Threshold Behavior of Partial-Wave Amplitudes in Quantum Field Theory*

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We show within the framework of Wightman field theory and the Haag-Ruelle scattering theory that the inelastic partial-wave amplitude for a scattering with two particles in the initial or final state has the classic threshold behavior $T\sim j^l$ and under additional assumptions the phase shifts for elastic scattering obey $\sin\delta_l\sim k^{2l+1}$ in the low-energy limit.

I. INTRODUCTION

IT is generally believed that the interactions of nuclear particles are mediated by short-range nuclear forces so that their long-range interactions are (neglecting Coulomb forces) dominated by the angular-momentum barrier. As a consequence it has become an article of faith among practitioners of phenomenological (or S matrix) nuclear physics that elastic-scattering phase shifts must have the threshold behavior[1]

$$\tan\delta_l\sim k^{2l+1} \tag{1}$$

for fixed barycentric momentum k and sufficiently high orbital angular momentum l.

We all know, of course, that the "initial" rate of growth described by Eq. (1) is predicted rigorously in a model where the scattering is described by a Schrödinger equation with a suitably behaved potential. In that case the long-range dominance of the angular-momentum barrier may be exhibited explicitly. There is, however, no assurance that such a simple model is adequate to describe the interactions of nuclear and subnuclear particles, and it becomes of interest to ask whether Eq. (1) cannot be made to follow from more general principles.

In order to answer this question we shall turn to axiomatic field theory on the grounds that it is the most general predictive theory presently available. Our intuition leads us to expect that the field-theoretic equivalent of the short-range nuclear force will be the presence in the theory of a finite gap in the mass spectrum or, equivalently, the absence of zero-mass particles. On the other hand, our experience with field theory suggests that even very obvious seeming results can be obtained only as a consequence of rather delicat. maneuvering. We would not have been amazed, therefore, to find it necessary to impose additional restrictions upon the theory (besides the existence of a mass gap) in order to obtain the threshold behavior of Eq. (1)e The purpose of the work described here is to see what, if any, additional restrictions might be required.[2]

The starting point for our deliberations will be the Haag-Ruelle[3,4] scattering theory which was erected upon the foundation of Wightman's relativistic quantum field theory.[5] The relevant axiomatic framework will be summarized at the beginning of Sec. III, but we should like to remark at this point upon the distribution character of the field operators of the theory. It will be recalled[6] that the field operators are conventionally assumed to be operator-valued *tempered* distributions. Although the temperateness assumption is convenient, in that it results in a symmetry between the configuration-space and momentum-space behavior of the field operators (Fourier transforms of tempered distributions are tempered distributions), it appears to be devoid of physical content and we feel uncomfortable in its presence.[7] It turns out that for our purposes we can settle for the somewhat less restrictive assumption to be made in Sec. III that permits rather wild behavior at infinite *momentum* of the Fourier transformed field operators. However, we are unable to say much (in Sec. V) concerning the question of whether this slight mathematical generalization is of any physical importance.

For our purposes it will be sufficient to discuss the theory of the ubiquitous neutral scalar field $A(x)$. The needed axioms and some of their consequences are recapitulated in Sec. III. Threshold behavior of production and elastic scattering amplitudes is derived in Sec. IV by making use of a Schrödinger-like representation of the relativistic scattering amplitude. In Sec. II we review the derivation of Eq. (1) from the Schrödinger equation, and in Sec. V we make some concluding remarks.

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[1] See, for example, the speculative remarks in Sec. IX of L. D. Roper, J. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

[2] One of the authors, in company with many of his friends, takes the viewpoint that Eq. (1) *must* hold in any reasonable theory, at least at sufficiently low energy.

[3] R. Haag, Phys. Rev. 112, 669 (1958).

[4] D. Ruelle, Helv. Phys. Acta 35, 147 (1962).

[5] For a review see R. F. Streater and A. S. Wrightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin and Company, New York, 1964).

[6] Reference 5, p. 98.

[7] See also the remarks of K. Bardakci and B. Schroer, J. Math. Phys. 7, 16 (1966).

II. THE SCHRÖDINGER-EQUATION ARGUMENT

Consider the partial wave Schrödinger equation[8]

$$\left\{-r^{-1}\frac{d^2}{dr^2}r+\frac{l(l+1)}{r^2}+v(r)-k^2\right\}R_l(k,r)=0 \qquad (2)$$

with a potential of finite ranger r_0. We recall that the radial wave function R_l is defined by requiring it to be regular at the origin. Outside of the range of the potential, the solution is a linear combination of spherical Bessel functions:

$$R_l(k,r)\sim j_l(kr)-\tan\delta_l\, n_l(kr). \qquad (3)$$

When k^2 becomes very small compared with the centrifugal term $l(l+1)r^{-2}$, R_l must approach its $k=0$ form[9] ($r>r_0$)

$$R_l(0,r)=a_lr^l+b_lr^{-l-1}. \qquad (4)$$

Equating the right-hand sides of Eqs. (3) and (4) and using the small argument approximations for the spherical Bessel functions (valid for $kr\ll l$) leads us to the desired result

$$\tan\delta_l \xrightarrow[k/l\to 0]{} (b_l/a_l)k^{2l+1} \qquad (5)$$

unless a_l happens to vanish. If this does happen for some $l\geqslant 1$, then $R_l(0,r)$ is square integrable so that R_l is the wave function of a zero-energy bound state. If a_0 happens to vanish, we say that the s-amplitude has a zero energy resonance.

We remark in passing that Carter[10] obtained the threshold behavior, Eq. (1), under the less restrictive assumption (and excluding the exceptional cases)

$$\int_0^\infty r^{2l+2}dr\,|V(r)|<\infty. \qquad (6)$$

Newton[11] showed that in the exceptional cases

$$\begin{aligned}\tan\delta_l&\sim\text{constant},\ k\to 0 \quad\text{if}\quad a_0=0,\\ &\sim k^{2l-1}, \qquad k\to 0 \quad\text{if}\quad a_l=0.\end{aligned} \qquad (7)$$

The preceding discussion exhibits in a very transparent manner the crucial nature of the requirement that the potential be "rapidly decreasing" at large distance. We see clearly how the rapid decrease permits the centrifugal barrier to "take over" at large distance and thereby determine the threshold behavior of the scattering amplitude. However, the approach to be taken in the present work will follow more closely along the lines of a discussion given by Goldberger and Watson[12] which it is instructive to reproduce. We write the coordinate space representation of the T matrix

$$\langle\mathbf{r}'|T|\mathbf{r}\rangle\equiv T=V+V[\epsilon(k)-H+i\eta]^{-1}V, \qquad (8)$$

and, in the limit $\epsilon(k)\to 0$, define

$$T_0=\lim_{\epsilon(k)\to 0}T\equiv\langle\mathbf{r}'|T_0|\mathbf{r}\rangle \qquad (9)$$

and

$$\langle r'|T_0{}^l|r\rangle=\int d\Omega'\int d\Omega\, Y_l{}^*(\Omega')\langle\mathbf{r}'|T_0|\mathbf{r}\rangle Y_l(\Omega). \qquad (10)$$

It is then a trivial matter to show that

$$T_l{}^0(k)=\frac{2}{\pi}\int_0^\infty r'^2dr'\int_0^\infty r^2dr\, j_l(kr')\langle r'|T_0{}^l|r\rangle j_l(kr) \qquad (11)$$

in the limit of vanishing k. If the integrand converges with sufficient rapidity, it follows again from the small argument behavior of the j_l's that

$$|T_l{}^0(k)|\equiv k^{-1}\sin\delta_l(k)\to\text{const}(k)^{2l}. \qquad (12)$$

The difficult part of the exercise is to determine the convergence properties of the integral over $\langle r'|T_0{}^l|r\rangle$. In Schrödinger theory, the rate of decrease of $\langle r'|T_0{}^l|r\rangle$ for large r and r' is closely related to the rate of decrease in the potential $V(r)$, as given by the condition in Eq. (6). In our work, however, similar properties will be expected to follow from rather general assumptions of field theory.

III. THE AXIOMATIC FRAMEWORK

In the course of the subsequent discussion we shall be concerned with the Wightman field theory of a neutral scalar field $A(x)$. Its axiomatic framework is as follows.

(A.) The field $A(x)$ is an operator-valued generalized function. Specifically, the Fourier transform $\tilde{A}(p)$ will be required to perform only upon test functions with compact support in Euclidean 4-space.[13] This test function space is referred to as K_4. The object $A(\phi)=\int d^4p\,\tilde{A}(p)\tilde{\phi}(p)$ with $\tilde{\phi}\in K_4$ is a well-defined operator on a dense invariant manifold of Hilbert space $\mathcal{H}$. $A(x)$ transforms under unitary representations of the inhomogeneous Lorentz group in a manner befitting a scalar field, namely, $U(a,\Lambda)A(x)U(a,\Lambda)^{-1}=A(\Lambda x+a)$. The spectrum of the momentum operator P_μ is assumed to lie in the forward light cone (positive energies). In addition we shall insist that (except for the vacuum) there is a lowest mass $m>0$ in the theory. Furthermore, $A(x)$ is a local field in the sense that[14]

$$[A(x),A(y)]=0,\quad (x-y)^2>0. \qquad (13)$$

[8] The succeeding argument is patterned upon that given in L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 403.

[9] It is easy to show that R_l is an analytic function of k^2.

[10] D. S. Carter as cited in Ref. 11, footnote 46.

[11] R. Newton, J. Math. Phys. **1**, 319 (1960).

[12] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), pp. 285–286.

[13] This is, matrix elements of $\tilde{A}(p)$ belong to the set K_4'. We use the language of I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), pp. 3–6.

[14] The metric is $x^2=\mathbf{x}\cdot\mathbf{x}-(x_0)^2$. Equation (13) needs interpretation since the set of test functions over which $A(x)$ is defined (the set Z of Ref. 13) do not permit localization of $A(\phi_1)$ and $A(\phi_2)$ in disjoint space-like regions. What is meant by Eq. (13) is the commutator can be made as small as we please by smearing the A's with admissible functions that have suitably sharp peaks at x and y.

(B.) It is crucial to our later discussion that $A(x)$ be the interpolating field corresponding to a mass that is isolated in the spectrum. If that mass is[15] m (the lowest mass), this condition may be stated concisely in the form

$$\langle 0|A(x)A(y)|0\rangle=\Delta_m^+(x-y)+\int_{2m}^{\infty} d\sigma(\mu)\Delta_\mu^+(x-y). \quad (14)$$

We shall be working with the *bona-fide* operators

$$B(x)=\int d^4y\,\phi(y-x)A(y) \quad (15)$$

where $\tilde{\phi}(p)$ has support in a suitable (for our later purposes) compact region of $\mathbf{p}$ space and also $-p^2$ restricted to a small region in the vicinity of the discrete mass m^2. The requirement of compact support in $\mathbf{p}$ space is not necessary to the remainder of the argument and is invoked only to emphasize that temperateness of $\tilde{A}(p)$ is not needed. We may also define the one-particle projection operators,

$$B(f_i,t)=\int d^3x\, f_i^*(x)\overleftrightarrow{\partial}_0 B(x)|_{x_0=t}, \quad (16)$$

where $f_i(x)$ is a solution of the Klein-Gordon equation for a particle of mass m with $\mathbf{p}$-space support in K_3 (assumed compatible with the support of $\tilde{\phi}$). However, in most of the subsequent discussion we will work in the limit

$$f_i(x)\rightarrow(2\pi)^{-3/2}(2\omega_p)^{-1/2}e^{ipx}, \quad (17)$$

and merely assure the reader here that the limit has been taken with due care.

As a result of the preceding assumptions, the following statements are all known to be true.

(I) $B^\dagger(x)|0\rangle\in H_{[m,0]}$, the space of single-particle spinless states of mass m.[4]

(II)$B^\dagger(f_i,t)|0\rangle$

$$=\int\frac{d^3p}{(2\omega_p)^{1/2}}\tilde{\phi}(\omega_p,\mathbf{p})\tilde{f}_i(\omega_p,\mathbf{p})a^\dagger(\mathbf{p})|0\rangle\equiv|f_i\rangle \quad (18)$$

is independent of t. Here $a^\dagger(\mathbf{p})$ is a free-particle creation operator.

(III) $\lim_{t\rightarrow\pm\infty}\prod_{j=1}^{m} B^\dagger(f_j,t)|0\rangle$

$$=|f_1\cdots f_m\text{ex}\rangle \text{ where ex}=\begin{pmatrix}\text{in}\\ \text{out}\end{pmatrix}$$

exists as a strong limit in $\mathcal{H}$.

(IV) $\lim_{x^2\rightarrow\infty}|x|^N\langle 0|B(x/2)J(-x/2)|\text{in}\rangle=0, \quad (19)$

[15] Equation (14) represents a nonessential simplification of the spectral conditions. The discrete mass could have an arbitrary value $m'\geqslant m$. In general the lower bound of the integral is m, and in fact, assume $m'>m$ when considering exothermic production processes.

and

(IV′) $\lim_{x^2\rightarrow\infty}|x|^N\langle\text{out}|J^\dagger(-x/2)B^\dagger(x/2)|2)|0\rangle=0, \quad (19')$

for all N, where $|\text{in}\rangle$ and $\langle\text{out}|$ denote, respectively, any in-state or out-state, and

$$(\Box-m^2)B(x)=J(x).$$

We note from (I) that $J^\dagger(x)$ annihilates the vacuum. Thus, to prove (IV), we write

$$\langle 0|B(x/2)J(-x/2)|\text{in}\rangle|^2\leqslant\langle\text{in}|\text{in}\rangle\times\langle 0|B(x/2)J(-x/2)J^\dagger(-x/2)B^\dagger(x/2)|0\rangle \quad (20)$$

by virtue of Schwarz's inequality. But the cluster decomposition theorem[4] states that the limit

$$\lim_{x^2\rightarrow+\infty}\langle 0|(x/2)J(-x/2)J^\dagger(-x/2)B^\dagger(x/2)0\rangle=\langle 0|B(0)B^\dagger(0)|0\rangle\times\langle 0|J(0)J^\dagger(0)|0\rangle \quad (21)$$

is approached faster than any inverse power of x^2. Evidently,

$$\langle 0|J(0)J^\dagger(0)|0=0, \quad (22)$$

which proves (IV). The proof for (IV′) is similar. Since (IV) does not depend on the particular choice of the in-state, it holds equally well in the sense of distributions for states with sharp momenta.

IV. THRESHOLD PROPERTIES OF RELATIVISTIC AMPLITUDES

At this point we are prepared to consider the T matrix between a two-particle initial state and an arbitrary final state of definite momentum. It is a well-known consequence of the properties listed in the preceding section that the T matrix may be written in the form

$$\tilde{\phi}_1(\mathbf{q}_1,\omega_1)\cdot T(f;\mathbf{q}_1\mathbf{q}_2)=+i(2\pi)^{5/2}(2\omega_1)^{-1/2}\delta^4(P_f-P_i)\langle f|J_1^\dagger(0)|\mathbf{q}_2\rangle, \quad (23)$$

where use is made of the one-step reduction formula. The factor $\tilde{\phi}_1$ is included to account for the fact that the interpolating fields $B_i(x)$ are obtained by convolution with the test functions $\phi_i(x-y)$. As long as $\phi_i(\mathbf{q},\omega)$ does not vanish in the region of interest to the subsequent discussion, the factor ϕ brings about a completely trivial complication which will be ignored for the sake of conciseness. Note that the fields B_i are labeled by their test functions ϕ_i.

Denoting by $\mathcal{T}$ the coefficient of the δ function in Eq. (23) and making use of the Eq. (18) we may write

$$\mathcal{T}(f;\mathbf{q}_1\mathbf{q}_2)=-(2\pi)(4\omega_1\omega_2)^{-1/2}\int d^3x\times\exp[iq_2\cdot x]\overleftrightarrow{\partial}_{0x}\langle f|J_1^\dagger(0)B_2^\dagger(x)|0\rangle. \quad (24)$$

In the barycentric system with total initial energy E_i, the preceding equation becomes

$$\mathcal{T}(f;\mathbf{k},-\mathbf{k})=-2\pi E_i^{-1}\exp[-\tfrac{1}{2}iE_ix_0]\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\left[\frac{\partial}{\partial x_0}+\frac{1}{2}iE_i\right]\times\langle f|J_1^\dagger(-\tfrac{1}{2}\mathbf{x},0)B_2^\dagger(\tfrac{1}{2}\mathbf{x},x_0)|0\rangle$$
$$=-2\pi i\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\times\langle f|J_1^\dagger(-\tfrac{1}{2}x)B_2^\dagger(\tfrac{1}{2}x)|0\rangle|_{x_0=0}. \quad (25)$$

The second part of Eq. (25) follows from the support properties of $\tilde{B}(p)$. Since only 1-particle intermediate states can occur between J^* and B^* the differentiation by $\partial/\partial x_0$ brings down a factor $\frac{1}{2}iE_i$, and x_0 can be chosen to be zero since $B^*(f,t)|0\rangle$ is time-independent.

Now choose a z axis along $\mathbf{k}$, let $\mathbf{p}$ be an appropriate vector characterizing the orientation of the final state system with respect to $\mathbf{k}$ (for example, the momentum of i final particle), and note that we can write

$$\langle f|J_1^\dagger(-x/2)B_2^\dagger(x/2)|0\rangle=A(|\mathbf{x}|^2,\mathbf{p}\cdot\mathbf{x},p,\alpha) \quad (26)$$

where α represents the remaining final-state quantum numbers. Then Eq. (25) may be expanded into the form

$$\mathcal{T}(f;q_1,q_2)=-2\pi_i\sum_l(2l+1)i^l\times P_l(\hat{k}\cdot\hat{p})\int_0^\infty r^2dr\; j_l(kr)A_l(r^2,p^2,\alpha)$$
$$\equiv-2\pi i\sum_l(2l+1)i^lP_l(\hat{k}\cdot\hat{p})\mathcal{T}_l(f;k,p;\alpha) \quad (27)$$

where

$$P_l(\hat{p}\cdot\hat{k})A_l(r^2,p^2,\alpha)=\int d\Omega_{\mathbf{x}}P_l(\hat{k}\cdot\hat{x})A(|\mathbf{x}|^2,\mathbf{p}\cdot\mathbf{x},p^2,\alpha) \quad (28)$$

and we note that for large r^2, $A_l(r^2,\cdots)$ must vanish faster than any inverse power of $|\mathbf{r}|$ as a result of property (IV). Thus, we have the interesting result

$$\lim_{k\to0}k^{-l}\mathcal{T}_l(f;k,p,\alpha)=[(2l+1)!!]^{-1}\int_0^\infty r^{l+2}dr\times A_l(r^2,p^2,\alpha)=\beta_l<\infty. \quad (29)$$

The achievement of Eq. (29) enables us to state two lemmas concerning production processes[16]: Lemma 1: For an exothermic reaction of the form

$$a+b\to a'+b'+\cdots,$$

where a and b are spinless particles, the lth partial-wave amplitude grows from threshold ($k=0$) *as* $|\mathbf{k}|^l$ *where* $\mathbf{k}$ *is the barycentric momentum of one of the initial particles*. A trivially obvious modification is Lemma 2: *For an endothermic reaction of the form* $a+b\to c+d$ *where c and d are spinless particles the lth partial-wave amplitude grows from threshold as* $|\mathbf{p}|^l$ *where* $\mathbf{p}$ *is the barycentric momentum of one of the final particles.*

The case we originally set out to consider is that of elastic scattering. In the barycentric system we may expand Eq. (25) into[17]

$$\mathcal{T}(\mathbf{p},\mathbf{k};E)\equiv\mathcal{T}(\mathbf{p},-\mathbf{p};\mathbf{k},-\mathbf{k})=-2\pi i\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\langle E,\mathbf{p}|J_1^\dagger(-x/2)B_2^\dagger(x/2)|0\rangle|_{x_0=0},$$
$$=(2\pi)^{-2}iE^{-1}\int d^3\xi\exp(-i\mathbf{p}\cdot\xi)\lim_{\eta_0\to\infty}\int d^3\eta\, e^{iE\eta_0}[\partial_{0\xi}{}^2-\tfrac{1}{4}(\partial_{0\eta}-iE)^2]\times\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\langle0|B_3(\eta+\xi/2)B_4(\eta-\xi/2)J_1^\dagger(-x/2)B_2^\dagger(x/2)|0\rangle|_{x_0=0}$$
$$\equiv(2\pi)^{-2}iE^{-1}\int d^3\xi d^3x\exp(-i\mathbf{p}\cdot\xi)\langle\xi|T_E|\mathbf{x}\rangle e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (30)$$

which exhibits the p and k dependence and the rotational invariance of $\langle\xi|T_E|\mathbf{x}\rangle$. Expanding the two exponentials in the last part of Eq. (30) in partial waves and carrying out the two angular integrations gives the result, analogous to Eq. (11),

$$\mathcal{T}(\mathbf{p},\mathbf{k};E)=i(2/E)\sum_{l=0}^\infty(2l+1)P_l(\hat{p}\cdot\hat{k})\mathcal{T}_l(p,k;E) \quad (31a)$$

where

$$\mathcal{T}_l(p,k;E)=\int_0^\infty\xi^2d\xi\times j_l(p\xi)\int_0^\infty x^2dx\; j_l(kx)\langle\xi|T_E{}^l|x\rangle \quad (31b)$$

and

$$2\langle\xi|T_E{}^l|r\rangle=(2\pi)^{-2}\int d\Omega_\xi\int d\Omega_xP_l(p\cdot\xi)\times\langle\xi|T_E|\mathbf{x}\rangle P_l(\mathbf{k}\cdot\mathbf{x}). \quad (31c)$$

What has been shown so far is that forp hysical E Eq. (31c) provides an off-mass-shell extrapolation in $\mathbf{p}$ and $\mathbf{k}$. Further, from the rapid decrease of $\langle\xi|T_E{}^l|x\rangle$

[16] Compare Brenig and R. Haag, Fortschr. Physik 7, 183 (1959).

[17] One could also derive expressions for $\langle|T_E|\mathbf{x}\rangle$ involving time-ordered or retarded products of J_4 and J_1.

in ξ *and* x (see Appendix) we find that

$$\lim_{p\to 0} p^{-l} \lim_{k\to 0} k^{-l} \mathcal{T}_l(p,k,E) = [(2l+1)!!]^{-2} \times \int_0^\infty \xi^{2l+2} d\xi \int_0^\infty x^{2l+2} dx \times \langle \xi | T_E{}^l | x \rangle < \infty \quad (32)$$

and therefore,

$$\mathcal{T}_l(k,k,E) = O(k^{2l}). \quad (33)$$

Assuming that the off-the-mass-shell amplitude $\mathcal{T}_l(k,k,E)$ is finite at the threshold energy $E_0 = 2m$, we arrive at the desired result:

$$\mathcal{T}_l(k,k,2(m^2+k^2))^{1/2} = O(k^{2l}), \quad (34)$$

Finally in order to relate the result of our discussion to the behavior of the phase shifts we remark that

$$i\mathcal{T}_l(k,k,E_k) = -4k^{-1}[\exp(2i\delta_l)-1], \quad (35)$$

where δ_l is the usual (possibly complex) scattering phase shift. Thus, we have shown that in axiomatic field theory (subject to the following remarks),

$$\sin\delta_l \underset{k\to 0}{\sim} k^{2l+1}. \quad (36)$$

Just as in the discussion in Sec. II we must now consider the posibility that the Eq. (36) may not hold in certain (we may hope) exceptional cases. The point is that in general we have to expect that $\langle \xi | T_E | \mathbf{x} \rangle$ and the offshell extrapolation $\mathcal{T}_l(k,k,E)$ are distributions in the energy E so that our result holds rigorously only for a smeared object like

$$\mathcal{T}_l(k,k,f) = \int dE\, f(E) \mathcal{T}_l(k,k,E). \quad (37)$$

Thus, if $\mathcal{T}_l(k,k,E)$ happens to have a pole at threshold energy the threshold behavior will be modified, so that

$$\lim_{k\to 0} k^{-n_l} \mathcal{T}_l(k,k,2(m^2+k^2)^{1/2}) < \infty \quad (38)$$

for some $n_l < 2l$. As in the case of potential scattering, this will be connected with the appearance of bound states at threshold energy. We offer no proof that such a pole must be of first order, if any. However, if the imaginary part of the scattering amplitude has a probabilistic interpretation, i.e., is a positive measure, the worst singularity can only be

$$\mathcal{T}_l(k,k,E) \sim \frac{1}{E-E_0-i\eta} \quad \text{for} \quad E \approx E_0 \quad l \geqslant 1 \quad (39)$$

which yields $n_l = 2l-1$, since

$$\frac{k^{2l+1}}{2(m^2+k^2)^{1/2}-2m-i\eta} \underset{k\to 0}{\longrightarrow} \frac{1}{m} k^{2l-1}. \quad (40)$$

In any case we want to emphasize that a deviation from the "usual" threshold behavior, Eq. (36), is of somewhat accidental nature even in field theory.

V. FINAL REMARKS

We began this work with the conjecture that our desired result, Eq. (36), would follow from the assumption of a finite "gap" in the mass-spectrum (absence of zero mass particles). This, we supposed, is the field-theoretic way of saying that the operative forces act only over finite ranges. As a consequence of the mass-spectral assumption we got property IV in Sec. III and, as expected, the threshold behavior followed in quite a straightforward manner.

It is well to remind ourselves, however, that there is at least one other very important assumption that has to do with the distribution character of our field operators underlying our result. Property III of Sec. III is derived[4] under the supposition that products of field operators behave as (operator-valued) tempered distributions in x space. Our slightly more general assumption concerning the $\tilde{A}(p)$ is harmless[18] insofar as property III is concerned, but one is left to wonder about the significance, if any, of this p-space freedom. In particular, one might wish to proceed to the case where the fields are localizable in x space, thereby interchanging the roles of $A(x)$ and $\tilde{A}(p)$ in our assumption (A), Sec. III. In this case we would lose the strong convergence property III, gaining at the same time an exponential rate of approach to the cluster decomposition limit[19] in property IV. We ask again whether these very mathematical considerations have anything to do with physics or whether they merely reveal our inadequate ability to describe the real world in mathematical terms.[20]

Let us close by remarking that the question of proceeding to the local limit in x space is not without interest in the present context. We have some preliminary indications that the sharper version of the cluster-decomposition theorem[19] adds enough to our knowledge of threshold behavior to permit continuation of the partial wave expansion to unphysical angles. If our present understanding is correct, the continuation will permit extension of the region of analyticity in

[18] See Ref. 13. Under our assumptions the $A(x)$ are defined on the set of entire functions that have rapid decrease on the real axis and (at most) exponential growth away from the real axis. The limitation on the rate of growth of $A(x)$ for real argument is the only tempered distribution property that is invoked in the proof of the cluster decomposition theorem and the asymptotic condition in Ref. 4. That is why our slightly more general definition of the $A(x)$ causes no difficulty with statements (III) and (IV) of Sec. III.

[19] H. Araki, K. Hepp, and D. Ruelle, Helv. Phys. Acta **35**, 164 (1962).

[20] Let us recall that the Froissart-Greenberg-Low bound on the rate of increase of total cross-sections at high energy makes use of temperateness in p space. Thus, it would appear that experimental violation of this bound could most easily be explained by dropping the assumption of temperateness as we have done in the present work. We refer, of course, to the work of M. Froissart, Phys. Rev. **123**, 1053 (1961) and O. W. Greenberg and F. E. Low, *ibid.* **124**, 2047 (1961).

momentum transfer, at low energies, to the point where a pole could occur corresponding to the particle with lightest mass that is exchanged. The continuation does not seem to depend upon the existence of any dispersion relations for the scattering amplitude or upon crossing symmetry, in contrast to the recent work of Martin.[21] We hope to present the detailed results in a forthcoming paper.

APPENDIX

The statement that $\langle\xi|T_E{}^l|x\rangle$ decreases rapidly in both ξ and x requires additional justification because of the limiting process $\eta_0\to\infty$ in Eq. (30). In order to discuss this it is convenient to rewrite the second Eq. (30) in the form

$$\mathcal{T}(\mathbf{p},\mathbf{k};E)=-iE(2\pi)^2\lim_{\eta_0\to\infty}e^{iE\eta_0}\int d^3\eta\int d^3\xi\int d^3x \times\exp(-ip\cdot\xi)\langle 0|B_3(\eta+\xi/2)B_4(\eta-\xi/2) J_1{}^\dagger(-x/2)B_2{}^\dagger(x/2)|0\rangle e^{i\mathbf{k}\cdot\mathbf{x}},\quad (A1)$$

where the effect of the differential operator in Eq. (30) is determined by inserting expansions in intermediate states. By translation invariance, this may also be written in the form

$$\mathcal{T}(\mathbf{p},\mathbf{k};E)=-iE(2\pi)^2\lim_{\eta_0\to-\infty}e^{-iE\eta_0}\int d^3\eta\int d^3\xi\int d^3x \times\exp(-i\mathbf{p}\cdot\xi)\langle 0|B_3(\xi/2)B_4(-\xi/2)J_1{}^\dagger(\eta-x/2) \times B_2{}^\dagger(\eta+x/2)|0\rangle e^{i\mathbf{k}\cdot\mathbf{x}}.\quad (A2)$$

Finally, we note that we may take the limit indicated in Eq. (A2) provided that E is the energy corresponding to the relative momentum $\mathbf{k}$. The result is

$$\mathcal{T}(\mathbf{p},\mathbf{k};E)\sim\int d^3\xi \times\exp(-i\mathbf{p}\cdot\xi)\langle 0|B_3(\xi/2)B_4(-\xi/2)|\mathrm{in}\rangle,\quad (A3)$$

where $|\mathrm{in}\rangle$ represents an asymptotic state of momentum $\mathbf{k}$.

We first show that the integrand in Eq. (A2) is of rapid decrease in $\mathbf{x}$ in the limit $\eta_0\to\infty$ (it clearly is for finite η_0). To see this we need only remark that

$$|\lim_{\eta_0\to\infty}e^{iE\eta_0}\langle B_3(\xi/2)B_4(-\xi/2)J_1{}^\dagger(\eta-x/2)B_2{}^\dagger(\eta+x/2)\rangle_0| \leqslant\|\langle 0|B_3(\xi/2)B_4(-\xi/2)\| \lim_{\eta_0\to\infty}{}^2J_1{}^\dagger(\eta-x/2)B_2{}^\dagger(\eta+x/2)|0\rangle\|.\quad (A4)$$

The first norm on the right is bounded. The second norm is independent of η and decreases for large $|\mathbf{x}|$ faster than any power of $|\mathbf{x}|^{-1}$ by virtue of Eq. (22).

[21] A. Martin, Nuovo Cimento **42A**, 930 (1966).

It is more tedious to show that the integrand in Eq. (A3) decreases rapidly for large $|\xi|$. We begin by considering the matrix element

$$\langle 0|B_1(f_1{}^{-a},t)B_2(f_2{}^a,t)|\hat{g}_1\hat{g}_2,\mathrm{in}\rangle\equiv\langle\Phi^a(t)|\mathrm{in}\rangle,\quad (A5)$$

where $B(f^a,t)=U(\mathbf{a})B(f,t)U(\mathbf{a})^{-1}$ and $U(\mathbf{a})$ is the translation operator. Taking note of statement III, Sec. III and applying the Schwarz inequality leads to the result that

$$|\langle\Phi^a(t)|\hat{g}_1\hat{g}_2,\mathrm{in}\rangle-\langle\hat{f}_1{}^{-a}\hat{f}_2{}^a,\mathrm{in}|\hat{g}_1\hat{g}_2,\mathrm{in}\rangle| =\left|\int_{-\infty}^t du\frac{d}{du}\langle\Phi^a(u)|\mathrm{in}\rangle\right| \leqslant\|\mathrm{in}\rangle\|\int_{-\infty}^t du\|\frac{d}{du}\langle\Phi^a(u)\|.\quad (A6)$$

However, it was shown by Haag[3] that $\|(d/du)\langle\Phi^a(u)\|$ vanishes for large u at least as fast as $u^{-3/2}$. Thus, using the cluster-decomposition theorem and Eq. (22) again we find that the right side of Eq. (A6) vanishes for large $|\mathbf{a}|$ faster than any power of $|\mathbf{a}|^{-1}$. On the other hand the $\langle\mathrm{in}|\mathrm{in}\rangle$ matrix element on the left side of Eq. (A6) is also rapidly decreasing in $|\mathbf{a}|$ since it is just a product of Fourier transforms with respect to $\mathbf{a}$ of test functions. We have shown, therefore, that

$$\lim_{|\mathbf{a}|\to\infty}|\mathbf{a}|\sigma\langle\Phi^a(t)|\mathrm{in}\rangle=0,\quad N=0,1,\cdots,\quad (A7)$$

and we can pass to the (distribution) limit of sequences of wave packets f_i and g_i to make the same assertion concerning

$$\int d^3x\int d^3y\exp[-i(p_1\cdot x+p_2\cdot y)]\overleftrightarrow{\partial}_{0x}\overleftrightarrow{\partial}_{0y} \times\langle 0|B_1(x-\mathbf{a})B_2(y-\mathbf{a})|\mathrm{in}\rangle=-(2\pi)^3E_fE_i\delta(\mathbf{p}_1+\mathbf{p}_2) \times\exp[i(E_f-E_i)\eta_0]\int d^3\xi\, e^{ik\cdot\xi} \times\langle 0|B_1(\xi/2-\mathbf{a})B_2(-\xi/2+\mathbf{a})|\mathrm{in}\rangle,\quad (A8)$$

where $x-\mathbf{a}$ means $x-(\mathbf{a},0)$, etc. In the last equation we have passed to the "barycentric" and relative coordinates η and ξ. The Lorentz frame was chosen so that the total momentum of the state $|\mathrm{in}\rangle$ is zero and energy E_i whereas E_f is the energy component of p_1+p_2 and $k=p_1-p_2$. Noting, finally, that the sequence (in $\mathbf{a}$) of quantities represented by the integral in (A8) tends to zero faster than any power of $|\mathbf{a}|^{-1}$ for arbitrary $\mathbf{k}$, we may assert the same for the Fourier transform with respect to $\mathbf{k}$ thereby achieving the desired result.

$$\lim_{|\xi|\to\infty}|\xi|^N\langle 0|B_1(\xi)B_2(-\xi)|\mathrm{in}\rangle=0.\quad N=0,1,\cdots.\quad (A9)$$