Zeros of the Partial-Wave Scattering Amplitude*

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Under the general assumptions of analyticity, unitarity, temperedness, and normal threshold behavior, the relation between the number of zeros and the high-energy behavior of the partial-wave scattering amplitude is studied. If the left-hand-cut discontinuity makes a finite number of sign changes, the maximum and minimum numbers of zeros can be determined by the high-energy upper and lower bound of the partial-wave respectively. In particular, for $C|s|^{-1+\epsilon}<|f_l(s)|<C$, the number of zeros is determined. After having determined the number of zeros, the number of subtractions as well as the sign of the scattering length is obtained for given number and character of zeros of the left-hand discontinuity $\Delta f_l(s)$.

I. INTRODUCTION

IN the application of the partial-wave dispersion relation, it is usual to regard the jump on the unphysical cut, $\Delta f_l(s)$, and the transmission factor $\eta_l(s)$ or inelasticity $1-\eta_l(s)$ as given information. Then, on using unitarity, the dispersion relation for $f_l(s)$ turns out to be an integral equation. In this approach, the partial-wave dispersion relation is compared to the nonrelativistic quantum theory, though the analogy is imperfect. In particular, the information on the jump over the left-hand cut which should correspond to the driving force is very scarce in a relativistic theory, though a simple-minded pole approximation could give reasonable results in the low-energy region. As to the inelasticity, a total neglect of this effect may also give a good approximation below the inelastic threshold. Rigorously speaking, there is no profound reason to treat the inelasticity as an input while keeping the elastic partial-wave amplitude as an unknown quantity to be determined by solving the integral equation.

Even if one takes this attitude of regarding the partial-wave dispersion relation as a closed integral equation, one must face the mathematical problem whether such an integral equation has a solution at all and whether such a solution is unique. These problems, however, have recently been studied in detail by Frye and Warnock[1] and by Balachandran.[2] In fact, among other things, they find conditions for $\Delta f_l(s)$ and $\eta_l(s)$ under which a solution exists and is unique. While it is well known that an unspecified number of the Castillejo-Dalitz-Dyson (CDD) zeros gives rise to the multiplicity of the solution,[3] the uniqueness condition on the solution entails certain restriction on the high-energy behavior.

In this paper, we address ourselves to the following problems. What is the relation between the number of zeros and the high-energy behavior of the partial-wave amplitude? How many subtractions are required for any given number of zeros? We assume only these general requirements: analyticity, unitarity, temperedness, and normal threshold behavior. In contrast to the conventional approach, we do not use any detailed information on $\Delta f_l(s)$ and $\eta_l(s)$, except that the left-hand-cut discontinuity does not change its sign infinitely many times. Consequently, our results are independent of the method of solution. Following a recent paper by A. Martin and one of us,[4] we use the Herglotz decomposition[5,6] of the partial-wave amplitude itself, instead of the Herglotz decomposition of the D function used in Refs. 1 and 2. Then $f_l(s)$ turns out to be a Herglotz function times an appropriate rational function which, among others, contains all of the zeros of $f_l(s)$ except at most one in the numerator, and contains at most one zero of $f_l(s)$ in the denominator. Then, by comparing the high-energy behavior of this decomposition of $f_l(s)$ with the given upper and lower bounds, one gets bounds on the number, as well as a classification, of zeros of $f_l(s)$. In particular, we impose the bound $C|s|^{-1+\epsilon}\leqslant|f_l(s)|\leqslant 1$ ($\epsilon>0$) and determine the number of zeros. Then it will be proved that if $\mu-l$ is odd, where μ is the number of sign changes of $\Delta f_l(s)/(s-4)^l$, the partial-wave dispersion relation does not need subtraction. Furthermore, using the classification of Herglotz functions appearing in the decomposition, the sign of the scattering length is also discussed. The results of our investigation are given in Tables I through IV.

The basic assumptions as well as the mathematical tools are introduced in Sec. II. In Sec. III, the Herglotz decomposition is explicitly carried out. In Sec. IV, the number of zeros is found, subject to given high-energy bounds on the partial-wave amplitudes. In Sec. V, the number of subtractions and the sign of the scattering length are discussed. Finally, in Sec. VI, our results are applied to the s- and p-wave amplitudes.

II. PRELIMINARIES

Let us consider the lth partial-wave amplitude $f_l(s)$ of the elastic scattering of two equal-mass spin-zero and isospin-zero particles normalized as

$$f_l(s)=(s/(s-4))^{1/2}e^{i\delta_l(s)}\sin\delta_l(s)\,, \tag{1}$$

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[1] G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).
[2] A. P. Balachandran, Ann. Phys. (N. Y.) 35, 209 (1965).
[3] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101, 453 (1956).
[4] Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964).
[5] K. Symanzik, J. Math. Phys. 1, 249 (1960), Appendix B.
[6] S. Weinberg, Phys. Rev. 124, 2049 (1961).

so that the unitarity reads

$$\mathrm{Im} f_l(s) \geqslant ((s-4)/s)^{1/2} |f_l(s)|^2. \qquad (2)$$

The inclusion of spin and isospin in our considerations is rather straightforward. However, the extension of our results to the case of unequal-mass scattering is not possible, unless one is prepared to neglect completely the well-known circular cut.

Our basic assumptions on the partial-wave amplitudes are as follows:

(I) $f_l(s)$ is analytic in the twice-cut s-plane with cuts $(-\infty,0]$ and $[4,\infty)$.

(II) $f_l(s)$ is bounded by a polynomial in arbitrary direction in the complex s plane, i.e.,

$$|f_l(s)| < C|s|^n.$$

(III) There are a finite number of sign changes of the absorptive part on the left-hand cut.

(IV) For $s \to 4$, $f_l(s) \to (s-4)^l$.

Though none of these assumptions has been rigorously proved, they are sufficiently weak to give us a general framework and to accommodate various models. (I) has been proved in every order of perturbation theory[7] and hence is a weaker assumption than the Mandelstam representation. (IV) can be shown, at least for $l \geqslant 2$, under the assumption of the one-dimensional dispersion relation in t. The assumption (II), in particular on the left-hand cut, may rather be doubtful if one assumes the Regge-cut in the l plane,[8] which would give rise to the behavior $f(-s,t) \sim s^{\sqrt{t}}$. There is no convincing theoretical argument to support assumption (III) except that it is one of the weakest assumptions one can make about the left-hand-cut discontinuity.

In what follows we shall work with the modified amplitude

$$\phi_l(s) = f_l(s)/(s-4)^l \qquad (3)$$

rather than $f_l(s)$ itself. Suppose the left-hand-cut discontinuity

$$\Delta\phi_l(s) = (1/2i)\{\phi_l(s+i\epsilon) - \phi_l(s-i\epsilon)\} \qquad (s \leqslant 0)$$

makes μ changes of sign (with respect to the sign of the right-hand cut) at $x_1 > x_2 > \cdots > x_\mu$. By introducing a new function

$$G_l(s) = \prod_{i=1}^{\mu} (s-x_i)\phi_l(s),$$

it is clear that

$$\mathrm{Im} G_l(s) \geqslant 0 \quad \text{for} \quad s \geqslant 4 \quad \text{and} \quad s \leqslant 0.$$

In general, the real part of the new function $G_l(s)$ will have either positive or negative slope at $s = x_i$. We shall call those real zeros where the real part has positive (negative) slope CDD (non-CDD) zeros. Let us further introduce a function

$$\psi_l(s) = \left(\sum_{i=1}^{\mu} (s-x_i)^{1-\epsilon_i}\right)^{-1} G_l(s),$$

where $\epsilon_i = +1$ or -1 depending on whether $s = x_i$ is a CDD or non-CDD zero of $G_l(s)$ on the left-hand cut. Thus we obtain

$$\psi_l(s) = \sum_{i=1}^{\mu} (s-x_i)^{\epsilon_i} \phi_l(s). \qquad (4)$$

Notice that

$$\mathrm{Im}\psi_l(s) \geqslant 0 \quad \text{for} \quad s \geqslant 4 \quad \text{and} \quad s \leqslant 0$$

and

$$|\psi_l(s)| < C|s|^{n+\nu-l}, \qquad (5)$$

where ν is the number of CDD zeros (ν_c) minus the number of non-CDD zeros (ν_n) of $G_l(s)$ on the left-hand cut, so that

$$\nu = \mu - 2\nu_n. \qquad (6)$$

For our discussion which follows, let us recall the properties of the Herglotz function.[9] A Herglotz function is a function analytic in $\mathrm{Im} z > 0$ with $\mathrm{Im} H(z) > 0$ for $\mathrm{Im} z > 0$. It admits the integral representation

$$H(z) = A + Bz + \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{(1+zx)\mathrm{Im} H(x)}{(1+x^2)(x-z)}, \qquad (7)$$

with

$$B \geqslant 0, \quad \mathrm{Im} H(z) \geqslant 0$$

and

$$\int_{-\infty}^{\infty} dx \frac{\mathrm{Im} H(x)}{1+x^2} < \infty. \qquad (8)$$

In addition, since $-H^{-1}(x)$ is also a Herglotz function,

$$\int_{-\infty}^{\infty} dx \frac{\mathrm{Im} H(x)}{(1+x^2)|H(x)|^2} < \infty \qquad (9)$$

and from (7) it follows that for $\epsilon < \arg z < \pi - \epsilon$,

$$C/|z| < |H(z)| < C'|z| \qquad (10)$$

and

$$B = \lim_{|z| \to \infty} H(z)/z. \qquad (11)$$

III. HERGLOTZ DECOMPOSITIONS

Let the zeros of $\psi_l(s)$ in the cut s plane, i.e., in the interior of the analyticity domain, be z_1, z_1^*, z_2, z_2^*, $\cdots z_p$, z_p^* (complex zeros) and $s_1 < s_2 < \cdots < s_q$ (real zeros). (Note that the analyticity does not put any restriction on possible zeros on the cut. However, if one considers, for instance,[4] an averaged function

$$\bar{\psi}_l(s) = \frac{1}{\Delta} \int_s^{s+\Delta} \psi_l(s') ds'$$

[7] J. G. Taylor, Nuovo Cimento **22**, 92 (1961); N. Nakanishi, Phys. Rev. **126**, 1225 (1962).

[8] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).

[9] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943).

with an arbitrary small Δ, there will be no zeros on the cut at all.) Following the discussion in Ref. 4, let us study the following three cases separately.

(1) q is even. In this case, remove all zeros and define

$$H_1(s)=\Big(\prod_{i=1}^{p}(s-z_i)(s-z_i^*)\prod_{j=1}^{q}(s-s_j)\Big)^{-1}\psi_l(s). \quad (12)$$

Then it turns out that $H_1(s)$ is a Herglotz function without zero. Hence from (10) it follows that

$$C|s|^{2p+q-1}<|\psi_l(s)|<C'|s|^{2p+q+1}. \quad (13)$$

By comparison with (5) it is clear that

$$2p+q\leqslant n+\nu-l+1. \quad (14)$$

If there is an odd number of real zeros and s_1 is a CDD (non-CDD) zero, then s_q is also a CDD (non-CDD) zero and there are $(q+1)/2$ CDD (non-CDD) and $(q-1)/2$ non-CDD (CDD) zeros. We consider the two cases separately.

(2) q is odd and s_1 is a CDD zero. Define

$$H_2(s)=\Big(\prod_{i=1}^{p}(s-z_i)(s-z_i^*)\prod_{j=2}^{q}(s-s_j)\Big)^{-1}\psi_l(s); \quad (15)$$

then $H_2(s)$ can be shown to be a Herglotz function with a zero at $s=s_1$. Thus we get

$$C|s|^{2p+q-2}<|\psi_l(s)|<C'|s|^{2p+q}, \quad (16)$$

and together with (5),

$$2p+q\leqslant n+\nu-l+2. \quad (17)$$

(3) q is odd and s_1 is a non-CDD zero. Define

$$H_3(s)=\Big((s-s_1)^2\prod_{i=1}^{p}(s-z_i)$$
$$\times(s-z_i^*)\sum_{j=2}^{q}(s-s_j)\Big)^{-1}\psi_l(s); \quad (18)$$

then $H_3(s)$ is a Herglotz function and has a pole at $s=s_1$ with a negative residue and has no zeros. Consequently

$$C|s|^{2p+q}<|\psi_l(s)|<C'|s|^{2p+q+2} \quad (19)$$

and by applying (5) we get

$$2p+q\leqslant n+\nu-l. \quad (20)$$

By generalizing these results, the partial-wave amplitude can be written as

$$f_l(s)=R(s)H(s), \quad (21)$$

where $R(s)$ is an appropriately chosen rational function of s and $H(s)$ is a Herglotz function. Therefore, for large s

$$f_l\sim|s|^N|H(s)|, \quad (22)$$

where N is an integer, and thus from (8) and (9) it follows that

$$\int_{-\infty}^{\infty}ds\frac{|s|^{-N}\,\mathrm{Im}f_l(s)}{1+s^2}<\infty \quad (23)$$

and

$$\int_{-\infty}^{\infty}ds\frac{|s|^{N}\,\mathrm{Im}f_l(s)}{(1+s^2)|f_l(s)|^2}<\infty. \quad (24)$$

By using the unitarity relation (2), which implies $|f_l(s)|<1$, we see that Eq. (24) entails

$$\int_{4}^{\infty}ds\frac{|s|^{N}\,\mathrm{Im}f_l(s)}{1+s^2}<\infty. \quad (25)$$

We now have the following three cases:

(a) $N\leqslant-1$ In this case (23) implies that an unsubtracted dispersion relation holds for $f_l(s)$.

(b) $N=0$. Since $f_l(s)$ is proportional to an H function, from (8) it is clear that for $f_l(s)$ one subtraction is sufficient.

(c) $N\geqslant1$. Although (25) entails

$$\int_{4}^{\infty}ds\frac{\mathrm{Im}f_l(s)}{s}<\infty \quad (26)$$

and hence no subtraction at least for the physical-cut dispersion integral, in general it does not necessarily imply that

$$\int_{4}^{\infty}ds\frac{\Delta f_l(s)}{s}<\infty, \quad (27)$$

where $\Delta f_l(s)$ denotes the discontinuity across the left-hand cut. This is due to the possible oscillation along either of the two cuts, i.e., the blowing up of

$$\frac{\lim_{s\to\pm\infty}\sup|f_l(s)|}{\lim_{s\to\pm\infty}\inf|f_l(s)|},$$

and this kind of trouble cannot be avoided, unless one assumes one form or another of the smoothness condition.

For this reason in what follows we shall assume that (V) $f_l(s)$ is a smooth function in s on both cuts.

As a definition of smoothness, we may take for instance those conditions for the Sugawara-Kanazawa theorem,[10] i.e., that $f_l(s\pm i\epsilon)$ has a finite limit for $s\to\infty$ and the discontinuity across the negative cut has a definite limit (finite or infinite) for $s\to-\infty$. Then one can show that (26) implies (27). Therefore, under the assumption (V), in the case $N\geqslant1$, i.e., (c), the dispersion relation for $f_l(s)$ does not require subtraction. On the other hand, we get from (22) that

$$C'|s|^{N+1}\geqslant|f_l(s)|\geqslant C|s|^{N-1}. \quad (28)$$

[10] M. Sugawara and A. Kanazawa, Phys. Rev. 123, 1895 (1961).

However, since $f_l(s)$ needs no subtraction for $N\geqslant 1$ under the assumption of smoothness, that assumption is inconsistent with (28). In particular, the cases $N>1$ will violate the unitarity condition $\text{Im}f_l(s)\geqslant |f_l(s)|^2$ as $s\to\infty$, and those cases are physically not allowed.

IV. NUMBER OF ZEROS OF THE PARTIAL-WAVE AMPLITUDE

As was mentioned in the foregoing section, it is an extremely difficult problem to relate the asymptotic behavior in the complex direction to that along the cut, unless one introduces one form or another of smoothness condition on the behavior of the function on the cut, i.e., on the boundary of the analyticity domain. Besides, we question very much the usefulness of the dispersion relation at all if there is such a pathologically divergent oscillation. For these reasons, in what follows we shall keep the assumption (V). By using unitarity and the Phragmén-Lindelöf theorem[11] it is then clear that

$$|f_l(s)|<C \tag{29}$$

in every direction in the s plane. Hence the results on the maximum number of zeros, i.e., (14), (17), and (20) are valid with $n=0$.

Without losing generality, let us now impose a lower bound which together with (29) gives

$$C'|s|^{-1+\epsilon}<|f_l(s)|<C, \tag{30}$$

where ϵ is an arbitrarily small positive number. In Eq. (30) l is fixed and finite, and the constant C' may depend on l. If one wants to consider any faster decrease of $f_l(s)$ in s, then one can certainly replace (30), for instance, by

$$C'|s|^{-m+\epsilon}<|f_l(s)|<C, \tag{31}$$

etc. Physically speaking, the bound (30) is general enough to include the high-energy behavior of $f_l(s)$ which follows from that of the scattering amplitude $f(s,t)$ in the near-forward region, i.e., $f(s,t)=sf(t)$ or $f(s,t)=\beta(t)s^{\alpha(t)}$ [with $\alpha(0)=1$], which gives $f_l(s)\sim$constant or $f_l(s)\sim(\ln s)^{-1}$, respectively.

Now let us compare the bounds (30) with those obtained from the Herglotz decompositions, (13), (16), and (19), with $n=0$. As the lower bound of (30) must be smaller than the upper bounds given in those equations, it is clear that the following inequalities hold.

(1) If q is even,

$$\nu-l-1\leqslant 2p+q\leqslant\nu-l+1. \tag{32}$$

(2) If q is odd and s_1 is a CDD zero,

$$\nu-l\leqslant 2p+q\leqslant\nu-l+2. \tag{33}$$

(3) If q is odd and s_1 is a non-CDD zero,

$$\nu-l-2\leqslant 2p+q\leqslant\nu-l. \tag{34}$$

[11] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1943), p. 179.

TABLE I. Total number of zeros, $2p+q$, of $f_l(s)$ for given ν, where $2p$ and q are the numbers of complex and of real zeros, respectively.

	$\nu-l$ is even	$\nu-l$ is odd
q is even	$2p+q=\nu-l$	$2p+q=\nu-l-1$
q is odd and s_1 is CDD	$2p+q=\nu-l+1$	$2p+q=\nu-l$
q is odd and s_1 is non-CDD	$2p+q=\nu-l-1$	$2p+q=\nu-l-2$

Had we used the bounds (31) instead of (30), in the lower limit of (32) through (34), we should only have had to replace $\nu-l$ with $\nu-l-m+1$.

At this point we should remark that in Ref. 4 it has been proved under the assumptions (I), (II), (III), and the threshold behavior (IV) that $\mu-l\geqslant 0$.[12] A more refined statement of this theorem is given by (14), (17), and (20) in Sec. III. Under our assumptions (I)–(IV), one can easily see that in general $\nu-l\geqslant -1$. The case $\nu-l=-1$, however, is not allowed under the assumption (V), as is discussed below. Thus we have $\mu-l\geqslant 2\nu_n$. Note that μ is the number of changes of sign of $\Delta\phi_l(s)$, so that $\Delta f_l(s)$ has μ or $\mu\pm 1$ changes of sign. We shall now discuss separately the possible number of zeros for the cases when $\nu-l$ is even and when $\nu-l$ is odd. Notice that if $\nu-l$ is even (odd) then $\mu-l$ is also even (odd). For instance, if $\nu-l$ is even and q is even, then from (32) it is clear that $2p+q=\nu-l$, since $\nu-l\pm 1$ is odd. Suppose $\nu-l$ is odd and q is even; then $2p+q=\nu-l\pm 1$, since $\nu-l\pm 1$ is even. Hence from (3), (4), and (13) one gets for large s

$$f_l(s)\sim|s|^{\pm 1}|H(s)|,$$

where $H(s)$ is an H function. If $f_l(s)\sim|s||H(s)|$, this is exactly the case (c) of Sec. III and hence should be rejected. Therefore, $2p+q=\nu-l+1$ is ruled out and we are left with $2p+q=\nu-l-1$. For the same reason, the cases $2p+q=\nu-l+2$ of (33) and $2p+q=\nu-l$ of (34) are not allowed. By completing these arguments, we obtain the results in Table I.

Before concluding this section, the following remark should be made. If the partial-wave amplitude has fewer zeros than shown in Table I, it automatically follows that $|f_l(s)|\leqslant|s|^{-1}$. On the other hand, should $f_l(s)$ have more zeros than shown in Table I, this would imply that $f_l(s)$ violates unitarity.

V. SCATTERING LENGTHS AND SUBTRACTIONS OF THE PARTIAL-WAVE DISPERSION RELATIONS

In the previous section we have fixed the total number of zeros of the partial-wave amplitude $f_l(s)$, for given ν of $\Delta\phi_l(s)$ along the left-hand cut. Having fixed the number of zeros, the Herglotz decomposition of the partial-wave amplitudes $f_l(s)$ given in Sec. III is unambiguous. For instance, if q is even and $\nu-l$ is even,

[12] It was also shown by P. Beckmann, Z. Physik **179**, 379 (1964).

TABLE II. Asymptotic behavior of $f_l(s)=R(s)H(s)$ for given ν. $H_1(s)$, $H_2(s)$, and $H_3(s)$ are Herglotz functions defined in Sec. III.

	$\nu-l$ is even	$\nu-l$ is odd
q is even	$f_l(s)\sim H_1(s)$	$f_l(s)\sim s^{-1}H_1(s)$
q is odd and s_1 is CDD	$f_l(s)\sim H_2(s)$	$f_l(s)\sim s^{-1}H_2(s)$
q is odd and s_1 is non-CDD	$f_l(s)\sim H_3(s)$	$f_l(s)\sim s^{-1}H_3(s)$

then from Table I $2p+q=\nu-l$. Hence from (3), (4), and (12)

$$f_l(s)=\left[(s-4)^l\prod_{i=1}^{p}(s-z_i)(s-z_i^*)\prod_{i=1}^{q}(s-s_i)\Big/\prod_{i=1}^{\mu}(s-x_i)^{\epsilon_i}\right]H_1(s)\underset{s\to\infty}{\approx}H_1(s),\quad(35)$$

where $H_1(s)$ is an H function with no zeros. If q is even and $\nu-l$ is odd, then $2p+q=\nu-l-1$ and one gets

$$f_l(s)=\left[(s-4)^l\prod_{i=1}^{p}(s-z_i)(s-z_i^*)\prod_{i=1}^{q}(s-s_i)\Big/\prod_{i=1}^{\mu}(s-x_i)^{\epsilon_i}\right]H_1(s)\underset{s\to\infty}{\approx}s^{-1}H_1(s).\quad(36)$$

By completing these arguments, we obtain Table II, in which the asymptotic behavior of $f_l(s)$ in different cases is given. In connection with Table II, the following remarks are in order.

(1) If $\mu-l$ is even, then the partial-wave dispersion relation may need one subtraction. Remember that even $\mu-l$ corresponds to even $\nu-l$.

(2) If $\mu-l$ is odd, then the partial-wave dispersion relation does not need subtraction.

(3) From (1) and (2), it is clear that if $f_l(s)\sim$constant or $f_l(s)\sim 1/\ln s$, then $\mu-l$ must be even. Thus, $f(s,t)\sim sf(t)$ or $f(s,t)\sim\beta(t)s^{\alpha(t)}$ implies that $\mu-l$ should be even.

(4) The scattering length

$$a_l=\lim_{s\to 4}[4/(s-4)]^l f_l(s)\quad(37)$$

has the same sign as that of $H_i(4)$. This is clear, e.g., from (35) or (36).

We shall now discuss the relation between the zeros and the scattering lengths. These Herglotz functions which we have introduced separately in Sec. II and in Table II, and which appear in our discussion, are of the following three types:

$H_1(s)$: has no zero and no pole.
$H_2(s)$: has one CDD zero and no pole.
$H_3(s)$: has one pole with negative residue and no zero.

The behavior of these H functions between the two branch points is shown in Fig. 1. Thus we have

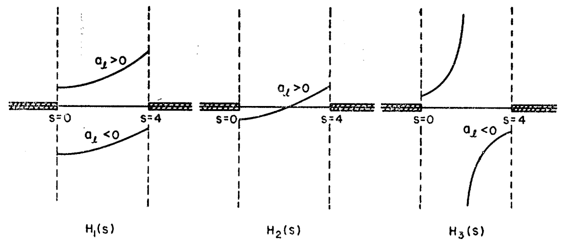

FIG. 1. The three types of the Herglotz function appear in our discussion.

$$H_1(4)H_1(0)>0,$$
$$H_2(4)>0,\quad H_2(0)<0,$$
$$H_3(4)<0,\quad H_3(0)>0,\quad(38)$$

where the sign of $H_i(4)$ is the same as that of the scattering length defined by (37). Furthermore, it is obvious that

$$H_i(0)\phi_l(0)>0,\quad(39)$$

where

$$\phi_l(0)=(-)^l 4^{-l}f_l(0)\quad(40)$$

and $i=1$, 2, and 3. Hence we get Table III in which the allowed situations are summarized.

For $l\geqslant 2$, however, Table III can be reduced further (see Table IV) by using a result obtained by Martin and one of us[13]: that $a_l>0$. This conclusion is based on the positiveness of the absorptive part $\mathrm{Im}f(s,t)$ and all of its derivatives with respect to t in $0\leqslant t<4$ for a completely (s,t,u)-symmetric amplitude. Thus, for even $l\geqslant 2$, $a_l<0$ should be ruled out. Hence the case with odd q and non-CDD s_1 can not occur. This is also true for the $I=0$ amplitude of $\pi\pi$ scattering, as is shown in the following section.

TABLE III. Allowed cases.

q is even	$a_l>0,\ \phi_l(0)>0$ or $a_l<0,\ \phi_l(0)<0$
	$\nu\geq\binom{l}{l+1}$ for $\binom{\text{even}}{\text{odd}}\nu-l$
q is odd and s_1 is CDD	$a_l>0,\ \phi_l(0)<0$
	$\nu\geq\binom{l}{l+1}$ for $\binom{\text{even}}{\text{odd}}\nu-l$
q is odd and s_1 is non-CDD	$a_l<0,\ \phi_l(0)>0$
	$\nu\geq\binom{l+2}{l+3}$ for $\binom{\text{even}}{\text{odd}}\nu-l$

TABLE IV. Allowed cases for even $l\geq 2$ when the total amplitude has s, t, u symmetry.

q is even	$a_l>0,\ f_l(0)>0$
q is odd, s_1 is CDD	$a_l>0,\ f_l(0)<0$

[13] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1375 (1964); A. Martin CERN report (unpublished).

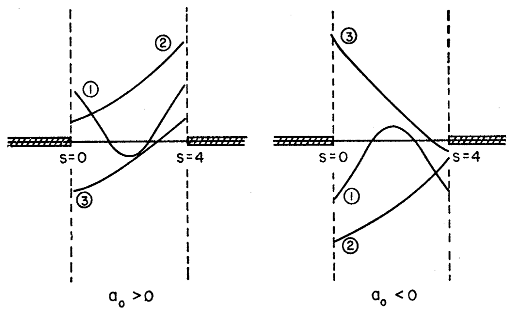

FIG. 2. The behavior of the s-wave amplitude between the two branch points when the left-hand discontinuity makes two changes of sign of the CDD type. ① represents ($p=0$, $q=2$), ② ($p=1$, $q=0$), and ③ ($p=1$, $q=1$ CDD or non-CDD).

It should be emphasized that this result holds for even $\nu-l$ as well as for odd $\nu-l$, and is consequently independent of the subtraction.

VI. THE *S*- AND *P*-WAVE AMPLITUDES

In this section, we shall apply the results obtained in Sec. V to the s- and p-wave amplitudes.

A. The *s*-Wave Amplitude

For the s wave, the number of sign changes in $\Delta\phi_0(s)$ along the left-hand cut is the same as that of $\Delta f_0(s)$. From the discussions of the preceding sections, the number ν in Eq. (6) should be equal to or larger than zero. This means that $\mu \geqslant 2\nu_n$. Notice that $\nu_c \geqslant \nu_n$.

(i) The case $\mu=0$. Then we have $\nu_n=\nu_c=\nu=0$. Here, from Table III, only three situations are possible. For $a_0>0$, either $2p+q=0$ or $2p=0$ with $q_{\text{CDD}}=1$. For $a_0<0$, $2p+q=0$. Thus if the s-wave amplitude has no sign changes of the discontinuity (with respect to the unitarity cut) along the left-hand cut, the amplitude can have no complex zeros except for at most one CDD zero. The s-wave amplitude may in this case need one subtraction, and will increase monotonically between $s=0$ and $s=4$. In particular, the solution should exhibit $a_0-f_0(0)\geqslant 0$.

(ii) The case $\mu=1$. Then we have $\nu_n=0$ and $\nu_c=\nu=1$. Again three situations are available from Table III. For $a_l>0$, $2p+q=0$, and $2p+q=1$, while for $a_0<0$, $2p+q=0$. Thus if the left-hand cut of the s-wave amplitude has one sign change (with respect to the unitarity cut), then it is a CDD zero of $G_0(s)$, and the amplitude can have no complex zeros and at most one CDD zero. The amplitude will need no subtractions in this case, but will have the same behavior between $s=0$ and $s=4$ as that of the case $\mu=0$.

(iii) The case $\mu=2$. Here we have either $\nu_n=\nu_c=1$ or $\nu_n=0$ and $\nu_c=2$, so that $\nu=0$ or $\nu=2$. The case $\nu=0$ is the same as the case $\mu=0$. For $\nu=2$, we have four cases. Namely, for $a_0>0$, $2p+q=2$ and $2p+1=3$; and for $a_0<0$, $2p+q=2$ and $2p+1=1$. This situation is explained in Fig. 2 and can be summarized as follows:

For $a_0>0$,

(1) no complex zeros and two real zeros;
(2) a pair of complex zeros and no real zeros;
(3) a pair of complex zeros and one CDD zero.

For $a_0<0$,

(1) no complex zero and two real zeros;
(2) a pair of complex zeros and no real zeros;
(3) no complex zeros and one non CDD zero.

The s-wave amplitude in this case may need one subtraction. The cases with more oscillations can be explored in a similar manner.

B. The p-Wave Amplitude

The left-hand-cut discontinuity $\Delta\phi_1(s)$ for p wave should have at least one sign change.

(i) The case $\mu=1$ for $\phi_1(s)$ is exactly the same as the case $\mu=0$ for the s-wave amplitude.

(ii) The case $\mu=2$ $\phi_1(s)$ is the same as the case $\mu=1$ for $f_0(s)$.

(iii) The case $\mu=3$ for $\phi_1(s)$ is again the same as the case $\mu=2$ for $f_0(s)$.

C. The *d*-Wave and Higher Partial-Wave Amplitudes

As is clearly seen from the above discussion, $\phi_l(s)$ can be discussed in the same manner as in the s wave case. In general the case $\mu-l=0$ for $\phi_l(s)$ will always be the same as the case $\mu=0$ for $f_0(s)$, while $\mu-l=1$ for $\phi_l(s)$ will be the same as $\mu=1$ for the s-wave amplitude, and so on. We remark again that $\mu-l\geqslant 2\nu_n$ and $\nu_c-l\geqslant\nu_n$.

However, as is mentioned in Sec. V, the scattering lengths for $l=2, 4,\cdots$ are always positive if the total amplitude has complete (s,t,u) symmetry. In particular, it was shown in Ref. 13 that

$$a_l=\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\frac{1}{\sqrt{\pi}}\int_4^\infty ds\frac{A_s(s,t=4)}{s^{l+1}} \tag{41}$$

or $l=2$, 4, etc.

We can also get similar results for the pion-pion scattering case. By starting with a t-fixed dispersion relation for $A^I(t,s)$ $(I=0, 1, 2)$

$$A^I(t,s)=\frac{1}{\pi}\int_4^\infty ds'\frac{A_s{}^I(t,s',u)}{s'-s}+\frac{1}{\pi}\int_4^\infty du'\frac{A_u{}^I(t,s,u')}{u'-u}, \tag{42}$$

where, from crossing symmetry,[14]

$$A_s{}^I(t,s,u)=\sum_{I'}\chi_{II'}A_s{}^{I'}(s,t,u) \tag{43}$$

and

$$A_u{}^I(t,s,u)=(-)^I\sum_{I'}\chi_{II'}A_u{}^{I'}(u,t,s), \tag{44}$$

[14] See, e.g., Kyungsik Kang, Phys. Rev. 134, B1324 (1964).

we have the Froissart-Gribov representation[15,16] for the partial waves for $l\geqslant 2$ and $0\leqslant t<4$. By taking $t\to 4$ in the representation

$$A_l^I(t)=\frac{1}{\pi k_t^2}\int_4^\infty dsQ_l\left(1+\frac{s}{2k_t^2}\right)\sum_{I'}\chi_{II'}A_s^{I'}(s,t,u)\,, \quad (45)$$

we can obtain an expression for the scattering lengths,

$$a_l^I=\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\frac{1}{\sqrt{\pi}}\int_4^\infty\frac{ds}{s^{l+1}}\sum_{I'}\chi_{II'}A_s^I(s,t=4)\,, \quad (46)$$

provided the s waves have normal threshold behavior. From the positiveness of the absorptive parts and of certain elements of the crossing matrix $(\chi_{II'})$, we conclude again that $a_l^{I=0}$ for $l\geqslant 2$ is always positive. In a similar manner, one can also get

$$a_l^{I=0}+2a_l^{I=2}\geqslant 0\,, \quad (47)$$

which means $a_l^{I=2}\geqslant -\frac{1}{2}a_l^{I=0}$ for $l=2, 4, \cdots$ etc.

VII. CONCLUDING REMARKS

Starting with general requirements of analyticity, unitarity, temperedness, and normal threshold behavior of the partial-wave scattering amplitude, we have fixed the number as well as the type of zeros of the amplitude. This information is used to investigate the high-energy behavior of the partial-wave amplitude. Instead of assuming the left-hand-cut discontinuity $\Delta f_l(s)$ and the transmission factor $\eta_l(s)$ as given, we have assumed that $\Delta f_l(s)$ has only a finite number of sign changes along the left-hand cut, while the inelastic effects are included in the general unitarity requirement. Our discussions are based on the Herglotz decomposition of the partial-wave amplitude, which is an extended version of the Symanzik theorem[5] in the twice-cut plane. To determine the necessary number of subtractions of $f_l(s)$ in the twice-cut plane, we are forced to assume the smoothness of $f_l(s)$ on both right- and left-hand cuts.

These rather general discussions have provided us much useful information on the partial-wave scattering amplitudes. The zeros of the amplitude have usually been taken less seriously than its poles. We have demonstrated the importance of zeros of the amplitude in connection with its high-energy behavior. Moreover, we feel it necessary to obtain information on the high-energy behavior of the amplitude, so that we can write down an appropriately subtracted (or unsubtracted) dispersion relation for the partial-wave amplitude. For then, investigation into the conditions for the existence and uniqueness of the solution to the partial-wave dispersion relation will be in order.

Zeros of the partial-wave scattering amplitude have also provided the sign of the scattering length as well as the behavior of the amplitude in the energy region between the two branch points. This information will will be useful when one deals with consistency problems. Thus, zeros of the partial-wave amplitude are related not only to the high-energy behavior but also to the low-energy behavior of the amplitude.

We mention again that if the partial-wave amplitude has fewer zeros than shown in Table I, then the amplitude will decrease faster than $|s|^{-1}$, while if it has more, then $f_l(s)$ will violate unitarity. There may of course be some solutions which will still give a good approxition to the amplitude in the low-energy region even if they do not satisfy our consistency conditions.

We remark that our assumption on the lower and upper bounds on the partial-wave amplitude is quite general enough to cover the cases that are consistent with the known behavior and narrow diffraction peak of $f(s,t)$.

Our approach to the partial-wave amplitude reflects the power and usefulness of Herglotz functions in dispersion theory. Since we have specified the zeros of the amplitude which in turn give information on both the high- and the low-energy behavior of the amplitude, we are now prepared to proceed to the method of solution of the partial-wave dispersion relations.[17]

Note added in proof: After submission of this paper for publication it came to our attention that parts of the results contained in this paper were also obtained by T. Kinoshitz, Phys. Rev. Letters **16**, 869 (1966). We received also an unpublished report by A. Martin, in which he proves $f_l(s)>0$ for $0\leq s<4$ by making more elaborate use of the positiveness of the absorptive part $\mathrm{Im}f(s,t)$. Hence, in Table IV, the case with q odd can further be dropped. His study reveals that the s-wave $f_0(s)$ cannot increase monotonically in $0\leq s<4$. This would imply that the cases with $\mu=0$ and $\mu=1$ for $f_0(s)$ should be excluded. We thank Professor A. Martin for informing us of his results.

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[15] M. Froissart, Proceedings of the La Jolla Conference on Weak and Strong Interactions, 1961 (unpublished).

[16] V. N. Gribov, Zh. Eksperim i Teor. Fiz. **41**, 1962 (1961) [English transl: Soviet Phys.—JETP **14**, 1395 (1962)].

[17] Y. S. Jin and Kyungsik Kang (to be published).