(s)$ will have poles if the denominator function $d(s)$ has zeros. Since such zeros of the $d(s)$ do not represent the physical particles, they will not appear as poles in $T(s)$. In other words, these poles in $t(s)$ should be canceled by the denominator function $\tilde{d}(s)$ of $A(s)$ in (29). Thus $\tilde{d}(s)$ must have poles at the zeros of $d(s)$, in addition to those poles corresponding to zeros of $A(s)$. While one may consider only those poles in $\tilde{d}(s)$ that come from zeros of $d(s)$,[22] we have simply assumed no zeros in $d(s)$, since this does not weaken the validity of our results on the compositeness condition.

In the next section we shall investigate the compositeness conditions in the form of $\det\mathbf{Z}=0$, and in particular, for the case of two particles with identical quantum numbers, we shall show that they can be explained as composites of other particles.

IV. COMPOSITENESS CONDITIONS

In the one-particle case, it has been proved by many authors[2,5,6] that a bound state as defined by the so-called bootstrap equations of the scattering amplitude, such as (31), is equivalent to an elementary particle with its wave-function renormalization constant set equal to zero, at least in the elastic unitarity approximation. In particular, it was shown in an earlier paper[6] that in order to obtain a definite relation between the mass and coupling constant of the particle, the scattering amplitude should have a certain high-energy lower bound and should not have any CDD zeros. Since we have constructed the scattering amplitude by requiring that $D(s)$ have no poles and that $N(s)$ and $D(s)$ have no common zeros, there will be no CDD poles in $D(s)$. Therefore, if what we have found in the one-particle problem is going to be true also in the present problem, there should be some connection between the high-energy behavior of the amplitude and the properties of $\mathbf{Z}$. As we mentioned earlier, this is indeed the case, and this can be seen easily from (55) and (56). Since the high-energy behavior of the denominator function of the amplitude $A(s)$ defined by (25) is related to $B\equiv-\lim_{s\to\infty}s^{-1}\times A^{-1}(s)$, as is discussed in Sec. II, the

[22] This assumption was made by M. Ida in the second reference of Ref. 20 for the single-particle case. If one adopts this attitude, then one can calculate the residue of the CDD pole from Eq. (29b).

high-energy lower bound of the amplitude $A(s)$ is related to the Z factors. It is interesting to notice that in the model field theories[2,6,23] which can be soluble exactly, one usually obtains an amplitude of the same type, i.e., no left-hand-cut contributions present, so that the high-energy behavior of the amplitude and the wave-function renormalization constant are rather directly related to each other. The amplitude $A(s)$ becomes the one in the Zachariasen[23] or the extended Lee model[2,6] in the absence of the left-hand cut in the single-particle irreducible part $t(s)$.

From (55), we get

$$\tilde{d}(\infty)=\frac{1}{g_i d(m_i^2)}\sum_{j=1}^{N} g_j D(m_j^2) Z_{ji} \quad (i=1,2,\cdots,N), \quad (59)$$

while from (56), it follows that

$$\frac{1}{\tilde{d}(\infty)}=\frac{1}{g_i D(m_i^2)}\sum_{j=1}^{N} g_j d(m_j^2)(Z^{-1})_{ji} \quad (i=1,2,\cdots,N). \quad (60)$$

We mention that in (59) and (60) $Z_{ij}=Z_{ji}$. If the matrix is not singular, i.e., det $\mathbf{Z}\neq 0$, then the inverse matrix exists and we have the relations

$$(Z^{-1})_{ji}=[(-)^{j+i}/\det\mathbf{Z}]D(j,i), \quad (61)$$

where $(-)^{j+i}D(j,i)$ is the cofactor of the element Z_{ji} in $\det\mathbf{Z}$. From (59), it is clear that for $\tilde{d}(\infty)=0$, the N simultaneous linear equations will have only $g_iD(m_i^2)=0 (i=1, 2, \cdots, N)$ if $\det\mathbf{Z}\neq 0$. Since all $D(m_i^2)$ cannot vanish, we state that if $\det\mathbf{Z}\neq 0$, then $\tilde{d}(\infty)\neq 0$. However, it is obvious from (60) that *if* $\tilde{d}(\infty)=0$, *then* $\det\mathbf{Z}=0$, because $D(m_i^2)\neq 0$ and $d(s)$ cannot have poles at $s=m_i^2$ $(i=1, 2, \cdots, N)$. Furthermore, this will enable $D(m_i^2)$ $(i=1, 2, \cdots, N)$ to have nonzero values in (59). The inverse of this statement is not true in general, because

$$\sum_{j=1}^{N}(-)^{j+1} g_j d(m_j^2) D(j,i)$$

can vanish for $N>1$ even if $\det\mathbf{Z}=0$. This is the case when we have mixture of some elementary and some composite particles, as will be seen later. We can accordingly say that *for* $N>1$, *if* $\det\mathbf{Z}=0$, *then* $\tilde{d}(\infty)=0$ *unless*

$$\sum_{j=1}^{N}(-)^{j+1} g_j d(m_j^2) D(j,i)=0.$$

One can easily check that whenever

$$\sum_{j=1}^{N}(-)^{j+i} g_j d(m_j^2) D(j,i)=0,$$

one has $\det\mathbf{Z}=0$, so that the right-hand side of (60) becomes indeterminate. In that case, $\tilde{d}(\infty)$ should be evaluated from (59).

At this point, we mention that a previous author[9] has claimed that "if one of the Z_{ij} tends to zero, then all the others also tend to zero" and that "if $\tilde{d}(\infty)=0$, then all the $Z_{ij}=0$." We see that neither of these statements is true in general.

We shall now investigate the connection between bound states defined by (30) and elementary particles with $\det\mathbf{Z}=0$ for both $\tilde{d}(\infty)=0$ and $\tilde{d}(\infty)\neq 0$. We shall briefly review the one-particle problem and discuss in detail the two-particle case.

A. The One-Particle Case

This case has been studied extensively by other authors,[5] and so we shall only sketch the main points. From (59), we get

$$\tilde{d}(m^2)Z=\tilde{d}(\infty), \quad (62)$$

which implies that if $\tilde{d}(\infty)\neq 0$, then $Z\neq 0$. This can also be seen by observing that if $\tilde{d}(\infty)$ is constant, then $D(\infty)$ is also constant, so that the integral in (58) converges. Remember that from the Herglotz property of $A(s)$, $\tilde{d}(\infty)$ can at worst be a positive-definite constant. If $Z=0$, then (59) gives

$$1=\frac{g^2}{\pi}\int_{s_t}^{\infty} ds\frac{\rho(s)}{(s-m^2)^2}\left|\frac{d(m^2)}{d(s)}\right|^2. \quad (63)$$

On the other hand, since $\tilde{d}(\infty)=0$, we can undo the subtraction in (27a) so that the scattering amplitude becomes

$$T(s)=t(s)+\left(\frac{d(m^2)}{d(s)}\right)^2\frac{g^2}{m^2-s}\times\left(-\frac{g^2}{\pi}\int_{s_t}^{\infty} ds'\frac{\rho(s')}{(s'-s)(m^2-s')}\left|\frac{d(m^2)}{d(s')}\right|^2\right)^{-1}, \quad (64)$$

which gives

$$\left(\frac{dT^{-1}(s)}{ds}\right)_{s=m^2}=-\frac{1}{\pi}\int_{s_t}^{\infty} ds\frac{\rho(s)}{(s-m^2)^2}\left|\frac{d(m^2)}{d(s)}\right|^2. \quad (65)$$

By comparing (65) with the bound-state condition (30b), we get the relation given by (63). Thus the elementary state becomes a composite state in the limit when $Z\to 0$, or equivalently when $B\to 0$. This is a generalized statement of what we have found before.[6] We remark that no CDD zeros are assumed for the amplitude $A(s)$ except those between the poles.

B. The Two-Particle Case

Let us first consider the situation in which both particles are elementary. The scattering amplitude is given

[23] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

by

$$T(s)=\frac{n(s)}{d(s)}+\frac{1}{d^2(s)}\frac{1}{\tilde{d}_2(s)}\times\left(\frac{d^2(m_1^2)g_1^2\tilde{d}_2(m_1^2)}{m_1^2-s}+\frac{d^2(m_2^2)g_2^2\tilde{d}_2(m_2^2)}{m_2^2-s}\right), \quad (66)$$

where

$$\tilde{d}_2(s)=\tilde{d}_2(m_j^2)-\frac{s-m_j^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)(s'-m_j^2)|d(s')|^2}\times\sum_{i=1}^{2}\frac{\tilde{d}_2(m_i^2)g_i^2d^2(m_i^2)}{m_i^2-s'}\quad (j=1 \text{ or } 2). \quad (67)$$

The wave-function renormalization constants can be worked out from (59), i.e.,

$$\begin{aligned}g_1D_2(m_1^2)Z_{11}+g_2D_2(m_2^2)Z_{21}&=g_1d(m_1^2)\tilde{d}_2(\infty),\\ g_1D_2(m_1^2)Z_{12}+g_2D_2(m_2^2)Z_{22}&=g_2d(m_2^2)\tilde{d}_2(\infty),\end{aligned} \quad (68)$$

where $D_2(s)=d(s)\tilde{d}_2(s)$. We mention that $Z_{12}=Z_{21}$ is related to Z_{11} and Z_{22} by

$$Z_{11}-\frac{g_1d(m_1^2)}{g_2d(m_2^2)}Z_{12}=1-\frac{m_1^2-m_2^2}{\pi}\times\int_{s_t}^{\infty}ds\frac{\rho(s)}{(s-m_1^2)^2(s-m_2^2)}\frac{g_1^2d^2(m_1^2)}{|d(s)|^2}, \quad (69)$$

and

$$Z_{22}-\frac{g_2d(m_2^2)}{g_1d(m_1^2)}Z_{12}=1-\frac{m_2^2-m_1^2}{\pi}\times\int_{s_t}^{\infty}ds\frac{\rho(s)}{(s-m_2^2)^2(s-m_1^2)}\frac{g_2^2d^2(m_2^2)}{|d(s)|^2}, \quad (70)$$

respectively. From (60) and (61), one gets

$$\begin{aligned}g_1D_2(m_1^2)/\tilde{d}_2(\infty)&=(g_1d(m_1^2)Z_{22}-g_2d(m_2^2)Z_{21})/\det\mathbf{Z},\\ g_2D_2(m_2^2)/\tilde{d}_2(\infty)&=(g_2d(m_2^2)Z_{11}-g_1d(m_1^2)Z_{12})/\det\mathbf{Z}.\end{aligned} \quad (71)$$

If $\det\mathbf{Z}\neq 0$, then we get from (71) that $\tilde{d}_2(\infty)\neq 0$ and finite. Also then it follows from (68) and (71) that

$$G_{1k}{}^{j}=G_{2k}{}^{j}, \quad (j, k=1, 2), \quad (72)$$

so that

$$Z_{22}=\frac{\tilde{d}_2(m_1^2)}{\tilde{d}_2(m_2^2)}Z_{11}+\left(\frac{g_2d(m_2^2)}{g_1d(m_1^2)}-\frac{g_1D_2(m_1^2)}{g_2D_2(m_2^2)}\right)Z_{12}, \quad (73)$$

where

$$G_{ik}{}^{j}=\frac{g_jD_2(m_j^2)}{g_iD_2(m_i^2)}\sum_{l=1}^{2}\frac{g_ld(m_l^2)}{g_kd(m_k^2)}(-)^{i+l}Z_{il}.$$

It is interesting to notice that in the elementary version, when the two particles have the same residue in the scattering amplitude (66), the diagonal elements are related to each other by a constant which contains only the values of the denominator function $\tilde{d}_2(s)$ evaluated at the masses of the particles. It is obvious from (68) and (71) that *neither* setting one of the Z_{ij} equal to zero *nor* letting $\tilde{d}_2(\infty)\to 0$ makes all the Z_{ij} simultaneously tend to zero, in general.

If $\det\mathbf{Z}=0$, and the right-hand sides of (69) and (70) are not zero, then $\tilde{d}_2(\infty)=0$. Again we can easily show in this case the equivalence between $\det\mathbf{Z}=0$ and the bound states defined by (30). Since $\det\mathbf{Z}=0$ in the present case gives $\tilde{d}_2(\infty)=0$, we find from (67) that

$$1=\frac{1}{\pi}\int_{s_t}^{\infty}ds\frac{\rho(s)}{m_1^2-s}\frac{1}{|d(s)|^2}\times\left(\frac{g_1^2d^2(m_1^2)}{m_1^2-s}+\frac{\tilde{d}_2(m_2^2)}{\tilde{d}_2(m_1^2)}\frac{g_2^2d^2(m_2^2)}{m_2^2-s}\right) \quad (74)$$

and

$$1=\frac{1}{\pi}\int_{s_t}^{\infty}ds\frac{\rho(s)}{m_2^2-s}\frac{1}{|d(s)|^2}\times\left(\frac{\tilde{d}_2(m_1^2)}{\tilde{d}_2(m_2^2)}\frac{g_1^2d^2(m_1^2)}{m_1^2-s}+\frac{g_2^2d^2(m_2^2)}{m_2^2-s}\right). \quad (75)$$

We notice from (58) that $\det\mathbf{Z}\to 0$ implies that the high-energy behavior of $D_2(s)$ is at most of the order of s^{-1}. Furthermore, it has been shown by Warnock[24] that the asymptotic behavior of $D_2(s)$ is given, possibly up to a logarithmic factor, by

$$\lim_{s\to\infty}D_2(s)=O(s^{n_c-n_b}), \quad (76)$$

where n_c is the number of CDD zeros and n_b the number of bound-state poles in the scattering amplitude. Since we are considering two bound states ($n_b=2$), there should be a CDD zero between two bound-state poles ($n_c=1$). Moreover, as we explained in Sec. II, there are no other CDD zeros if we assume an unsubtracted dispersion relation for $T(s)$ and that $T(s_t)<0$. Thus the asymptotic behavior $O(s^{-1})$ of $D_2(s)$ is consistent with having two bound states, i.e., two zeros of the denominator function.

Furthermore, the scattering amplitude $T(s)$ of (66), after undoing the subtraction in $\tilde{d}_2(s)$ [since $\tilde{d}_2(\infty)=0$],

[24] R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

can be rewritten as

$$\frac{N_2(s)}{D_2(s)}=\frac{n(s)}{d(s)}+\left(\frac{d(m_1^2)}{d(s)}\right)^2\frac{g_1^2}{m_1^2-s}\left[-\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)|d(s')|^2}\left(\frac{g_1^2d^2(m_1^2)}{m_1^2-s'}+\frac{\tilde{d}_2(m_2^2)}{\tilde{d}_2(m_1^2)}\frac{g_2^2d^2(m_2^2)}{m_2^2-s'}\right)\right]^{-1}+\left(\frac{d(m_2^2)}{d(s)}\right)^2\frac{g_2^2}{m_2^2-s}$$
$$+\left[-\frac{1}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s)|d(s')|^2}\left(\frac{\tilde{d}_2(m_1^2)g_1^2d^2(m_1^2)}{\tilde{d}_2(m_2^2)(m_1^2-s')}+\frac{g_2^2d^2(m_2^2)}{m_2^2-s'}\right)\right]^{-1}. \quad (77)$$

The poles come from the second and third terms on the right of Eq. (77). We get also from (77) that

$$\left[\frac{dT^{-1}(s)}{ds}\right]_{s=m_i^2}=-\frac{1}{g_i^2}\left[-\frac{1}{\pi}\int_{s_t}^{\infty}ds\frac{\rho(s)}{(s-m_i^2)|d(s)|^2}\sum_{j=1}^{2}\frac{g_j^2d^2(m_j^2)}{m_j^2-s}\left(\frac{\tilde{d}_2(m_j^2)}{\tilde{d}_2(m_i^2)}\right)\right] \quad (i=1,2). \quad (78)$$

By comparing this with (30b), we see that (74) and (75) are reproduced for $i=1$ and 2, respectively. What we have shown here is that the two particles can be exhibited as composite states of other particles by letting $\det\mathbf{Z}\to 0$ so as to have $\tilde{d}_2(\infty)\to 0$. Thus we agree with Houard and LeGuillou[11] that $\det\mathbf{Z}=0$ is a natural generalization of the compositeness condition for two elementary particles with the same quantum numbers. However, these authors obtained this result for a special field-theoretic model and did not attempt to make both particles composite. We also mention that recently Broido and Taylor[12] discussed field-theoretic mechanisms by which two particles having identical quantum numbers can be exhibited as composites of other particles.

Next, let us consider the situation in which only one of the two particles, say m_1, is elementary, and the other particle, say m_2, is composite. Then the scattering amplitude is given by

$$T(s)=\frac{n(s)}{d(s)}+\left(\frac{d(m_1^2)}{d(s)}\right)^2\frac{g_1^2}{m_1^2-s}\frac{\tilde{d}_1(m_1^2)}{\tilde{d}_1(s)}, \quad (79)$$

where

$$\tilde{d}_1(s)=\tilde{d}_1(m_1^2)\times\left[1+\frac{s-m_1^2}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')g_1^2}{(s'-s)(s'-m_1^2)^2}\left|\frac{d(m_1^2)}{d(s')}\right|^2\right], \quad (80)$$

and generates dynamically a bound state at $s=m_2^2$ given by (30). A similar case, but with no left-hand cut, was considered by some previous authors.[7] For this purpose, let us recall the N/D decomposition of $T(s)$, i.e.,

$$T(s)=N_1(s)/D_1(s), \quad (81)$$

where

$$N_1(s)=\frac{n(s)}{d(s)}D_1(s)+\frac{1}{d(s)}\frac{\tilde{d}_1(m_1^2)g_1^2d^2(m_1^2)}{m_1^2-s} \quad (82)$$

and

$$D_1(s)=d(s)\tilde{d}_1(s). \quad (83)$$

Then the Eqs. (30) defining the bound state can be rewritten as

$$D_1(m_2^2)=0 \quad (84a)$$

and

$$\left[\frac{N_1(s)}{D_1'(s)}\right]_{s=m_2^2}=-g_2^2. \quad (84b)$$

From (83) and (84a), we obtain

$$1=\frac{m_1^2-m_2^2}{\pi}\int_{s_t}^{\infty}ds\frac{\rho(s)g_1^2}{(s-m_2^2)(s-m_1^2)^2}\left|\frac{d(m_1^2)}{d(s)}\right|^2 \quad (85a)$$

or equivalently

$$Z_{11}=\frac{g_1d(m_1^2)}{g_2d(m_2^2)}Z_{12}, \quad (85b)$$

while from the condition (84b), it follows that

$$1=\frac{m_2^2-m_1^2}{\pi}\int_{s_t}^{\infty}ds\frac{\rho(s)g_2^2}{(s-m_1^2)(s-m_2^2)^2}\left|\frac{d(m_2^2)}{d(s)}\right|^2, \quad (86a)$$

so that from (70),

$$Z_{22}=[g_2d(m_2^2)/g_1d(m_1^2)]Z_{12}. \quad (86b)$$

Although the conditions of (84) make the matrix $\mathbf{Z}$ singular, they do not necessarily make $\tilde{d}_2(\infty)=0$, because (85b) and (86b) make the right-hand sides of (71) indeterminate. We notice that (85) and (86) are symmetric with respect to particles 1 and 2. Moreover, we would have obtained (86a) and (85a), respectively, corresponding to (84a) and (84b), had we assumed that m_2 is elementary and tried to generate a bound state at $s=m_1^2$. Contrary to what was asserted by the previous authors,[7,9] we have achieved one elementary and one bound state by letting $\det\mathbf{Z}=0$ but with $\tilde{d}_2(\infty)\neq 0$. Thus, making one of the particles composite does not automatically force the other particle to become composite. From our discussions, it is clear that $\det\mathbf{Z}=0$ makes at least one of the particles composite, and this allows us to define the compositeness concept coherently.

For the case of more than two particles, we can proceed in the same way as for the two-particle case. Instead of going into more examples, we shall just state the main results. First of all, if $\det\mathbf{Z}\neq 0$, then $\tilde{d}(\infty)\neq 0$

and all the particles remain in the elementary states. Secondly, if the single-particle reducible part of the amplitude decreases as fast as s^{-1} at high energies, so that $\tilde{d}(\infty)=0$, then $\det\mathbf{Z}=0$. In this case, all the particles can be exhibited as composites of all other particles. All the elementary particles with $\det\mathbf{Z}=0$ become equivalent to the bound states generated dynamically. Thirdly, even if $\det\mathbf{Z}=0$, there are cases in which $\tilde{d}(\infty)\neq 0$ because of

$$\sum_{j=1}^{N}(-)^{j+i}g_j d(m_j^2)D(j,i)=0 \quad \text{for} \quad N>1.$$

We have seen above that this is exactly the case where one elementary and one bound state occur in the two-particle problem. In particular, it turns out that if

$$\sum_{j=1}^{N}(-)^{j+n}g_j d(m_j^2)D(j,n)=0,$$

then the nth particle becomes composite and can be expressed by $D(m_n^2)=0$, while all others remain in the elementary state. We shall report on the three- and more-particle problems in detail elsewhere.[25]

V. CONCLUSIONS

We have shown that in the presence of many particles with identical quantum numbers, $\det\mathbf{Z}=0$ allows us to obtain composite states from the elementary particles and they are equivalent to the bound states generated dynamically. In the limit of $\det\mathbf{Z}=0$, there are two situations, i.e., either $\tilde{d}(\infty)=0$ or $\tilde{d}(\infty)\neq 0$. In the former case, the one-particle reducible part $A(s)$ of the amplitude does not decrease as fast as s^{-1} at high energies, and all the particles can be exhibited as composites of other particles; while in the latter case, $B\equiv\lim_{s\to\infty}[-s^{-1}A^{-1}(s)]$ is a positive-definite constant, and mixture of elementary and composite particles is possible. The condition $\det\mathbf{Z}=0$, in any case, makes at least one of the elementary particles become composite. Thus $\det\mathbf{Z}=0$ is a natural generalization of the compositeness condition in the case of many elementary particles with the same quantum numbers.

We have seen that letting one of the Z_{ij} vanish does not, in general, make all the Z_{ij} simultaneously tend to zero. Nor does it make $\tilde{d}(\infty)\to 0$, in general. In the two-particle problem, when $\det\mathbf{Z}=0$ with $B\neq 0$, we obtain one bound and one elementary state, and the off-diagonal element Z_{12} is related to the diagonal elements by (85b) and (86b). Thus if one sets one of the Z_{ij} in this case equal to zero, then all of the Z_{ij} vanish. But then the composite particle is indistinguishable from the elementary one. We remark again that the condition $\det\mathbf{Z}=0$ with $\tilde{d}(\infty)\neq 0$, not setting one of the Z_{ij} equal to zero, is the mechanism used to obtain one bound and one elementary particle in this case. We have just stated the results for the many-particle case at the end of the last section.

It should be mentioned that the condition $\det\mathbf{Z}=0$ with either $\tilde{d}(\infty)=0$ or $\tilde{d}(\infty)\neq 0$ would result, in general, in relations which contain parameters other than those of masses and coupling constants of the particles, unless one neglects all the CDD zeros except those between the particle poles in the amplitude $A(s)$ defined by (25). We have assumed no other CDD zeros of $A(s)$ in our discussions. Including the possible CDD zeros in the amplitude $A(s)$, however, does not change anything in our discussion of the compositeness condition $Z=0$.

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[25] K. Kang, Nuovo Cimento (to be published).