Particles and Sources

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It is proposed that the phenomenological theory of particles be based on the source concept, which is abstracted from the physical possibility of creating or annihilating any particle in a suitable collision. The source representation displays both the momentum and the space-time characteristics of particle behavior. Topics discussed include: spin and statistics, charge and the Euclidean postulate, massless particles, and SU_3 and spin. It is emphasized that the source description is logically independent of hypotheses concerning the fundamental nature of particles.

INTRODUCTION

THE particle is presently the central concept in interpreting the raw data of high-energy experimental physics. The discovery of the meson η^* (960 MeV) by studying mass correlations among five π mesons[1] is a recent example of the experimental procedure that defines a particle primarily by energy and momentum characteristics. The latter aspect of particle behavior is of such obvious significance that many theoretical studies in particle phenomenology concentrate on it completely. But the particle has another, complementary, aspect—a degree of spatial localizability. Is it possible to give a useful phenomenological definition or characterization of particles that does not emphasize unduly either of these complementary facets of the particle concept?

Any particle can be created in a collision, given suitable partners before and after the impact to supply the appropriate values of the spin and other quantum numbers, together with enough energy to exceed the mass threshold. In identifying new particles it is a basic experimental principle that the specific reaction is not otherwise relevant. Then let us abstract from the physical presence of the additional particles involved in creating a given one, and consider them simply as the *source* of the physical properties that are carried by the created particle. The ability to give some localization in space and time to a creation act may be represented by a corresponding coordinate dependence of a mathematical source function, $S(x)$. The effectiveness of the source in supplying energy and momentum may be described by another mathematical source function, $S(p)$. The complementarity of these source aspects can then receive its customary quantum interpretation, as illustrated by

$$S(p)=\int (dx)e^{-ipx}S(x).$$

The source of a particular particle must also have the multiplicity necessary to represent its spin and those internal quantum numbers appropriate to the dynamical level of description that is used.

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[1] G. R. Kalbfleisch *et al.*, Phys. Rev. Letters **12**, 527 (1964); M. Goldberg *et al.*, *ibid.* **12**, 546 (1964).

For simplicity, in the following we shall only consider a restricted time scale such that the possible instability of any particle is not significant. This restriction can always be removed. Particles of zero mass will receive special attention, and it is otherwise understood that the mass does not vanish.

SPINLESS PARTICLES

As a first step in supplying a quantitative framework for the source concept consider a spinless particle of mass m, without internal quantum numbers. We begin with the vacuum state $|0\rangle$, and then let a weak source operate in some space-time region. The probability amplitude for the generation of a single particle in a state specified by a small momentum range about the momentum $\mathbf{p}$ will be written

$$\langle 1_p|0_-\rangle^S=\left[\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\right]^{1/2} i\int (dx)e^{-ipx}S(x),$$
$$p^0=+(\mathbf{p}^2+m^2)^{1/2},$$

which is an invariant expression if $S(x)$ is transformed as a scalar. The subscript on the vacuum state indicates the time sense—this is the vacuum state before the source has acted. The probability amplitude for the inverse process appears as

$$\langle 0_+|1_p\rangle^S=\left[\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\right]^{1/2} i\int (dx)e^{ipx}S(x),$$

which refers to the vacuum state after the source, now functioning as a sink, has acted to annihilate the particle. The two processes are related by the "TCP" operation of complex conjugation and space-time coordinate reflection. The factors of i have been inserted to make these expressions consistent with the further restriction of $S(x)$ to be a real function. The latter property symbolizes the reciprocity between creation and annihilation mechanisms.

These postulated representations of the creation and annihilation aspects of a source can be united by considering the vacuum probability amplitude

$$\langle 0_+|0_-\rangle^S,$$

where

$$S=S_1+S_2$$

and S_2, effectively localized in time prior to S_1, creates a particle which is subsequently annihilated by S_1. The notion of weak source, which remained unexplained before, means that only single-particle exchange between the sources is numerically significant. This is expressed by the probability-amplitude composition law

$$\langle 0_+|0_-\rangle^S \cong \langle 0_+|0_-\rangle^{S_1}\langle 0_+|0_-\rangle^{S_2}+\sum_p \langle 0_+|1_p\rangle^{S_1}\langle 1_p|0_-\rangle^{S_2}$$
$$\cong 1+\int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\int (dx)(dx')iS_1(x) \times e^{ip(x-x')}iS_2(x').$$

The second form involves a further, but temporary simplification, which is to retain only contributions that are linear in each partial source. A vacuum amplitude like $\langle 0_+|0_-\rangle^{S_2}$, which is restricted by the probability condition

$$1\cong |\langle 0_+|0_-\rangle^S|^2+\sum_p |\langle 1_p|0_-\rangle^S|^2,$$

should deviate from unity by terms that are quadratic functions of S_2. These effects are reinstated by writing $\langle 0_+|0_-\rangle^S$ as a functional of S, rather than of the constituent sources S_1 and S_2. That is accomplished by the expression

$$\langle 0_+|0_-\rangle^S \cong 1+\tfrac{1}{2}i\int (dx)(dx')S(x)\Delta_+(x-x')S(x'),$$

where

$$\Delta_+(x-x')=\Delta_+(x'-x)$$
$$=i\int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{ip(x-x')},\quad x^0>x^{0\prime}.$$

The function $\Delta_+(x)$ is invariant with respect to the transformations of the proper, orthochronous, homogeneous Lorentz group. The additional symmetry,

$$\Delta_+(-x)=\Delta_+(x),$$

is also conveyed by the statement of invariance under the attached Euclidean rotation group ($x_4=ix^0$).

The quadratic terms in S_1 and S_2 that appear in $\langle 0_+|0_-\rangle^S$ reproduce the structure of the product of $\langle 0_+|0_-\rangle^{S_1}$ and $\langle 0_+|0_-\rangle^{S_2}$. Furthermore, the property that

$$-\mathrm{Re}\, i\Delta_+(x-x')=\mathrm{Re}\int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{ip(x-x')},$$

for all $x-x'$, leads immediately to the necessary relations

$$|\langle 0_+|0_-\rangle^S|^2 \cong 1-\sum_p |\langle 1_p|0_-\rangle^S|^2,$$
$$\cong 1-\sum_p |\langle 0_+|1_p\rangle^S|^2.$$

When the production source S_2 and the detection source S_1 are located within certain macroscopic volumes, the trajectory of the exchanged particle is correspondingly limited. All this defines a region associated with the specified particle. Outside that region other independent acts of creation and detection can be considered, and similarly represented. Thus, the restriction to weak sources, or single-particle states, is easily removed subject to the limitation that the various particles remain physically independent, in virtue of their spatial separation. This situation is described by multiplying the vacuum probability amplitudes associated with the various independent source pairs,

$$\langle 0_+|0_-\rangle^S=\prod\left[1+\tfrac{1}{2}i\int (dx)(dx')S(x)\Delta_+(x-x')S(x')\right].$$

To represent this as a property of a single source, uniting the several spatially isolated parts, we have only to write

$$\langle 0_+|0_-\rangle^S=\exp\left[\tfrac{1}{2}i\int (dx)(dx')S(x)\Delta_+(x-x')S(x')\right].$$

Here is the simplest example of our answer to the problem of giving a phenomenological particle representation in which both localization and momentum aspects receive proper attention. The corresponding momentum-space formula is

$$\langle 0_+|0_-\rangle^S=\exp\left[\tfrac{1}{2}i\int \frac{(dp)}{(2\pi)^4}S(-p)\frac{1}{p^2+m^2-i\epsilon}S(p)\right],$$

which involves the limiting process $\epsilon\to +0$.

It is a reasonable extrapolation to apply this result under conditions for which the interactions among the particles are still not significant but the particles need not be macroscopically isolated, so that microscopic quantum interference effects come into play. Consider again a production and detection source S_2 and S_1, respectively, and write

$$\langle 0_+|0_-\rangle^S=\langle 0_+|0_-\rangle^{S_1}\times\exp\left[i\int (dx)(dx')S_1(x)\Delta_+(x-x')S_2(x')\right]\times\langle 0_+|0_-\rangle^{S_2}$$
$$=\sum_{\{n\}}\langle 0_+|\{n\}\rangle^{S_1}\langle\{n\}|0_-\rangle^{S_2}.$$

To identify the individual multiparticle probability amplitudes, let us note that

$$i\int (dx)(dx')S_1(x)\Delta_+(x-x')S_2(x')=\sum_p iS_{1p}{}^*iS_{2p},$$

where

$$iS_p=\left[\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\right]^{1/2}iS(p)$$

is just the quantity designated by $\langle 1_p|0_-\rangle^S$ in the discussion of a weak source. We now find that

$$\langle\{n\}|0_-\rangle^S=\langle 0_+|0_-\rangle^S\prod_p (iS_p)^{n_p}/[n_p!]^{1/2}$$

and

$$\langle 0_+|\{n\}\rangle^S = \langle 0_+|0_-\rangle^S \prod_p (iS_p^*)^{n_p}/[n_p!]^{1/2},$$

in which the possible values of each $n_p=0, 1, 2, \cdots$. Probability normalization conditions are satisfied in consequence of the property

$$|\langle 0_+|0_-\rangle^S|^2 = \exp[-\sum_p |S_p|^2].$$

The significant conclusion is that the system under investigation is a Bose-Einstein (B.E.) ensemble of indistinguishable particles. It will be recognized that this characteristic has been introduced implicitly by regarding the source function as an ordinary numerical quantity. The latter assumption now emerges as the mathematical representation of B.E. statistics.

Even though an application to spinless particles is inappropriate, it can be appreciated that an analogous representation of Fermi-Dirac (F.D.) statistics demands that the source have such properties that $(S_p)^2=0$, for all p. This implies that the source functions, $S(x)$, of F.D. particles are totally anticommutative quantities, as realized by the elements of an exterior algebra.[2] The general correspondence between particle statistics and the mathematical nature of the source is thus expressed by

$$\text{B.E.:}\quad [S(x),S(x')]=0,$$
$$\text{F.D.:}\quad \{S(x),S(x')\}=0.$$

Let us also note here a consequence of the implicit reference to a source defined in a certain spatiotemporal region. Suppose the source is rigidly displaced, in the sense that

$$S(x)\to S(x+X).$$

Then

$$S_p \to e^{ipX}S_p$$

and

$$\langle\{n\}|0_-\rangle^S \to e^{iPX}\langle\{n\}|0_-\rangle^S,$$

which identifies the total energy and momentum of this state

$$P^\mu = \sum_p n_p p^\mu.$$

PARTICLES WITH SPIN

The extension of the preceding discussion to include particles with spin can be done in a variety of ways for specific values. If we wish a uniformly applicable description, however, the unique source representation is the multispinor. We consider sources $S_{\zeta_1\cdots\zeta_n}(x)$, where each ζ_κ is a four-valued Dirac spin index, and S has a definite symmetry pattern with respect to permutations of the n spin indices. The totally symmetric choice suffices for most purposes, but other possibilities will be considered.

[2] There is a related discussion of exterior algebras in J. Schwinger, Proc. Natl. Acad. Sci. **48**, 603 (1962).

The general expression for the vacuum amplitude is taken to be

$$\langle 0_+|0_-\rangle^S = \exp\left[\tfrac{1}{2}i\int (dx)(dx')S(x)G_+(x-x')S(x')\right].$$

In order that this source representation be an invariant one, the matrix function $G_+(x)$ must have the transformation properties

$$G_+(x)=L^TG_+(lx)L,$$

where

$$L=\prod_{\kappa=1}^n L_\kappa(l)$$

and each $L_\kappa(l)$ produces the individual spin transformation that accompanies the proper orthochronous coordinate transformation

$$\bar{x}^\mu = l^\mu{}_\nu x^\nu.$$

We recall the transformation properties (the index κ is omitted)

$$L^T\beta L=\beta,$$
$$L^T\alpha^\mu L = l^\mu{}_\nu\alpha^\nu,$$

where the real matrices α^μ and β are, respectively, symmetrical and antisymmetrical. In another notation,

$$-i\beta=\gamma^0,\quad \alpha^\mu=\gamma^0\gamma^\mu.$$

The matrix function $G_+(x-x')$ must be constructed by multiplying the invariant function $\Delta_+(x-x')$ with matrices that satisfy the covariance properties of G_+. The result should describe a particle of definite spin and parity in its rest coordinate system. As we shall verify, this is accomplished by

$$G_+(x-x')=\prod_{\kappa=1}^n [\gamma^0(m-\gamma^\mu(1/i)\partial_\mu)]_\kappa\Delta_+(x-x').$$

The individual factors here are antisymmetrical under transposition of the matrices, combined with the exexchange of x with x', and

$$G_+(x'-x)^T=(-1)^nG_+(x-x').$$

Since this operation effectively interchanges the two source factors in the vacuum amplitude we learn that even n demands B.E. statistics and odd n, F.D. statistics. The polynomial factor of G_+ is also represented by

$$x^0>x^{0\prime}:\quad G_+(x-x')=i\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}\times e^{ip(x-x')}\prod_{\kappa=1}^n[\gamma^0(m-\gamma p)]_\kappa.$$

We proceed to identify single-particle states, labeled by momentum and spin, as implied by the effect of a weak production and detection source,

$$\sum_{p\lambda}\langle 0_+|1_{p\lambda}\rangle^{S_1}\langle 1_{p\lambda}|0_-\rangle^{S_2}$$
$$=\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}iS_1(-p)\prod_\kappa[\gamma^0(m-\gamma p)]_\kappa S_2(p).$$

The individual Hermitian matrix combinations have the projection property

$$[\gamma^0(m-\gamma p)]^2=2p^0[\gamma^0(m-\gamma p)].$$

This permits us to write

$$\sum_\lambda\langle 0_+|1_{p\lambda}\rangle^{S_1}\langle 1_{p\lambda}|0_-\rangle^{S_2}=\sum_{\zeta_1\cdots\zeta_n}iS_1{}^{(-)}{}_{p\zeta_1\cdots\zeta_n}iS_2{}^{(+)}{}_{p\zeta_1\cdots\zeta_n},$$

where the two spinors are

$$S_p{}^{(+)}=\left[\frac{(d\mathbf{p})}{(2\pi)^3}(2p^0)^{n-1}\right]^{1/2}\prod_\kappa\left[\frac{\gamma^0(m-\gamma p)}{2p^0}\right]_\kappa S(p),$$
$$S_p{}^{(-)}=\left[\frac{(d\mathbf{p})}{(2\pi)^3}(2p^0)^{n-1}\right]^{1/2}S(-p)\prod_\kappa\left[\frac{\gamma^0(m-\gamma p)}{2p^0}\right]_\kappa.$$

In the rest system, $p^0=m$, the projection factor $\frac{1}{2}(1+\gamma^0)$ selects $\gamma^0=+1$ for each of the n Dirac indices, and the source thus produces particles of definite parity. It is purely conventional to have all values of γ^0 be $+1$, and this can be altered by redefining the source function with suitable γ_5 factors.

Now let us consider a totally symmetrical spinor source. In the rest system the restriction to $\gamma_\kappa{}^0=+1$, $\kappa=1,\cdots n$, effectively reduces the source to a totally symmetrical function of two-valued spin indices, which can be identified with the eigenvalues of $\sigma_{3\kappa}$. A totally symmetric combination of n spins has a definite spin angular momentum,

$$s=\tfrac{1}{2}n.$$

Thus, all possible half-integral spin values can be represented by a suitable odd integer n, and all possible integer spins can be represented by an even integer n, except $s=0$. For the latter one has to take $n=2$ and use an antisymmetrical spinor source. The relation between statistics and the even-odd nature of n will be recognized as the connection between spin and statistics. Note that more complicated symmetry patterns give equivalent descriptions, to the extent that a definite spin appears in the rest system. Consider, for example, $n=3$ with the requirement of antisymmetry in a pair of Dirac indices. The latter contributes zero spin in the rest system and we have a possible description of an $s=\frac{1}{2}$ F.D. particle.

The individual single-particle amplitudes are easily identified. If we use a totally symmetric spinor, for example, the extreme values of the spin magnetic quantum number correspond to identical choices for the n component spins, as suggested by

$$\langle 1_{ps}|0_-\rangle^S=iS_{p++\cdots}{}^{(+)}$$
$$\langle 0_+|1_{ps}\rangle^S=iS_{p++\cdots}{}^{(-)},$$

and the other states can be generated from these. There is a simple relation between the two spinors $S_p{}^{(\pm)}$:

$$S_p{}^{(-)}=S_p{}^{(+)*}.$$

This conveys the Hermitian nature of the matrices $\gamma^0(m-\gamma p)$, and the reality of the source $S(x)$. For B.E. particles, the physically necessary property

$$|\langle 0_+|0_-\rangle^S|^2=\exp[-\sum_p S_p{}^{(+)*}S_p{}^{(+)}]<1$$

is deduced on remarking that (n is even)

$$-\mathrm{Re}\,iG_+(x-x')=\mathrm{Re}\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}$$
$$\times e^{ip(x-x')}\prod_\kappa[\gamma^0(m-\gamma p)]_\kappa,$$

which can be expressed as

$$-\mathrm{Re}\,i\int(dx)(dx')S(x)G_+(x-x')S(x')$$
$$=\sum_p S_p{}^{(+)*}S_p{}^{(+)}>0.$$

An analogous positiveness statement exists for F.D. particles. Let $\epsilon(x-x')$ state the algebraic sign of $x^0-x^{0\prime}$ and, for $x^0-x^{0\prime}\neq 0$, consider the function (n is odd)

$$-\mathrm{Re}\,i\epsilon(x-x')G_+(x-x')$$
$$=\mathrm{Re}\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0}e^{ip(x-x')}\prod_\kappa[\gamma^0(m-\gamma p)]_\kappa,$$

which we shall define by the right-hand side for all $x-x'$. Then, if $S_1(x)$ is a real *commutative* spinor, we have

$$-\mathrm{Re}\,i\int(dx)(dx')S_1(x)\epsilon(x-x')G_+(x-x')S_1(x')$$
$$=\sum_p S_{1p}{}^{(+)*}S_{1p}{}^{(+)}>0.$$

CHARGED PARTICLES

The specification of the mass of a particle (stable or unstable) defines it uniquely, within present experimental knowledge, except when the particle carries electrical or nucleonic charge. These are exactly conserved quantities. The *TCP* theorem then assures us of the existence of two particles with opposite charges, and identical masses. This situation can be represented by supplying the real source function with another index, upon which the appropriate charge acts as a matrix,

$$l=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

The diagonalization of this matrix defines the complex sources of charged particles.

The introduction of the antisymmetrical charge matrix puts the connection between spin and statistics in a new light. It should be evident that the construction of $G_+(x-x')$ in the previous section was essentially uniquely determined by the various space-time requirements imposed upon it. The resulting structure possesses a symmetry that is specified completely by n, the number of Dirac spin indices. But now we have the option of inserting the Lorentz-invariant matrix l as a factor in G_+, which would alter the symmetry and destroy the spin-statistics connection. It is the physical positiveness properties that enable one to reject this possibility, and retain the correlation between spin and statistics. That invites us to re-examine the physical basis of these positiveness assertions.

We cannot improve on the B.E. situation in which sources appear directly in probability statements. For F.D. particles, however, the positiveness property emerged as an algebraic observation without direct reference to F.D. sources, which are a more abstract concept. In this circumstance we must turn to the physical system with which the source is coupled. We shall be able to treat both statistics in a unified way, however. Let the real multicomponent source $S(x)$ be coupled to the Hermitian operators $\Psi(x)$, which we assume to transform contragradiently to $S(x)$ under Lorentz transformations. It is immaterial for the following discussion whether F.D. sources anticommute or commute with $\Psi(x)$. To have a uniform treatment of both statistics we assume commutativity, and take the source term in a phenomenological Lagrange function as

$$\mathcal{L}_{\text{source}}=\Psi(x)S(x)=S(x)\Psi(x).$$

The action principle supplies the differential statement

$$\delta_S\langle 0_+|0_-\rangle^S=i\langle 0_+|\int(dx)\delta S(x)\Psi(x)|0_-\rangle^S$$

and the repetition of this operation gives

$$\begin{aligned}\delta_S{}^2\langle 0_+|0_-\rangle^S &=-\langle 0_+|\int(dx)(dx')(\delta S(x)\Psi(x)\delta S(x')\Psi(x'))_+|0_-\rangle^S\\ &=-\int(dx)(dx')(\delta S(x)\delta S(x'))_+\\ &\qquad\times\langle 0_+|(\Psi(x)\Psi(x'))_+|0_-\rangle^S.\end{aligned}$$

We have not written out possible additional terms, which occur when some of the components of $\Psi(x)$ are explicit functions of $S(x)$ or its coordinate derivatives. The overt reference to statistics appears on writing

$$(\delta S(x)\delta S(x'))_+=\begin{cases}\text{B.E.}: & \delta S(x)\delta S(x')\\ \text{F.D.}: & \epsilon(x-x')\delta S(x)\delta S(x').\end{cases}$$

A comparison with the general source representation identifies the vacuum expectation values,

$$\langle(\Psi(x)\Psi(x'))_+\rangle=\begin{cases}\text{B.E.}: & -iG_+(x-x')\\ \text{F.D.}: & -i\epsilon(x-x')G_+(x-x'),\end{cases}$$

apart from possible additional delta-function terms, which do not contribute for $x\neq x'$. Now we observe that

$$2\,\mathrm{Re}\langle(\Psi(x)\Psi(x'))_+\rangle=\langle\{\Psi(x),\Psi(x')\}\rangle$$

is a positive matrix structure. The positiveness properties we have noted thus emerge as necessary quantum consequences of the coupling between source and physical system.

EUCLIDEAN POSTULATE

It is a remarkable fact that all F.D. particles carry some kind of charge. The experimental proof of nonidentity between electron and muon neutrinos[3] confirms an early suggestion[4] that neutrinos would be no exception to that rule. A representation of this regularity is given by the following abstract Euclidean postulate: The vacuum probability amplitude must be transformable into the attached Euclidean space in such a way that the original time axis cannot be identified. We first illustrate this for B.E. particles.

The basic Euclidean transformations are

$$\begin{gathered} i(dx)\to(d_4x)\;=dx_1\cdots dx_4,\\ -i\Delta_+(x)\to\Delta_E(x)=\int(d_4p)\frac{e^{ip_\mu x_\mu}}{p^2+m^2},\\ p^2=p_\mu p_\mu>0.\end{gathered}$$

The transformation of B.E. sources is taken to be

$$S(x)\to\exp[\tfrac{1}{4}\pi i(\prod_{\kappa=1}^{n}\gamma_\kappa{}^0+1)]S_E(x).$$

The product of the even number of γ^0 matrices is symmetrical, and

$$-S(x)[\prod_\kappa\gamma_\kappa{}^0]S(x)\to S_E(x)S_E(x).$$

For each spinor index κ, we define the matrices

$$\alpha_\mu=\exp[-\tfrac{1}{4}\pi i\prod\gamma_\kappa{}^0](-i)\gamma_\mu\exp[\tfrac{1}{4}\pi i\prod\gamma_\kappa{}^0],\quad \mu=1\cdots 4$$

which maintain the commutativity of those referring to different indices. The new set has the algebraic property

$$\tfrac{1}{2}\{\alpha_\mu,\alpha_\nu\}=\delta_{\mu\nu}.$$

Each of the α_μ is imaginary and antisymmetrical, which places all of them on the same footing. The resulting Euclidean transformation of the vacuum amplitude is

$$\langle 0_+|0_-\rangle^S\to$$

$$\exp\left[-\frac{1}{2}\int(d_4x)(d_4x')S_E(x)G_E(x-x')S_E(x')\right]$$

[3] G. Danby *et al.*, Phys. Rev. Letters **9**, 36 (1962).

[4] J. Schwinger, Ann. Phys. **2**, 407 (1957). See also K. Nishijima, Phys. Rev. **108**, 907 (1957). This is the introduction of a leptonic charge not related to neutrino helicity.

where

$$G_E(x-x')=\prod_{\kappa=1}^{n}(m-\alpha_\mu\partial_\mu)_\kappa\Delta_E(x-x').$$

This is a Euclidean form in which all coordinate axes are indeed indistinguishable.

There is an alternative version which is produced by the further transformation

$$S_E(x)\to\prod_\kappa[\exp(\tfrac{1}{4}\pi i(\alpha_5-1))]_\kappa S_E(x),$$

where

$$\alpha_5=\alpha_1\alpha_2\alpha_3\alpha_4=\alpha_5{}^*=\alpha_5{}^T.$$

Now the function

$$G_E(x-x')=\prod_{\kappa=1}^{n}(\alpha_5 m+i\alpha_\lambda\partial_\lambda)_\kappa\Delta_E(x-x')$$

is real. If the associated Euclidean sources are taken to be real, so also is this Euclidean transcription of the vacuum probability amplitude.

It is interesting to examine further the algebraic basis of the Euclidean transformation, at least in the simplest B.E. situation, $n=2$. Let α_μ, $\mu=1\cdots5$ be a set of anticommuting matrices of unit square such that the product $\alpha_1\cdots\alpha_5$ is the unit matrix. With the latter, the five α_μ and the ten $\alpha_\mu\alpha_\nu$, $\mu<\nu$, comprise 16 independent matrices. If there be n_s symmetrical matrices among the α_μ, and $n_a=5-n_s$ antisymmetrical ones, the decomposition of the 16 matrices into symmetrical and antisymmetrical members is counted as

$$N_s=1+n_s+n_sn_a=10-(n_s-3)^2,$$
$$N_a=n_a+\tfrac{1}{2}n_s(n_s-1)+\tfrac{1}{2}n_a(n_a-1)=6+(n_s-3)^2.$$

Two independent systems of this type comprise 256 matrices, which are equivalent to the full 16×16 matrix algebra if there are just $\frac{1}{2}\,16\times15$ antisymmetrical matrices. This requires that $N_sN_a=60$, or that

$$(n_s-3)^2=0,\ 4.$$

The first possibility, $n_s=3$, $n_a=2$, is the one realized in the Minkowski metric where the three symmetrical matrices are $-i\gamma_k$, $k=1, 2, 3$, and the remaining two are the antisymmetrical matrices γ^0, $-i\gamma_5$. The second possibility contains two alternatives, $n_s=1$, $n_a=4$, or $n_s=5$, $n_a=0$. This is the Euclidean realization, in which the α_μ, $\mu=1\cdots4$, are antisymmetrical and α_5 is symmetrical, or alternatively $\alpha_\mu'=i\alpha_\mu\alpha_5$, $\mu=1\cdots4$ and $\alpha_5'=\alpha_5$ are all symmetrical. Only in the Minkowski metric, however, is any one set of 16 matrices equivalent to a 4×4 matrix algebra. The latter contains six antisymmetrical matrices, whereas the Euclidean realizations require ten such matrices.

We now consider both statistics. In response to the Euclidean source transformation

$$S(x)\to RS_E(x),$$

the function G_E emerges as

$$G_E(x-x')=-R^T[\textstyle\prod_\kappa\gamma_\kappa{}^0]R\times\prod_\kappa(m-R^{-1}(1/i)\gamma_\mu R\partial_\mu)_\kappa\Delta_E(x-x').$$

The matrix factor $-R^T[\prod_\kappa\gamma_\kappa{}^0]R$ must be a Euclidean scalar. It is symmetrical for B.E. particles, and antisymmetrical for F.D. particles. The matrices $R^{-1}(1/i)\gamma_\mu R$, $\mu=1\cdots4$, are required to have a common symmetry. We shall let α_μ, $\mu=1\cdots4$, specifically designate antisymmetrical matrices, and construct the alternative symmetrical matrices as $i\alpha_\mu\alpha_5$. The appropriate Euclidean scalar matrix must commute with the α_μ and anticommute with the $i\alpha_\mu\alpha_5$. The latter requirement can be satisfied by the factor $\prod_\kappa\alpha_{5\kappa}$, and either choice requires the existence of a Euclidean scalar matrix that commutes with all the α_μ and has a definite symmetry, as demanded by the statistics. The $\alpha_{\mu\kappa}$, $\kappa=1\cdots n$ generate an algebra of dimensionality 16^n. If the latter is the full $4^n\times4^n$ matrix algebra, only the symmetrical unit matrix can commute with all elements. This is the B.E. situation. The antisymmetrical scalar matrix of F.D. statistics can be realized only by adjoining such a matrix, l, to the algebra generated by the α_μ. The necessity for this additional element is also evident in the remark that 4×4 matrices do not permit a representation of Euclidean symmetries.

A suitable Euclidean transformation matrix for F.D. statistics is

$$R=\exp[\tfrac{1}{4}\pi i(l\prod_{\kappa=1}^{n}\gamma_\kappa{}^0+1)]$$

and

$$G_E(x-x')=l\textstyle\prod_\kappa(m-\alpha_\mu\partial_\mu)_\kappa\Delta_E(x-x'),$$

which makes explicit the charge carried by every kind of F.D. particle. The alternative Euclidean representation

$$G_E(x-x')=l\textstyle\prod_\kappa(\alpha_5m+i\alpha_\mu\partial_\mu)_\kappa\Delta_E(x-x')$$

is produced by the transformation

$$S_E(x)\to\textstyle\prod_\kappa[\exp(\tfrac{1}{4}\pi i(\alpha_5-1))]_\kappa S_E(x).$$

The new G_E function is imaginary. Nevertheless real Euclidean source functions imply a real Euclidean transcription of the vacuum probability amplitude, according to the definition of complex conjugation in an exterior algebra,[2]

$$(S(x)S(x'))^*=S(x')S(x)=-S(x)S(x').$$

MASSLESS PARTICLES

In order to deal with the special circumstances posed by particles of zero mass, we return to the stage of identifying single-particle states and consider, for $m=0$, the individual projection matrices

$$\gamma^0(-\gamma p)/2p^0=\tfrac{1}{2}(1-i\gamma_5\sigma_p).$$

Here σ_p is the component of $\boldsymbol{\sigma}$ that is parallel to the momentum of the particle. The isolated occurrence of the γ_5 matrices demands a further specification of particle sources. It is sufficient to consider symmetrical spinors that are subject to the single-invariant constraint

$$\left[\left(\frac{1}{n}\sum_{\kappa=1}^{n} i\gamma_{5\kappa}\right)^2-1\right]S(x)=0.$$

The latter requires that each $i\gamma_{5\kappa}$ have the same eigenvalue, which becomes the common eigenvalue of $-\sigma_{p\kappa}$. This is the familiar statement that massless B.E. particles have only two helicity states, those for which the component of angular momentum parallel to the linear momentum equals $\pm s$, $s=\frac{1}{2}n$. The photon and graviton are represented by $n=2$ and $n=4$, respectively. Incidentally, the Euclidean version of the constraint is

$$\left[\left(\frac{1}{n}\sum_{\kappa} \alpha_{5\kappa}\right)^2-1\right]S_E(x)=0.$$

A massless particle obeying F.D. statistics can be described by a real source which is a symmetrical spinor obeying one of the two constraints:

$$\left[l\frac{1}{n}\sum_{\kappa=1}^{n} i\gamma_{5\kappa}-1\right]S(x)=0$$

or

$$\left[l\frac{1}{n}\sum_{\kappa=1}^{n} i\gamma_{5\kappa}+1\right]S(x)=0.$$

In either situation each of the $i\gamma_{5\kappa}$ has the same eigenvalue, which is the common eigenvalue of $-\sigma_{p\kappa}$, and is also equal to the eigenvalue of l, or $-l$, according to the nature of the particle. A given particle has two helicity states, of angular momentum $\pm s$, $s=\frac{1}{2}n$. The e and μ neutrinos, presumed to be massless, are represented by the two alternatives available for $n=1$.

The Euclidean transcription of the F.D. constraints requires an additional source transformation in order to be compatible with the reality of Euclidean sources. This transformation involves the fermionic charge reflection matrix r_l, a real symmetrical matrix, which can be chosen as

$$r_l=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

An alternative charge reflection matrix is $ir_l l$. With the further transformation

$$S_E(x)\rightarrow \exp(\tfrac{1}{4}\pi r_l l)S_E(x),$$

the constraint equations become

$$[r_l(1/n)\sum_{\kappa}\alpha_{5\kappa}\mp 1]S_E(x)=0$$

while leaving unaltered the imaginary structure

$$G_E(x-x')=l\prod\nolimits_{\kappa}(i\alpha_\mu\partial_\mu)_\kappa\Delta_E(x-x').$$

SU_3 AND SPIN

The rapid increase in experimental information concerning massive, strongly interacting particles has brought a provisional answer to the long quest for a particle classification scheme. It is supplied by the multiplets of the internal symmetry group SU_3. Let the two types of three-valued unitary indices be designated by a and a^*. The various unitary multiplets are labeled by symmetrical functions of indices $a_1, \cdots a_r$, and of indices $a_{r+1}^*, \cdots a_\nu^*$, which are irreducible with respect to contraction of an index of type a with an a^* index. Thus, the various particle sources can be indicated by

$$S_{\zeta_1\cdots\zeta_n;a_1\cdots a_r,a_{r+1}^*\cdots a_\nu^*}(x).$$

It should be emphasized that the unitary indices are a means of supplying the quantum numbers appropriate to a given particle, which may still need to be identified through its mass value, and that no presumption exists concerning the masses of the different particles which are united in a particular multiplet.

As we have already remarked in connection with the spin indices, there are alternative ways, of designating sources, that employ more complicated symmetry patterms. While an antisymmetrical pair of spin indices is affectively inert, an antisymmetrical combination of similar unitary indices is equivalent to a complex-conjugate index. In this way, other symmetry patterns can be reduced to the totally symmetric structures (zero spin is an exception, of course). Several different sources may appear in consequence of the reduction process. The possibility thus suggested of a more inclusive classification can be illustrated by considering symmetric functions of the combined indices

$$A=\zeta,\, a\quad A^*=\zeta, a^*.$$

The generalized source

$$S_{A_1\cdots A_r;A_{r+1}^*\cdots An^*}(x),$$

a symmetrical function of $A_1\cdots A_r$, and of $A_{r+1}^*\cdots A_n^*$, unites the sources of particles, with various spins and a common parity, that belong to several unitary multiplets. Thus, the source with $r=1$, $n=2$, $S_{\zeta_1a_1;\zeta_2a_2^*}$, describes spin 0 and spin 1 meson multiplets, each containing a unitary octuplet and singlet. This source can be applied to the well-established 0^- and 1^- mesons, both of which form octuplet + singlet families. The baryon multiplets comprised in $S_{\zeta_1a_1,\zeta_2a_2,\zeta_3a_3}$ have spin $\frac{1}{2}$ and $\frac{3}{2}$, forming a unitary octuplet and a decuplet, respectively. The latter is applicable to the known system of $\frac{1}{2}^+$ baryons and $\frac{3}{2}^+$ baryon resonances. These wider classifications have been achieved by a combinatorial union of spin and unitary indices. There is no reference to a continuous group of transformations on all the indices.

CONCLUSION

The source concept emerges as a valid and useful phenomenological particle description. It displays the complementary aspects of particle behavior, and supplies the connection between spin and statistics in a simple and direct way. The sources also serve as carriers of quantum numbers and give concrete expression to family relations among particles.

There is, in this description, no commitment to any specific view of particle structure. The extreme S-matrix attitude can be introduced by insisting that no source function $S(p)$ is defined for momentum values that do not obey $-p^2=m^2$. But if it is considered meaningful to define $S(p)$ over a wider momentum range, one has admitted a concept of matter more fundamental than that of the particle. It seems desirable to have a phenomenological description of particles which is logically independent of hypotheses concerning the deeper nature of these objects. Such hypotheses may then be suggested by the formal representation of observed regularities.

Note added in proof. The complete independence of the phenomenological source description from other formulations needs more emphasis. The text contains reference to the field-theoretic version of the *TCP* theorem, and to operator-based positiveness properties. Neither is required. The essential tool is the formal expression of completeness for the multi-particle states in the two forms

$$\sum_n \langle 0_-|n\rangle^S\langle n|0_-\rangle^S=1$$

$$\langle 0_-|n\rangle^S=\langle n|0_-\rangle^{S*},$$

and

$$\sum_n \langle 0_+|n\rangle^S\langle n|0_+\rangle^S=1$$

$$\langle n|0_+\rangle^S=\langle 0_+|n\rangle^{S*},$$

where n symbolizes the whole set of occupation numbers. Using the multispinor representation, which unifies all spins and associated statistics, the consideration of a production and detection source gives

$$\langle n|0_-\rangle^S=\langle 0_+|0_-\rangle^S\prod_{p\lambda}(iS_{p\lambda})^{n_{p\lambda}}/(n_{p\lambda}!)^{1/2}$$

$$\langle 0_+|n\rangle^S=\langle 0_+|0_-\rangle^S\prod_{p\lambda}{}'(iS_{p\lambda}^*)^{n_{p\lambda}}/(n_{p\lambda}!)^{1/2}$$

where Π and Π' refer, respectively, to some standard multiplication order and its inverse. Here

$$S_{p\lambda}=\left[\frac{(d\mathbf{p})}{(2\pi)^3}\frac{(2m)^n}{2p^0}\right]^{1/2}u_{p\lambda}^*(\prod_\kappa\gamma_\kappa^0)S(p)$$

and

$$\prod_\kappa(m-\gamma p/2m)_\kappa=\sum_\lambda u_{p\lambda}u_{p\lambda}^*(\prod\gamma_\kappa^0),$$

with

$$u_{p\lambda}^*(\Pi\gamma_\kappa^0)u_{p\lambda'}=\delta_{\lambda\lambda'}.$$

It is the comparison of the two completeness expressions that indicates the necessity for the general rule

$$(S(x)S(x'))^*=S(x')S(x).$$

Then either form gives

$$1=|\langle 0_+|0_-\rangle^S|^2\exp[\sum_{p\lambda}S_{p\lambda}^*S_{p\lambda}].$$

The usual connection between spin and statistics assures the reality of the combination

$$S(x)(\Pi_\kappa\gamma_\kappa^0)S(x'),$$

which is the basis of the direct evaluation

$$|\langle 0_+|0_-\rangle^S|^2=\exp[-\sum_{p\lambda}S_{p\lambda}^*S_{p\lambda}].$$

An attempt to reverse the connection between spin and statistics by introducing an antisymmetrical matrix q into the general source structure will now founder on the contradiction between the direct evaluation, which recognizes the indefinite nature of the q spectrum, and the summation over all particle states, which involves only the magnitudes of the q eigenvalues. Similarly, any reference to internal properties is limited to a symmetrical positive matrix which can always be transformed into the unit matrix. Here is the assurance that a charged particle has an oppositely charged counterpart of equal mass.

The source representation of the invariance operation designated as *TCP* combines space-time reflection with transposition—the reversal in multiplication order of all sources. The former is performed through the attached Euclidean group and leaves the vacuum amplitude unchanged, but at the expense of making half-integer spin sources imaginary. The actual sign reversal of a product of such sources is then compensated by the transposition of the sources, in virtue of the connection between spin and statistics.

We have built the source description on a principle of the unity of the source; sources that are effective in different space-time regions are constituents of one general source. Perhaps it should be pointed out that only the two usual statistics are admitted by this principle. In the consideration of a production and detection source that leads to the identification of the particle states, it is necessary to give an ordered form to the product of sources. This demands the existence of relations of the type

$$S(x')S(x)=\lambda S(x)S(x')$$

which exhibit the algebraic properties of the source function, and limit λ to one of the alternatives, ±1.