we use Glauber's P representation[5] for the density operator:

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha=\frac{\pi}{h}\int\int P(\alpha)|\alpha\rangle\langle\alpha|dp'dq'.$$

With the P representation and a few steps of manipulation, we can show that

$$(\Delta p')^2(\Delta q')^2=\hbar^2+\frac{\pi}{h}\int\int(q'-q_0)^2(p'-p_0)^2P(\alpha)dp'dq'.$$

Now, we compute $f(\alpha^*)$ by Eq. (4.11) of Glaubers'[5] with the choice

$$|f\rangle=\tfrac{1}{2}i\hbar[(a^\dagger-\alpha_0^*)^2-(a-\alpha_0)^2]|\alpha\rangle.$$

Substituting the result for $|f(\alpha^*)|^2$ just obtained into Eq. (7.9) of Glauber's, we obtain

$$\frac{\pi}{h}\int\int(q'-q_0)^2(p'-p_0)^2P(\alpha)dp'dq'\geq 0,$$

where the equality holds if $P(\alpha)=\delta^2(\alpha-\alpha_0)=(h/\pi)\times\delta(q'-q_0)\delta(p'-p_0)$. Therefore, $(\Delta p')^2(\Delta q')^2\geq\hbar^2$ and the equality (minimum) holds if the state immediately before the ideal simultaneous measurement is a coherent state, i.e., if $P(\alpha)=\delta^2(\alpha-\alpha_0)$ or $\rho=|\alpha_0\rangle\langle\alpha_0|$.

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Characteristic States of the Electromagnetic Radiation Field

M. M. MILLER AND E. A. MISHKIN
Polytechnic Institute of Brooklyn, Brooklyn, New York
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It has been argued that the positive-frequency part of the quantized electromagnetic field is the "observable" that one would most naturally associate with field measurements using quantum photodetectors. However, since it is possible in principle to make field measurements via the process of stimulated emission, the question of the possible solutions of the characteristic-value equation for the creation operator $a^\dagger$ is examined. Various proofs are given to demonstrate that the characteristic kets of $a^\dagger$ are not physically admissible states of the radiation field. The possible existence of other useful basis states besides $|n\rangle$, $|\alpha\rangle$, and states generated from these by unitary transformations is then considered. It is shown that when certain restrictions are placed on the correspondence between Hermitian combinations of the arbitrary non-normal operators b and $b^\dagger$ and the harmonic-oscillator variables x, p, and H, then the only possible basis states are the coherent states $|\alpha\rangle$ and the number states $|n\rangle$. A λ-dependent variation on the photon annihilation operator a is also considered. Its characteristic states for $-1\leq\lambda\leq 1$ are derived, and shown to form a complete set.

I. INTRODUCTION

THE recent development[1–7] of a quantum-mechanical theory of optical coherence has demonstrated the utility of the characteristic states of the non-Hermitian, non-normal boson annihilation operator a, the quasiclassical or coherent states. For a single-mode radiation field the coherent state vector $|\alpha\rangle$ satisfies the characteristic value equation,

$$a|\alpha\rangle=\alpha|\alpha\rangle, \tag{1}$$

with α as its corresponding complex characteristic value. Although the $|\alpha\rangle$ states are not orthogonal, that is,

$$|\langle\alpha|\beta\rangle|^2=\exp(-|\alpha-\beta|^2), \tag{2}$$

they can be normalized to unity, $\langle\alpha|\alpha\rangle=1$. The $|\alpha\rangle$ states also constitute a basis for the representation of arbitrary states and operators of the radiation field since the nonorthogonal projection operators $|\alpha\rangle\langle\alpha|$ satisfy a completeness relation of the form[8]

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \tag{3}$$

where $d^2\alpha=d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$ is the real element of area, and the integration extends over the entire complex plane. Because of their nonorthogonality, expansions in terms of coherent states are in general not unique unless additional restrictions are placed upon the expansion coefficients.[4] In contrast to the infinite complete sequence of occupation-number states $|n\rangle$, $n=0, 1, 2, \cdots, \infty$ which form an orthonormal basis for the field state vectors, the basis formed by the $|\alpha\rangle$ characteristic states constitutes a complete nondenumerable infinity of normalized characteristic vectors which are not

[1] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
[2] E. C. G. Sudarshan, *Quantum Optics, Lecture Notes* (University of Bern, Bern, Switzerland, 1963).
[3] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[4] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[5] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
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linearly independent. It is possible, at least in principle, to extract from the continuous set α, discrete subsets of complex numbers α_n, $n=0, 1, 2, \cdots, \infty$ such that the vectors $|\alpha_n\rangle$ span the Hilbert space of the $|n\rangle$ states if the subset α_n is convergent. In this sense the coherent states are "super-complete."[9] The physical significance of the coherent states is evidenced by the fact that the observed Poisson counting statistics of an unimodal amplitude-stabilized laser and the interference fringes and beats[10–12] produced by the superposition of two independent laser beams can be explained by assuming that the ideal laser field is represented by a pure coherent state or by a random-phase ensemble of coherent states.[13,14] A theoretical analysis[15] also shows that a pure coherent state is one possible solution of the nonlinear equations of motion describing the interaction of the laser source and its emitted radiation field.

It should be noted that the basis formed by the coherent states will be the "natural" basis to use in quantum electrodynamics, not only in optical-coherence theory but in all problems in which the number of photons involved is not well-defined. In optical-coherence theory the number of photons emitted by a typical radiation source is both very large and not well-defined, so that the coherent states $|\alpha\rangle$ with $|\alpha|^2 \gg 1$ are the states which have the appropriate classical limit in the sense that they minimize the uncertainty product $\Delta n \Delta \varphi$ and hence correspond to states of the field with a well-defined phase.[16] However, the coherent states are also useful in situations in which the number of photons need not be large. For an application of these states to the infrared-divergence problem in quantum electrodynamics, see Ref. 17.

The fundamental importance of the characteristic states of the annihilation operator in optical-coherence theory arises from the fact that a measurement of the moments or correlations of the field by photoelectric detectors is accomplished by the annihilation or absorption of the field's photons by the atomic detectors or counters. In this sense, the operator a is sometimes referred to as an "observable," and the theory is formulated using the coherent characteristic states of this "observable" as a basis. It is well known that non-Hermitian operators cannot be associated with any physical observables of a quantum system since their characteristic values are complex and the measurement of their real and imaginary parts are, in general, incompatible. The radiation-field quantities actually measured in the laboratory are the photon-counting rates which are proportional to transition probabilities such as $\Sigma_f \langle\psi_i|E^{(-)}(\mathbf{r},t)|\psi_f\rangle\langle\psi_f|E^{(+)}(\mathbf{r},t)|\psi_i\rangle = \langle\psi_i|E^{(-)}(\mathbf{r},t)E^{(+)} \times(\mathbf{r},t)|\psi_i\rangle$, where $|\psi_i\rangle$, $|\psi_f\rangle$ are the initial and final states of the interacting field-detector system. The true observable is then the Hermitian positive-definite product E^-E^+ and $\langle\psi_i|E^-E^+|\psi_i\rangle$ defines its expected value when the system is in state ψ_i. With the state of the system defined by the density matrix $\rho=\Sigma_i|\psi_i\rangle P_i\langle\psi_i|$, where P_i is the probability that the system is in the state $|\psi_i\rangle$, the counting rate, or the expected value of the observable E^-E^+, is proportional to the trace $\mathrm{Tr}(\rho E^-E^+)$.[4] $E^+(\mathbf{r},t)$ the positive-frequency part of the electric field at the space-time point $\{\mathbf{r},t\}$ is directly proportional to the set of annihilation operators $\{\mathbf{a}_{\mathbf{k},\lambda}\}$ which characterize the field in the plane-wave expansion

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\frac{i}{L^{3/2}}\sum_{\{\mathbf{k},\lambda\}}(\tfrac{1}{2}h\omega_k)^{1/2}a_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}; \quad (4)$$

$a_{\mathbf{k},\lambda}$ is the photon annihilation operator associated with the wave vector $\mathbf{k}$ and polarization index $\lambda=1, 2$, $\mathbf{e}_{\mathbf{k},\lambda}$ is the complex unit polarization vector satisfying the usual transverse field relations

$$\mathbf{e}_{\mathbf{k},\lambda}{}^{*}\cdot\mathbf{e}_{\mathbf{k},\lambda'}=\delta_{\lambda,\lambda'}; \quad \mathbf{k}\cdot\mathbf{e}_{\mathbf{k},\lambda}=0, \quad (5)$$

and $\{\mathbf{k},\lambda\}$ is the set of all the modes contained in the normalization volume L^3. For a multimode radiation field, Eq. (1) becomes

$$a_{\mathbf{k},\lambda}|\alpha_{\mathbf{k},\lambda}\rangle=\alpha_{\mathbf{k},\lambda}|\alpha_{\mathbf{k},\lambda}\rangle, \quad (6)$$

so that the multimode coherent state defined by $|\{\alpha_{\mathbf{k},\lambda}\}\rangle = \prod_{\{\mathbf{k},\lambda\}}|\alpha_{\mathbf{k},\lambda}\rangle$ is a characteristic state of $\mathbf{E}^{(+)}(\mathbf{r},t)$; that is,

$$\mathbf{E}^{+}(\mathbf{r},t)|\{\alpha_{\mathbf{k},\lambda}\}\rangle=\boldsymbol{\mathcal{E}}(\mathbf{r},t)|\{\alpha_{\mathbf{k},\lambda}\}\rangle, \quad (7)$$

and

$$\langle\{\alpha_{\mathbf{k},\lambda}\}|\mathbf{E}^{(-)}(\mathbf{r},t)=\langle\{\alpha_{\mathbf{k},\lambda}\}|\boldsymbol{\mathcal{E}}^{*}(\mathbf{r},t), \quad (8)$$

where

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t)=\frac{i}{L^{3/2}}\sum_{\{\mathbf{k},\lambda\}}(\tfrac{1}{2}h\omega_k)^{1/2}\alpha_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}. \quad (9)$$

Although the usual quantum-mechanical optical detector operates by stimulated absorption of photons, it is also possible to make field measurements via the process of stimulated emission, that is, by using an amplifier as a detector. It is therefore of interest to inquire into the possible existence and properties of the characteristic states of the creation operator $a^\dagger$. This problem is considered in Sec. II, where it is shown that the characteristic states of $a^\dagger$ are outside the Hilbert space of the radiation field state vectors and hence are not physically admissible states of the radiation field. In Sec. III, the possible existence of other bases for the field besides the $|\alpha\rangle$ and $|n\rangle$ and their unitary transformations is considered. A variation on the creation and annihilation operators $a^\dagger$, a is treated in Sec. IV.

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II. THE CHARACTERISTIC STATES OF THE CREATION OPERATOR

The complete sets of normalizable characteristic vectors $|n\rangle$ and $|\alpha\rangle$ of the number operator N and photon annihilation operator a are equally acceptable coordinate systems for the Hilbert space of the state vectors of the electromagnetic radiation field. The characteristic states of the creation operator $a^\dagger$ are not physically admissible states of the radiation field, since the characteristic vectors $|\beta\rangle\rangle$ [18] of $a^\dagger$,

$$a^\dagger|\beta\rangle\rangle=\beta|\beta\rangle\rangle \tag{10}$$

are not normalizable, and lie therefore outside the Hilbert space spanned by the basis $|n\rangle$ or $|\alpha\rangle$. This statement is a result of the following considerations: Since Eq. (10) with $\beta=0$ is not compatible with the defining equation for the photon creation operator

$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle, \tag{11}$$

when $n=0$, it is obvious that the state $|0\rangle\rangle$ does not coincide with the vacuum state defined by $a|0\rangle=0$.

A more serious difficulty is made evident by evaluating the expectation value of the operator $aa^\dagger$ in the state $|0\rangle\rangle$;

$$\begin{aligned}\langle\langle 0|aa^\dagger|0\rangle\rangle=0&=\langle\langle 0|\{1+a^\dagger a\}|0\rangle\rangle\\&=\langle\langle 0|0\rangle\rangle+\|a|0\rangle\|^2.\end{aligned} \tag{12}$$

In the positive-definite metric space spanned by the $|n\rangle$ states, both terms on the right-hand side of the last equation are non-negative. The only possible solution then of Eq. (12) is

$$|0\rangle\rangle=0. \tag{13}$$

Moreover, since

$$a^\dagger e^{-\beta a}|0\rangle\rangle=e^{-\beta a}a^\dagger|0\rangle\rangle+\beta e^{-\beta a}|0\rangle\rangle=\beta e^{-\beta a}|0\rangle\rangle, \tag{14}$$

we see by comparison with Eq. (10) that the state $|\beta\rangle\rangle$ with β arbitrary is related to the state $|0\rangle\rangle$ by the relation

$$|\beta\rangle\rangle=e^{-\beta a}|0\rangle\rangle, \tag{15}$$

indicating thereby that Eq. (10) does not have any non-null solutions.[19]

The nonexistence of the $|\beta\rangle\rangle$ states may also be verified directly by assuming the expansion

$$|\beta\rangle\rangle=\sum_n|n\rangle\langle n|\beta\rangle\rangle \tag{16}$$

to be valid. We calculate now the expansion coefficient $\langle n|\beta\rangle$ by taking the scalar product of both sides of Eq. (10) with the bra vector $\langle n|$. Using the Hermitian-adjoint of the defining equation of the annihilation operator

$$a|n\rangle=n^{1/2}|n-1\rangle, \tag{17}$$

we find the recursion relation

$$\beta\langle n|\beta\rangle\rangle=n^{1/2}\langle n-1|\beta\rangle\rangle. \tag{18}$$

Applying Eq. (18) n times we have,

$$\langle n|\beta\rangle\rangle=(n!)^{1/2}/\beta^n\langle 0|\beta\rangle\rangle, \tag{19}$$

and

$$|\beta\rangle\rangle=\langle 0|\beta\rangle\rangle\sum_n\frac{(n!)^{1/2}}{\beta^n}|n\rangle. \tag{20}$$

The norm

$$\langle\langle\beta|\beta\rangle\rangle=|\langle 0|\beta\rangle\rangle|^2\sum_n\frac{n!}{|\beta|^{2n}} \tag{21}$$

diverges for all finite values of β and the $|\beta\rangle\rangle$ states cannot be viewed as physically admissible states of the radiation field.

It might also be of interest to inquire into the coordinate and momentum spaces representatives of the $|\beta\rangle$ vectors. Expressing the operators a, $a^\dagger$ in terms of the one-dimensional harmonic-oscillator observables x and p, we have

$$a=[i/(2m\hbar\omega)^{1/2}](p-im\omega x), \tag{22a}$$

$$a^\dagger=[-i/(2m\hbar\omega)^{1/2}](p+im\omega x). \tag{22b}$$

The wave functions $\psi_\alpha(x)$ and $\psi_\beta(x)$ are solutions of the linear first-order differential equations of the operators a and $a^\dagger$,

$$a\psi_\alpha=\alpha\psi_\alpha;\quad a^\dagger\psi_\beta=\beta\psi_\beta,$$

$$\psi_\alpha(x)=\langle x|\alpha\rangle=\frac{e^{-\alpha_i^2}}{(2\pi x_0^2)^{1/4}}\exp\left[-\left(\frac{x}{2x_0}-\alpha\right)^2\right];$$

$$\alpha_i=\mathrm{Im}\alpha, \tag{23a}$$

$$\psi_\beta(x)=\langle x|\beta\rangle\rangle=\frac{e^{-\beta_i^2}}{(2\pi x_0^2)^{1/4}}\exp\left[+\left(\frac{x}{2x_0}-\beta\right)^2\right];$$

$$\beta_i=\mathrm{Im}\beta, \tag{23b}$$

where $x_0=(\hbar/2m\omega)^{1/2}$.[20] While the normalized wave functions $\psi_\alpha(x)$ are the well-known minimum uncertainty wave packets of a $2x_0\alpha$ displaced harmonic oscillator ground state and thus have a Gaussian probability density dependence, the non-normalizable $|\beta\rangle\rangle$ states in the position (and momentum) representation display a divergent behavior of the supposedly oscillating mass of the abstracted linear harmonic oscillator.

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[18] The kets $|\beta\rangle\rangle$ differ from the kets $|\alpha\rangle$, even when $\alpha=\beta$. The notation $|\rangle\rangle$ is to point out this distinction. It should be noted also that the kets $|n\rangle$ and $|\alpha\rangle$, when $\alpha=n+j0$ is a real number, are not the same. While the former are orthogonal to each other, the latter are not.

[19] If one considers an indefinite metric space, however, both eigenvalue equations, (1) and (10) may be satisfied simultaneously; see M. G. Gundzik, J. Math. Phys. **7**, 641 (1966).

[20] Clearly, the $\psi_\beta(x)$ function is not a square integrable one, and the coefficient $(2\pi x_0^2)^{-1/4}e^{-\beta_i^2}$ is introduced here for reasons of symmetric appearance only and is in agreement with Eq. (32b) below for $\lambda=1$.

III. POSSIBLE BASIS STATES FOR THE RADIATION FIELD

We have noted in Sec. I that an arbitrary state of the radiation field may be referred to the $|n\rangle$ or $|\alpha\rangle$ coordinate system. Other bases may of course be formed by means of unitary transformations; for example, the state $D(\alpha)|n\rangle$, where $D(\alpha)=e^{\alpha a^\dagger-\alpha^* a}$ is a unitary displacement operator in the complex α plane, is a characteristic state of the Hermitian operator $(a^\dagger-\alpha^*)(a-\alpha)$ with characteristic value n, that is,

$$(a^\dagger-\alpha^*)(a-\alpha)D(\alpha)|n\rangle=nD(\alpha)|n\rangle. \tag{24}$$

Equation (24) can be easily verified as follows. The characteristic value equation of the number operator $N=a^\dagger a$ is

$$a^\dagger a|n\rangle=n|n\rangle. \tag{25}$$

Multiplying Eq. (25), from the left, by $D(\alpha)$ and by virtue of the unitarity of the $D(\alpha)$ operator, $D(\alpha)D^\dagger(\alpha)=1$,

$$D(\alpha)a^\dagger aD^\dagger(\alpha)D(\alpha)|n\rangle=nD(\alpha)|n\rangle. \tag{26}$$

Using the Baker-Hausdorff identity,

$$\begin{aligned}D(\alpha)&=\exp(\alpha a^\dagger-\alpha^* a)\\&=\exp(\alpha a^\dagger)\exp(-\alpha^* a)\exp(-\tfrac{1}{2}|\alpha|^2),\end{aligned} \tag{27}$$

we have

$$\begin{aligned}D(\alpha)a^\dagger aD^\dagger(\alpha)=&\exp(\alpha a^\dagger)\exp(-\alpha^* a)a^\dagger a\\&\times\exp(\alpha^* a)\exp(-\alpha a^\dagger).\end{aligned} \tag{28}$$

From the commutation relations of a and $a^\dagger$ it can be shown that

$$\exp(-\alpha^* a)a^\dagger a\exp(\alpha^* a)=(a^\dagger-\alpha^*)a, \tag{29}$$

$$\exp(\alpha a^\dagger)(a^\dagger-\alpha^*)a\exp(-\alpha a^\dagger)=(a^\dagger-\alpha^*)(a-\alpha),$$

and Eq. (24) follows.

The question now arises as to the possible existence of other useful bases for the radiation field besides $|n\rangle$, $|\alpha\rangle$ and states generated from these by unitary transformations.

First, let us consider possible bases formed by the characteristic states of Hermitian operators which characterize the radiation field. Classically, the basic observables are the electric and magnetic fields at each space-time point $\{\mathbf{r},t\}$, $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and the field energy $H=\frac{1}{2}\int(\mathbf{E}^2+\mathbf{B}^2)d\mathbf{r}$. Quantum-mechanically, only values of $\mathbf{E}$ and $\mathbf{B}$ averaged over finite regions of space-time have a physical meaning in the sense of being measurable quantities.[21,22] However, although this difficulty can be circumvented by formulating the theory in terms of the "smeared" fields which are well-defined Hermitian operators, the real spectra of these operators are continuous and their characteristic states lie outside the Hilbert space spanned by the $|n\rangle$ states. This is strictly analogous to the difficulty with the characteristic states of the position operator in ordinary quantum mechanics. Hence, if we restrict our attention to the basic field observables of the free radiation field and functions of these observables, then the only complete set of characteristic states are those belonging to the number operator N. Next, let us suppose that there exists a non-Hermitian field operator b whose characteristic states $|\gamma\rangle$ form a basis for the field. The simplest forms of Hermitian operators which can be generated from b and its adjoint $b^\dagger$ are $(b+b^\dagger)$, $i(b-b^\dagger)$, and $b^\dagger b$. If we require that these fundamental operators be associated with the linear harmonic oscillator observables, x, p, and H, or equivalently, with the field observables $\mathbf{E}$, $\mathbf{B}$, and H, since each normal mode of the radiation field has the same quantum-mechanical structure as the one-dimensional harmonic oscillator,[23] then it follows that b must be equivalent to the annihilation operator a, so that $|\gamma\rangle\equiv|\alpha\rangle$. For, if $(b+b^\dagger)$ is equivalent to x within a constant, then $b\equiv a$, and $i(b-b^\dagger)\equiv p$; $b^\dagger b\equiv H$ and $|\gamma\rangle\equiv|\alpha\rangle$. The possibility that $b=a^\dagger$ must be ruled out since we have seen that the $|\beta\rangle\rangle$ states are not physically admissible. The correspondence $(b+b^\dagger)\equiv p$, $i(b-b^\dagger)\equiv x$, amounts to a relabeling of the x and p variables and leads to no new result. The only possible correspondence $b^\dagger b\equiv x$ implies a Hamiltonian which leads to incorrect equations of motion for x and p. Therefore, under the assumption that the basic operator triplet $(b+b^\dagger)$, $i(b-b^\dagger)$, and $b^\dagger b$ corresponds to x, p, and H, the only possible useful basis beside the characteristic number states $|n\rangle$ are the coherent states $|\alpha\rangle$. If we place less restrictive conditions on b and consider in particular non-normal operators, e.g., $b=(x+ip^3)$, then the situation becomes much more difficult since no spectral theorem exists for this class of operators. Indeed, there was no other *a priori* reason to expect the set of characteristic states of the annihilation operator to be complete, apart from the fact that the positive-frequency part of the electric field $\mathbf{E}^{(+)}(\mathbf{r},t)$ appears to be the "observable" one would naturally associate with field measurements using photodetectors at optical frequencies (see Sec. I). Despite the fact that the possible existence of other overcomplete[24] families of states cannot be ruled out in principle, the $|n\rangle$ and $|\alpha\rangle$ bases seem to be the only one useful in all theoretical and applied considerations of the radiation field.

IV. A VARIATION ON THE PHOTON CREATION AND ANNIHILATION OPERATOR a, $a^\dagger$

In the previous section it was shown that with the restriction placed on the correspondence between simple Hermitian combinations of a non-normal operator b, whose characteristic kets form a basis for the radiation field, and the harmonic oscillator variables x, p, and H, there are no other useful basis states besides $|n\rangle$ and $|\alpha\rangle$. This restriction is a severe one, however, and we

[21] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Matt. Fys. Medd **12**, 8 (1933), Sec. II.

[22] J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965), p. 36.

[23] G. Barton, *Introduction to Advanced Field Theory* (Interscience Publishers, Inc., New York, 1963), p. 118.

[24] J. R. Klauder, J. Math. Phys. **4**, 1055 (1963).

might note that it is possible to generalize the definition of the annihilation operator Eq. (22a) in a relatively simple manner and thereby define a new non-normal operator A whose characteristic functions computed below form a basis for the field for all values of a dimensionless parameter λ, in the range $0<\lambda\leq 1$.[25] We define the new operators $A, A^\dagger$ as follows:

$$A=[i/(2m\hbar\omega)^{1/2}](p-i\lambda m\omega x), \tag{30a}$$

$$A^\dagger=[-i/(2m\hbar\omega)^{1/2}](p+i\lambda m\omega x) \tag{30b}$$

where λ may vary continuously from -1 to $+1$. When $\lambda=1$, $A, A^\dagger$ reduce to the annihilation and creation operators $a, a^\dagger$. Considering Eqs. (22a) and (22b) the operators A and $A^\dagger$ can be expressed in terms of the usual photon annihilation and creation operators a and $a^\dagger$,

$$A=\tfrac{1}{2}[(1+\lambda)a+(-1+\lambda)a^\dagger], \tag{31a}$$

$$A^\dagger=\tfrac{1}{2}[(-1+\lambda)a+(1+\lambda)a^\dagger]. \tag{31b}$$

The commutator $[A,A^\dagger]=\lambda$ follows directly from the last equations and $[a,a^\dagger]=1$. For $\lambda=1$, A coincides with a and only the characteristic kets of A exist (see Sec. II). For $\lambda=-1$, $A\rightarrow -a^\dagger$ and only the characteristic bras exist. For $\lambda=0$, A is the anti-Hermitian operator $\frac{1}{2}(a-a^\dagger)$, and therefore has both bras and kets. These results have been noted previously. The characteristic kets of A also exist for $0<\lambda<1$. The normalized wave function $\psi_\gamma(x)$ is a solution of first-order characteristic-value equation of the operator A,

$$A\psi_\gamma=\gamma\psi_\gamma, \tag{32a}$$

and

$$\psi_\gamma(x)=\langle x|\gamma\rangle=\left(\frac{\lambda}{2\pi x_0^2}\right)^{1/4} e^{-\gamma_i^2/\lambda}e^{-\lambda(x/2x_0-\gamma/\lambda)^2}; \quad \gamma_i=\mathrm{Im}\gamma. \tag{32b}$$

The last expression of $\psi_\gamma(x)$ coincides with formulas $\psi_\alpha(x), \psi_\beta(x)$ for $\lambda=\pm 1$.

[25] This form of the operator A was suggested to one of the authors (E. A. M.) by Prof. R. J. Glauber.

It is of interest to note that the characteristic states of the operator A form a complete set, in the sense that the unit operator can be represented as an integral over all the projection operators of the γ states, for all values of λ, including those negative values for which $\psi_\gamma(x)$ is not a square-integrable function. To prove this point we may write the analogous equation to Eq. (3),

$$\frac{1}{\pi\lambda}\int|\gamma\rangle\langle\gamma|d^2\gamma=1, \tag{33}$$

with the integral extending over the whole complex plane γ. Multiplying the last equation, from the left by $\langle x|$ and the right by $|x'\rangle$, we obtain,

$$\langle x|x'\rangle=\delta(x-x')=\frac{1}{\lambda\pi}\int\langle x|\gamma\rangle\langle\gamma|x'\rangle d^2\gamma, \tag{34a}$$

or

$$\frac{1}{\pi\lambda}\int\psi_\gamma^*(x')\psi_\gamma(x)d^2\gamma=\delta(x-x'). \tag{34b}$$

Substitution of expression (32b) for $\psi_\gamma(x)$ into Eq. (34b) factorizes the integral on the left-hand side of the equation into integrals over all real and imaginary values of γ,

$$\frac{1}{\pi\lambda}\int\psi_\gamma^*(x')\psi_\gamma(x)d^2\gamma=\frac{1}{(2\pi\lambda x_0^2)^{1/2}} \times\exp\left[-\frac{\lambda}{4x_0^2}(x^2+x'^2)\right]\int\exp\left[\frac{x-x'}{x_0}\gamma_r-\frac{2}{\lambda}\gamma_r^2\right]d\gamma_r \times\int\exp\left(i\frac{x-x'}{x_0}\gamma_i\right)d\gamma_i, \tag{35}$$

$$\gamma_r=\mathrm{Re}\gamma;\quad \gamma_i=\mathrm{Im}\gamma.$$

It is easily seen that the last integrals lead to the delta function $\delta(x-x')$, proving thereby the completeness relations (34b) or (33).