Spin-Matrix Polynomial Development of the Hamiltonian for a Free Particle of Arbitrary Spin and Mass*

S. A. WILLIAMS, J. P. DRAAYER,† AND T. A. WEBER
Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa
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The Hamiltonian and other relevant physical operators for a free particle of arbitrary spin and mass, based upon the generalized Foldy-Wouthuysen transformation, are developed in terms of the spin-matrix polynomials. The general theory of these polynomials, given in an earlier paper, is reviewed briefly. It is a general feature of these expansions that the number of terms is determined by the spin of the particle, but the coefficients are spin-independent. The massless limit is examined in some detail and the invariant metric operator is discussed.

A. INTRODUCTION

IN an earlier paper[1] (hereafter referred to as I) a technique was developed for the expansion of an arbitrary analytic function of a spin matrix in terms of spin-matrix polynomials (SMP). These polynomials are so constructed as to incorporate the Cayley-Hamilton theorem. Therefore, for fixed spin s the expansion is automatically given as a polynomial of degree $2s$ in the spin matrix, since the SMP of higher degree then vanish identically. In I, the rotation operator for arbitrary but fixed spin was derived as an application of the technique. It is the purpose of this paper to apply the technique to the derivation of the Hamiltonian and other related operators for a particle of fixed, but arbitrary, spin and mass following the theory of Good, Hammer, and Weaver.[2] In Sec. B the equations based upon the results of I are given in order to establish a modified notation. In Sec. C the explicit forms of the generalized Foldy-Wouthuysen transformation and its inverse are given, together with the Hamiltonian and polarization operators. Finally, in Sec. D the massless limit is examined and the invariant metric is given.

B. EXPANSIONS IN TERMS OF THE SMP

The SMP defined in I may be conveniently written in terms of Appell's[3] symbol as

$$W_{2n+1}(z)=(z-n,2n+1)\,, \tag{1}$$

which are applicable to integral spin, and as

$$W_{2n}(z)=(z-n+\tfrac{1}{2},2n)\,, \tag{2}$$

which are applicable to half-integral spin. The Appell symbol of Eqs. (1) and (2) is defined as

$$(a,k)=a(a+1)(a+2)\cdots(a+k-1)\,, \tag{3}$$

for integral k and complex a. When k is a negative integer, $-\kappa$, the symbol is not defined for $a=1,2,\cdots,\kappa$.

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† NASA Fellow.
[1] T. A. Weber and S. A. Williams, J. Math. Phys. **6**, 1980 (1965).
[2] D. L. Weaver, C. L. Hammer, and R. H. Good, Jr., Phys. Rev. **135**, B241 (1964).
[3] P. Appell, Compt. Rend. **90**, 286 (1880).

The Appell symbol satisfied the following useful relationships:

$$(a,k+1)=(a,k)(a+k,1)\,, \tag{4a}$$

$$(a,k)=(-1)^k/(1-a,-k)\,, \tag{4b}$$

$$\sum_{k=0}^{n}\frac{(a,k)(b,n-k)}{(1,k)(1,n-k)}=\frac{(a+b,n)}{(1,n)}\,. \tag{4c}$$

An arbitrary analytic function is decomposed into its even and odd parts and expanded in terms of the SMP as

$$f_e(z)=f_e(0)+\sum_{n=0}a(n)zW_{2n+1}(z)\,, \tag{5a}$$

$$f_0(z)=\sum_{n=0}b(n)W_{2n+1}(z)\,, \tag{5b}$$

which is suitable for integral spin, while for half-integral spin we have

$$f_e(z)=\sum_{n=0}\alpha(n)W_{2n}(z)\,, \tag{6a}$$

$$f_0(z)=\sum_{n=0}\beta(n)zW_{2n}(z)\,. \tag{6b}$$

Using the ascending difference operator of unit step, it was shown in I [Eqs. (9) and (17)] that

$$b(n)=\sum_{k=0}^{2n+1}\frac{(-1)^k}{(1,2n+1-k)(1,k)}f_0(n-k)\,. \tag{7}$$

It is also possible to include in these coefficients an arbitrary constant c. This is accomplished by rewriting Eq. (7) as

$$b(n)=\sum_{k=0}^{n}\frac{(-1)^k}{(1,2n+1-k)(1,k)}f_0(n-k)+\sum_{k=n+1}^{2n+1}\frac{(-1)^k}{(1,2n+1-k)(1,k)}f_0(n-k)\,. \tag{8}$$

In Eq. (8), replace k by $n-k$ in the first sum and by $n+k$ in the second. Then, by noting that $1/(1,-1)=0$, and that f_0 denotes an odd function of its argument, one may write Eq. (8) as

$$b(n)=\sum_{k=1}^{n+1}\frac{2k(-1)^{n+1+k}}{(1,n+1-k)(1,n+1+k)}f_0(k).$$

Finally, from Eq. (4c) this may be written as

$$b(n)=\frac{c(-1)^n}{(2n+1)(1,n+1)(1,n)}+\sum_{k=1}^{n+1}\frac{2k(-1)^{n+1+k}}{(1,n+1+k)(1,n+1-k)}[f_0(k)-c], \quad (9)$$

where c is an arbitrary constant. The proof that the added and subtracted terms in Eq. (9) are equal is somewhat involved, and is given in the Appendix.

In a similar manner we find

$$a(n)=\frac{c(-1)^{n+1}}{(1,n+1)(1,n+1)}+\sum_{k=1}^{n+1}\frac{2(-1)^{n+1+k}}{(1,n+1+k)(1,n+1-k)}\times[f_e(k)-f_e(0)+c], \quad (10)$$

$$\alpha(n)=c\delta_{n0}+\sum_{k=1}^{n+1}\frac{(2k-1)(-1)^{n+1+k}}{(1,n+1-k)(1,n+k)}[f_e(k-\tfrac{1}{2})-c], \quad (11)$$

and

$$\beta(n)=\frac{2c(-1)^n}{(2n+1)(1,n)(1,n)}+\sum_{k=1}^{n}\frac{2(-1)^{n+1+k}}{(1,n+1-k)(1,n+k)}[f_0(k-\tfrac{1}{2})-c]. \quad (12)$$

C. APPLICATION TO PARTICLES OF ARBITRARY SPIN AND MASS

Apart from a multiplicative constant, the generalized Foldy-Wouthuysen transformation operator S, which effects the transformation from the rest to laboratory frames, is given by Weaver, Hammer, and Good[2] as

$$S=\exp[s\epsilon\boldsymbol{\alpha}\cdot\mathbf{p}/p\tanh^{-1}(p/E)], \quad (13)$$

where s is the magnitude of the particle's spin, $\mathbf{p}$ is its momentum with magnitude p, and E is its energy. The theory is relativistic and therefore the rest mass is defined by the usual $E^2=p^2+m^2$. In Eq. (13) the quantity ϵ is the sign operator ± 1, and the matrices β and $\boldsymbol{\alpha}$ are of order $2(2s+1)$, being given by

$$\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad (14a)$$

$$\boldsymbol{\alpha}=\frac{1}{s}\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix}. \quad (14b)$$

In Eq. (14b), $\mathbf{s}$ is the spin operator (s_x,s_y,s_z) and it is obvious that $s\boldsymbol{\alpha}\cdot\mathbf{p}/p$ has the same eigenvalues as $\mathbf{s}\cdot\mathbf{p}/p$. We define

$$z=s\boldsymbol{\alpha}\cdot\mathbf{p}/p, \quad (15a)$$

and

$$\theta=\tanh^{-1}(p/E). \quad (15b)$$

Then, Eq. (13) may be rewritten as

$$S=\cosh(\theta z\epsilon)+\sinh(\theta z\epsilon). \quad (16a)$$

The time-independent operator, obtained from this by replacing ϵ by β, has the same effect as Eq. (16a) when operating upon the rest-frame wave functions.[2] Since $[z,\beta]_+=0$ and $\beta^2=1$, it is clear that Eq. (16a) with ϵ replaced by β may be written as

$$S=\cosh(\theta z)-\beta\sinh(\theta z). \quad (16b)$$

Since S is not unitary in the usual sense, we formally write its inverse as

$$S^{-1}=[\cosh(\theta z)-\beta\sinh(\theta z)]^{-1},$$

or as

$$S^{-1}=\cosh(\theta z)\,\mathrm{sech}(2\theta z)+\beta\sinh(\theta z)\,\mathrm{sech}(2\theta z). \quad (17)$$

Then, in terms of S and S^{-1} the Hamiltonian for a particle of arbitrary spin is given by

$$H/E=S\beta S^{-1},$$

which we may write as

$$H/E=\tanh(2\theta z)+\beta\,\mathrm{sech}(2\theta z). \quad (18)$$

In the following, we shall restrict the discussion to the expansion in SMP of $\tanh(2\theta z)$ for the case of integral spin and merely state the remaining results.

We shall for the moment regard z as a complex variable. From Eqs. (5b) and (9) it follows that

$$\tanh(2\theta z)=\sum_{n=0}\left\{\frac{c(-1)^n}{(2n+1)(1,n+1)(1,n)}+\sum_{k=1}^{n+1}\frac{2k(-1)^{n+1+k}}{(1,n+k+1)(1,n+1-k)}[\tanh(2\theta k)-c]\right\}W_{2n+1}. \quad (19)$$

Then from Eq. (15b) we have

$$\exp\theta=(E+p)/m \quad (20)$$

so that

$$\tanh(2\theta k)=(e^{2k\theta}-e^{-2k\theta})/(e^{2k\theta}+e^{-2k\theta})=\frac{(E+p)^{4k}-m^{4k}}{(E+p)^{4k}+m^{4k}}. \quad (21)$$

To separate out the terms which are mass independent we set $c=1$ and using Eq. (21) find that Eq. (19) becomes

$$\tanh(2\theta z)=\sum_{n=0}\left\{\frac{(-1)^n}{(2n+1)(1,n+1)(1,n)}+\sum_{k=1}^{n+1}\frac{4k(-1)^{n+k}}{(1,n+k+1)(1,n+1-k)}\frac{m^{4k}}{(E+p)^{4k}+m^{4k}}\right\}W_{2n+1}. \tag{22}$$

By similarly expanding the other functions of interest for both integral and half-integral spin and by employing the expansions for $\cos(\theta z)$ and $\sin(\theta z)$ as given in I, we find for integral spin

$$S=1+\sum_{n=0}\frac{[2(E-m)/m]^{n+1}}{(1,2n+2)}zW_{2n+1}(z)-\beta\sum_{n=0}\frac{(p/m)[2(E-m)/m]^n}{(1,2n+1)}W_{2n+1}(z), \tag{23a}$$

$$S^{-1}=1+\sum_{n=0}\left\{\frac{(-1)^{n+1}}{(1,n+1)(1,n+1)}+\sum_{k=1}^{n+1}\frac{2(-1)^{n+k+1}}{(1,n+k+1)(1,n+1-k)}[m(E+p)]^k\frac{(E+p)^{2k}+m^{2k}}{(E+p)^{4k}+m^{4k}}\right\}zW_{2n+1}(z)$$
$$+\beta\sum_{n=0}\left\{\sum_{k=1}^{n+1}\frac{2k(-1)^{n+1+k}}{(1,n+k+1)(1,n+1-k)}[m(E+p)]^k\frac{(E+p)^{2k}-m^{2k}}{(E+p)^{4k}+m^{4k}}\right\}W_{2n+1}(z), \tag{23b}$$

and

$$H/E=\sum_{n=0}\left\{\frac{(-1)^n}{(2n+1)(1,n+1)(1,n)}+\sum_{k=1}^{n+1}\frac{4k(-1)^{n+k}}{(1,n+1+k)(1,n+1-k)}\frac{m^{4k}}{(E+p)^{4k}+m^{4k}}\right\}W_{2n+1}(z)$$
$$+\beta\left[1+\sum_{n=0}\left\{\frac{(-1)^{n+1}}{(1,n+1)(1,n+1)}+\sum_{k=1}^{n+1}\frac{4(-1)^{n+k+1}}{(1,n+1+k)(1,n+1-k)}\frac{m^{2k}(E+p)^{2k}}{(E+p)^{4k}+m^{4k}}\right\}zW_{2n+1}(z)\right]. \tag{23c}$$

For half-integral spin the corresponding results are

$$S=\sum_{n=0}\left\{\frac{[(E+m)/(2m)]^{1/2}[2(E-m)/m]^n}{(1,2n)}\right\}W_{2n}(z)-\beta\sum_{n=0}\frac{[2(E-m)/m]^{n+1/2}}{(1,2n+1)}zW_{2n}(z), \tag{24a}$$

$$S^{-1}=\sum_{n=0}\sum_{k=1}^{n+1}\frac{(2k-1)(-1)^{n+1+k}}{(1,n+k)(1,n-k+1)}[m(E+p)]^{k-1/2}\frac{[(E+p)^{2k-1}+m^{2k-1}]}{[(E+p)^{4k-2}+m^{4k-2}]}W_{2n}(z)$$
$$+\beta\sum_{n=0}\sum_{k=1}^{n+1}\frac{2(-1)^{n+1+k}}{(1,n+k)(1,n-k+1)}[m(E+p)]^{k-1/2}\frac{[(E+p)^{2k-1}-m^{2k-1}]}{[(E+p)^{4k-2}+m^{4k-2}]}zW_{2n}(z), \tag{24b}$$

and

$$H/E=\sum_{n=0}\left\{\frac{2(-1)^n}{(2n+1)(1,n)(1,n)}-\sum_{k=1}^{n+1}\frac{(-1)^{n+1+k}}{(1,n-k+1)(1,n+k)}\frac{2m^{4k-2}}{(E+p)^{4k-2}+m^{4k-2}}\right\}zW_{2n}(z)$$
$$+\beta\sum_{n=0}\sum_{k=1}^{n+1}\frac{(2k-1)(-1)^{n+1+k}}{(1,n+k)(1,n-k+1)}\frac{2m^{2k-1}(E+p)^{2k-1}}{(E+p)^{4k-2}+m^{4k-2}}W_{2n}(z). \tag{24c}$$

If z is actually a complex variable, the upper limit of the sum on n in Eqs. (23) and (24) is ∞. For $z=s(\boldsymbol{\alpha}\cdot\mathbf{p})/p$, it is unnecessary to specify the upper limit because of the termination property of the SMP.

Mathews[4] has obtained a series expansion of H/E in terms of projection operators derived from the infinitisimal generators of the Poincaré group. His result can be shown to be equivalent to that given in Eqs. (23c) and (24c) by expanding the projection operators in terms of SMP. The coefficients in the expansion of H/E in terms of projection operators are also spin independent but the explicit form of a given projection operator depends upon the spin.

The polarization operator $\mathbf{O}$ is defined by

$$\mathbf{O}=S\beta\mathbf{s}S^{-1}. \tag{25}$$

Since $\mathbf{s}\cdot\mathbf{p}$ commutes with S, the projection of $\mathbf{O}$ onto the momentum direction is particularly simple and may be written as

$$\mathbf{O}\cdot\hat{p}=-\gamma_5 zH(z)/E. \tag{26}$$

[4] P. M. Mathews, Phys. Rev. **143**, 987 (1966).

Here, γ_5 in the representation of Eqs. (14) is given by

$$\gamma_5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{27}$$

If one uses Eq. (23c) or (24c) for $H(z)/E$ in Eq. (26), the resulting expression for $\mathbf{O}\cdot\mathbf{p}$ will be of one higher degree in $z=s(\boldsymbol{\alpha}\cdot\mathbf{p})/p$ than is necessary. This can be remedied by application of the recursion relationships

$$z^2W_{2n+1}(z)=W_{2n+3}(z)+(n+1)^2W_{2n+1}(z), \tag{28a}$$

and

$$z^2W_{2n}(z)=W_{2n+2}(z)+(n+\tfrac{1}{2})^2W_{2n}(z). \tag{28b}$$

Using Eqs. (28), it is a simple matter to express $\mathbf{O}\cdot\mathbf{p}$ in the standard form of Eq. (5) or Eq. (6); the result will not be given here.

D. THE MASSLESS LIMIT

Let the eigenfunctions for a particle of arbitrary mass be denoted by $v_{\epsilon k}$ in the rest frame. The $v_{\epsilon k}$ satisfy an energy eigenvalue equation

$$\beta v_{\epsilon k}=\epsilon v_{\epsilon k}, \tag{29}$$

and a polarization eigenvalue equation

$$\beta\mathbf{s}\cdot\hat{p}v_{\epsilon k}=kv_{\epsilon k}, \tag{30}$$

where we have taken the polarization axis along the momentum direction. The laboratory-frame eigenfunctions follow from the $v_{\epsilon k}$ by the application of the generalized Foldy-Wouthuysen transformation S. In the massless limit, however, S as defined by Eq. (13) does not exist, but m^sS does. Therefore, the laboratory functions are taken to be

$$\psi_{\epsilon k}=m^sSv_{\epsilon k}, \tag{31}$$

which are well defined even in the massless limit. One cannot, however, perform the inverse transformation back to the rest frame after the limit is taken, since the inverse $m^{-s}S^{-1}$ does not exist in the limit.

To demonstrate that in the massless limit the wave functions have only two components we shall use the functional form of S given by Eq. (16). It will be convenient to change from the variable $z=s\boldsymbol{\alpha}\cdot\hat{p}$ to the variable $x=\beta\mathbf{s}\cdot\hat{p}$. These are related by

$$z=-\gamma_5\beta x. \tag{32}$$

Then we have

$$\cosh(\theta z)=\cosh(-\theta\gamma_5\beta x)=\cosh(\theta x), \tag{33}$$

where we made use of the facts that the hyperbolic cosine is an even function of its argument and that $[\gamma_5\beta,x]_+=0$. Similarly,

$$\sinh(\theta z)=-\sinh(\theta\gamma_5\beta x)=-\gamma_5\beta\sinh(\theta x). \tag{34}$$

Then, from Eq. (16) we have finally

$$S=\cosh(\theta x)-\gamma_5\sinh(\theta x). \tag{35}$$

Apply S to $v_{\epsilon k}$ given by Eq. (30) and one has

$$Sv_{\epsilon k}=[\cosh(\theta k)-\gamma_5\sinh(\theta k)]v_{\epsilon k}. \tag{36}$$

But by using Eq. (20) we have, for example,

$$\cosh(\theta k)=\tfrac{1}{2}(e^{\theta k}+e^{-\theta k})=\tfrac{1}{2}\{[(E+p)/m]^k+[(E+p)/m]^{-k}\}. \tag{37}$$

It then follows that

$$Sv_{\epsilon k}=\tfrac{1}{2}\{[(E+p)/m]^k+[(E+p)/m]^{-k}-\gamma_5[(E+p)/m]^k+\gamma_5[(E+p)/m]^{-k}\}v_{\epsilon k},$$

or

$$Sv_{\epsilon k}=\tfrac{1}{2}\{(1-\gamma_5)[(E+p)/m]^k+(1+\gamma_5)[(E+p)/m]^{-k}\}v_{\epsilon k}. \tag{38}$$

Thus

$$\psi_{\epsilon k}=\tfrac{1}{2}m^s\{(1-\gamma_5)[(E+p)/m]^k+(1+\gamma_5)[(E+p)/m]^{-k}\}v_{\epsilon k}. \tag{39}$$

Now, in the limit as $m\to 0$, it is clear that only the terms with $k=\pm s$ are nonvanishing. Therefore we have in the limit

$$\psi_{\epsilon k}=\tfrac{1}{2}(2p)^s[(1-\gamma_5)\delta_{k,s}+(1+\gamma_5)\delta_{k,-s}]v_{\epsilon k} \tag{40}$$

which agrees with Good, Hammer, and Weaver.[2]

It is to be noted that this result can also be derived by using the explicit form of S in terms of the SMP. In that case, the only polynomials which remain in m^sS after the limit is taken are the projection operators for the states $k=\pm s$.

For other than the massless limit, Eq. (39) gives the helicity states, that is, states with the polarization along the direction of momentum. It is a simple matter to construct states of polarization along other than the momentum direction. For example, if $u_{\epsilon k}$ is defined by

$$\beta u_{\epsilon k}=\epsilon u_{\epsilon k},$$
$$\beta\mathbf{s}\cdot\hat{e}u_{\epsilon k}=ku_{\epsilon k}, \tag{41}$$

where $\hat{e}$ is some arbitrary direction, then clearly

$$u_{\epsilon k}=R(s,\boldsymbol{\varphi})v_{\epsilon k}, \tag{42}$$

where

$$R(s,\boldsymbol{\varphi})=e^{i\mathbf{s}\cdot\boldsymbol{\varphi}} \tag{43}$$

is the rotation operator for a rotation of $\varphi=\cos^{-1}(\hat{p}\cdot\hat{e})$ about the direction $\hat{p}\times\hat{e}$; i.e., a rotation from polarization along the $\hat{p}$ direction to polarization along the $\hat{e}$ direction. Under the generalized Foldy-Wouthuysen transformation, R becomes SRS^{-1}. Therefore, states of polarization along $\hat{e}$ may be written in terms of the helicity states, given in Eq. (39), by

$$\Psi_{\epsilon k}(\mathbf{p},s,\hat{e})=SRS^{-1}\psi_{\epsilon k}(\mathbf{p},s,\hat{p}). \tag{44}$$

The SMP expansion of R has already been given in I.

One can use the functional form of H given by Eq. (18) to show that in the massless limit the effective

Hamiltonian may be written as

$$H=-\gamma_5 \mathbf{s}\cdot\mathbf{p}/s. \tag{45}$$

As is evident from Eq. (40), the eigenfunction decomposes into two functions, each having $2s+1$ components and from the form of Eq. (41), the $2s+1$ square Hamiltonian is $\mathbf{s}\cdot\mathbf{p}/s$.

To derive Eq. (45) we shall use the polynomial form of H rather than the functional form. In particular, we shall illustrate the proof by doing explicitly the case of integral spin. From Eq. (23c) using Eq. (32) we have in the massless limit

$$H/E=\beta\gamma_5\sum_{n=0}\frac{(-1)^n}{(2n+1)(1,n+1)(1,n)}W_{2n+1}(x)+\beta\left[1+\sum_{n=0}\frac{(-1)^{n+1}}{(1,n+1)(1,n+1)}xW_{2n+1}(x)\right]. \tag{46}$$

Since x and γ_5 anticommute we have that

$$[W_{2n+1}(x),\gamma_5]_+=0, \tag{47a}$$

and

$$[xW_{2n+1}(x),\gamma_5]=0. \tag{47b}$$

Thus

$$H/E(1\pm\gamma_5)v_{\epsilon,\mp s}=\mp\beta\gamma_5(1\mp\gamma_5)\sum_{n=0}\frac{(-1)^nW_{2n+1}(s)}{(2n+1)(1,n+1)(1,n)}v_{\epsilon,\mp s}+\beta(1\pm\gamma_5)\left[1+\sum_{n=0}\frac{(-1)^{n+1}sW_{2n+1}(s)}{(1,n+1)(1,n+1)}\right]v_{\epsilon,\mp s}. \tag{48}$$

A little algebra using Eqs. (2) and (4) shows that

$$\sum_{n=0}\frac{(-1)^nW_{2n+1}(s)}{(2n+1)(1,n+1)(1,n)}=1, \tag{49a}$$

and

$$\sum_{n=0}\frac{(-1)^{n+1}sW_{2n+1}(s)}{(1,n+1)(1,n+1)}=-1. \tag{49b}$$

Therefore we have

$$H/E(1\pm\gamma_5)v_{\epsilon,\mp s}=\beta\gamma_5(1\mp\gamma_5)(\mp v_{\epsilon,\mp s}),$$

or

$$H/E(1\pm\gamma_5)v_{\epsilon,\mp s}=\beta\gamma_5(1\mp\gamma_5)(x/s)v_{\epsilon,\mp s}=\beta\gamma_5(x/s)(1\pm\gamma_5)v_{\epsilon,\mp s}. \tag{50}$$

Thus, when operating upon $\psi_{\epsilon k}$ the effective Hamiltonian in the massless limit is

$$H=p\beta\gamma_5x/s=-\gamma_5\mathbf{s}\cdot\mathbf{p}/s. \tag{51}$$

It is amusing to notice that if one merely replaces x by $\pm s$ in Eq. (46), Eq. (51) immediately follows.

We have used plane waves to investigate the massless limit. In the more general case, one would want to work with properly normalizable wave functions which may be written as[2]

$$\psi(\mathbf{r},t)=(2\pi)^{-3/2}m^s\int d^3pE^{-1}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})Sv_{\epsilon k}\times\exp[i(\mathbf{p}\cdot\mathbf{r}-\epsilon Et)], \tag{52a}$$

or defining the rest-frame function $\Phi(\mathbf{r},t)$ by

$$\Phi(\mathbf{r},t)=(2\pi)^{-3/2}\int d^3pE^{-1/2}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})v_{\epsilon k}\times\exp[i(\mathbf{p}\cdot\mathbf{r}-\epsilon Et)], \tag{52b}$$

we have

$$\psi(\mathbf{r},t)=E^{-1/2}m^sS\Phi(\mathbf{r},t), \tag{52c}$$

which is properly defined even in the massless limit. For each fixed spin, then, we can define an inner product

$$(\psi^{(l)},\psi^{(n)})=\int d^3pE^{-1}\sum_{\epsilon k}A_{\epsilon k}{}^{(l)*}(\mathbf{p})A_{\epsilon k}{}^{(n)}(\mathbf{p}). \tag{53}$$

This specifies a positive definite Hilbert space, and furthermore this inner product is Lorentz invariant. As shown in Ref. 2 this inner product is written as

$$(\psi^{(l)},\psi^{(n)})=m^{-2s}\int d^3r\psi^{(l)*}E(S^{-1})^\dagger S^{-1}\psi^{(n)}, \tag{54}$$

so that $m^{-2s}E(S^{-1})^\dagger S^{-1}$ plays the role of an invariant metric operator. Although $m^{-s}S^{-1}$ does not exist in the limit as $m\to 0$, this invariant metric must, as may be readily displayed. From Eq. (17) we have immediately that

$$\mathcal{G}\equiv m^{-2s}E(S^{-1})^\dagger(S^{-1})=m^{-2s}E\,\mathrm{sech}(2\theta z). \tag{55}$$

In passing we note that since

$$[\beta,\tanh 2\theta z]_+=0, \tag{56}$$

we have

$$[H,\beta]_+=2E\,\mathrm{sech}(2\theta z), \tag{57}$$

so that

$$\mathcal{G}=(1/2m^{2s})[H,\beta]_+. \tag{58}$$

It is easy to express the metric $\mathcal{G}$ in terms of SMP by noting that $\mathrm{sech}(2\theta z)$ appears in the Hamiltonian given in Eq. (18) with the coefficient β. Thus, in terms of SMP, $\mathrm{sech}(2\theta z)$ is just the factor multiplying β in Eqs. (23c) and (24c).[5]

Recently, it has been shown[6] that the metric $[H,\beta]_+$ gives an invariant integral by using the fact that the wave function ψ satisfies the Schrödinger-Klein-Gordon equation and the assumption that the Hamiltonian H

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[5] Matthews has also noted that $\mathcal{G}$ is proportional to the coefficient of β in H. See P. M. Matthews, Phys. Rev. 143, 985 (1966).

[6] D. Shay, H. S. Song, and R. H. Good, Jr., Suppl. Nuovo Cimento 3, 455 (1966).

is Hermitian in the usual way. That H is Hermitian is apparent from our representation in terms of SMP. Furthermore, Eqs. (55) and (58) establish the equivalence of the metric $[H,\beta]_+$ and the metric $m^{-2s}E(S^{-1})^\dagger(S^{-1})$ used in Ref. 2.

The metric for the case of the massless limit,

$$\mathcal{g}=(2p)^{1-2s}, \tag{59}$$

is particularly simple in that it contains no matrices. In deriving this result, one must use the fact that k can have only the values $\pm s$ in this limit.

E. CONCLUSION

We have shown how the Hamiltonian and other relevant physical operators for a free particle of arbitrary spin and mass may be formulated in terms of the spin matrix polynomials. The principle advantage of this formulation is that while the number of terms in the expansion is determined by the spin, the coefficients are spin-independent. The massless limit becomes particularly simple in that after the limit is taken only those SMP which project out the states of $\pm$ helicity remain in the transformation.

The SMP have of course been developed for the cases of a single fixed value of spin. Thus, they are applicable to problems involving a single representation of $SU(2)$ or $R(3)$. It is interesting to speculate about the use of suitable generalizations of these polynomials to problems involving higher groups wherein a single irreducible representation of the group may contain several irreducible representations of $SU(2)$.

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APPENDIX

In order to complete the derivation of Eq. (9) it is necessary to show that

$$D\equiv\sum_{k=1}^{n+1}\frac{2k(-1)^{k+1}}{(1,n+1+k)(1,n+1-k)}=\frac{1}{(2n+1)(1,n+1)(1,n)}. \tag{A1}$$

First the summation is changed to go from 0 to n and Eq. (4b) with $a=n+3$ is used to eliminate $(-1)^k$.

$$D=\sum_{k=0}^{n}\frac{2(k+1)(-1)^k}{(1,n+2+k)(1,n-k)}=\sum_{k=0}^{n}\frac{2(n+3,k)(-2-n,-k)(1,k+1)}{(1,n+2+k)(1,n-k)(1,k)}. \tag{A2}$$

From Eq. (4a) one obtains

$$(1,n+2+k)=(1,n+2)(n+3,k), \tag{A3}$$

$$(-2-n,-k)=(-2-n,-n)(-2-2n,n-k), \tag{A4}$$

and from the definition of the Appell symbol

$$(1,k+1)=(2,k). \tag{A5}$$

Making these substitutions, Eq. (A2) can be written as

$$D=\frac{2(-2-n,-n)}{(1,n+2)}\sum_{k=0}^{n}\frac{(-2-2n,n-k)(2,k)}{(1,n-k)(1,k)}=\frac{2(-2-n,-n)(-2n,n)}{(1,n+2)(1,n)}, \tag{A6}$$

the summation following from Eq. (4c). Next using Eq. (4b) we set

$$(-2n,n)=\frac{(-1)^n}{(1+2n,-n)}, \tag{A7}$$

and

$$(-2-n,-n)=\frac{(-1)^n}{(3+n,n)}, \tag{A8}$$

to obtain

$$D=2[(1,n+2)(1,n)(1+2n,-n)(3+n,n)]^{-1}. \tag{A9}$$

Now setting

$$(n+3,n)=(2n+2)(2n+1)(n+3,n-2), \tag{A10}$$

and applying Eq. (4a) twice

$$(n+3,n-2)(1,n+2)=(1,2n), \tag{A11}$$

$$(1,2n)(1+2n,-n)=(1,n), \tag{A12}$$

we have

$$D=1[(2n+1)(n+1)(1,n)(1,n)]^{-1}. \tag{A13}$$

Then Eq. (A1) follows with the substitution

$$(n+1)(1,n)=(1,n+1). \tag{A14}$$