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Simultaneous Measurement of Noncommuting Observables

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The state of a quantum-mechanical system after a simultaneous measurement of conjugate variables such as coordinate and momentum is derived. In the case of the minimum-uncertainty, or ideal, simultaneous measurement, the state of the system after the measurement is a coherent state corresponding to that representing a minimum-uncertainty wave packet. An operator formalism to describe simultaneous measurements of noncommuting observables is introduced which parallels that familiar for a single measurement, and an expression for the joint probability distribution predicting the results of an ideal simultaneous measurement is derived and applied to the example of a harmonic oscillator. It is found that the minimum uncertainty in a simultaneous measurement of noncommuting variables has two causes: (1) the unavoidable perturbations introduced by the measuring process, and (2) the unavoidable lack of precision in the state of the system itself. For position and momentum measurements, these two independent uncertainties contribute to give a net minimum uncertainty of $\Delta q'\Delta p'=\hbar$. It is also shown that the formalism of simultaneous measurement leads to a well-defined quantum-mechanical phase space.

INTRODUCTION

THE formalism of the quantum theory has long made use of the concept of the ideal accurate measurement of a single physical variable. That this concept is a mathematical idealization which cannot be realized in practice has long been recognized.[1] Although the subject of simulatneous measurement is of some interest since the introduction of quantum theory, only recently, with the advent of the laser and the possibility of optical communication systems where the quantum effects on the accuracy of measurement play a role, has there been any real need to investigate the practical consequences of this mathematical idealization. In particular, since in communication systems one usually makes a simultaneous measurement of both amplitude and phase or other quantities from which these two variables can be inferred, one would like fully to understand the quantum-mechanical consequences of a simultaneous measurement of noncommuting observables. The uncertainty principle which immediately comes to mind when simultaneous measurement arises, actually sheds very little light on the subject, for it is concerned with two separate ideal single measurements applied to separate members of an ensemble of identical systems. In order to treat the problem of simultaneous measurement of noncommuting observables, one needs to develop a formalism and an interpretation as has been done in the case of single ideal measurements. In particular, one needs to know the state of the system after a simultaneous measurement has been carried out. Recently, Arthurs and Kelly[2] have considered this problem using the approach of von Neumann. They allow two meter systems to interact with the system under observation and then apply an ideal single-variable measurement to each of the meters. With certain assumptions as to the initial conditions of the meter systems, their result is equivalent to the special case of the "ideal" simultaneous measurement developed here. Once having obtained the state of the system corresponding to an ideal simultaneous measurement, we can in the usual manner define the probability of a simultaneous-measurement result in terms of the overlap of this state with the actual state of the system.

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1958), pp. 48–99.

[2] E. Arthurs and J. L. Kelly, Jr., Bell System Tech. J. 44, 725 (1965).

IDEAL SINGLE MEASUREMENTS

The practical unrealizability of the precise single measurement comes about in several ways. First, such a measurement would require an infinite amount of precision and from an information-theory standpoint would require an infinite amount of information. Moreover, a precise measurement of one variable leaves the value of its conjugate variable completely undetermined. Thus a precise measurement of position leaves all momentum values equally probable or, conversely, a precise measurement of momentum leaves all locations of the system equally probable. Neither of these conditions is physically realistic, the first because of limitations of energy; and the second, of size. In practice, when one makes a measurement of particle momentum, for example, he knows the particle is at least located in the laboratory and indeed, in the vicinity of the measurement apparatus. Or, if a measurement of particle position is made, one knows that the particle cannot have infinite energy, and in fact, the possible energy range of the particle is invariably delimited by the response characteristics of the detector. These considerations suggest that in actuality, all measurements are simultaneous measurements of noncommuting observables. The nearest approach to a single measurement would then be a simultaneous measurement in which one variable is rather precisely measured while the other variable is determined only inaccurately.

In order to point up the parallel formalism of a simultaneous measurement of noncommuting observables developed here, we shall reproduce some of the postulates of quantum mechanics as they refer to measurement of single observables. First, the act of measurement of a real dynamical variable ξ which results in a measured value ξ' forces the system into the state $|\xi'\rangle$, an eigenstate of the variable ξ having an eigenvalue equal to the measured result. Second, the probability interpretation of the quantum theory postulates that the probability of a measurement of the variable ξ yielding a result ξ' given the fact that the system is in a state $|\psi\rangle$ is proportional to $|\langle\xi'|\psi\rangle|^2$. In terms of the description of the system by a density operator ρ, this probability is proportional to $\langle\xi'|\rho|\xi'\rangle$. For simplicity, we shall limit our discussions to a system characterized fully by its position variable q and conjugate momentum p, and develop the formalism of the simultaneous measurement in terms of these variables.

THE STATE OF THE SYSTEM AFTER A SIMULTANEOUS MEASUREMENT

In order to develop a formalism for the simultaneous measurement of noncommuting observables, let us first investigate the state of the system after a simultaneous measurement of position and momentum has been made. We shall adopt the technique of Jaynes,[3] whereby we develop the "unbiased" description of the system obtained by maximizing the system entropy (minimizing the information content) subject to our knowledge of the system.

What do we know about the system after the simultaneous measurement of p and q? First, we know the measured values p_0 and q_0. Moreover, since the simultaneous measurements cannot yield precise knowledge of both p and q, we know there must be some uncertainty Δp and Δq. Presumably, the particular form of the measuring apparatus sets Δp and Δq. We shall see later that for the ideal simultaneous measurement, what is important is really the relative accuracies of the two measurements, that is, the ratio of Δp and Δq.

Our problem then is to maximize

$$S=-k\,\mathrm{Tr}\rho\ln\rho\,,$$

subject to the constraints

$$\begin{aligned}\langle p\rangle&=\mathrm{Tr}p\rho=p_0\,,\\ \langle q\rangle&=\mathrm{Tr}q\rho=q_0\,,\\ \langle(p-p_0)\rangle^2&=\mathrm{Tr}(p-p_0)^2\rho=(\Delta p)^2\,,\\ \langle(q-q_0)^2\rangle&=\mathrm{Tr}(q-q_0)^2\rho=(\Delta q)^2\,.\end{aligned}$$

The last two equations define what we mean by Δq and Δp. By the method of Lagrange mutlipliers, one can immediately determine that the density matrix ρ which describes the state of the system after the simultaneous measurement must be of the form

$$\rho=\exp(\lambda_0+\lambda_1q^2+\lambda_2q+\lambda_3p^2+\lambda_4p)\,.$$

The coefficients are evaluated in Appendix A by making use of two new operators

$$\begin{aligned}a&=(2\hbar\beta)^{-1/2}(q+i\beta p)\,,\\ a^\dagger&=(2\hbar\beta)^{-1/2}(q-i\beta p)\,,\end{aligned}$$

where

$$\beta=\Delta q/\Delta p\,.$$

These operators are reminiscent of the familiar annihilation and creation operators, since $[a,a^\dagger]=1$. The resulting density matrix may be written in one of several forms, such as

$$\rho=\hbar[(\Delta p)^2(\Delta q)^2-\tfrac{1}{4}\hbar^2]^{-1/2}\times\exp\left\{-\frac{\Delta p\Delta q}{2\hbar}\left[\ln\left(\frac{\Delta p\Delta q+\frac{1}{2}\hbar}{\Delta p\Delta q-\frac{1}{2}\hbar}\right)\right]\times\left[\left(\frac{q-q_0}{\Delta q}\right)^2+\left(\frac{p-p_0}{\Delta p}\right)^2\right]\right\};$$

$$\rho=\frac{\hbar}{\Delta p\Delta q+\frac{1}{2}\hbar}\times\exp\left\{-\left[\left(\ln\frac{\Delta p\Delta q+\frac{1}{2}\hbar}{\Delta p\Delta q-\frac{1}{2}\hbar}\right)(a^\dagger-\alpha^*)(a-\alpha)\right]\right\}.$$

$$\rho=\frac{\hbar}{\Delta p\Delta q+\frac{1}{2}\hbar}\mathfrak{N}\left\{\exp\left[-\frac{\hbar}{\Delta p\Delta q+\frac{1}{2}\hbar}(a^\dagger-\alpha^*)(a-\alpha)\right]\right\}.$$

[3] E. T. Jaynes, Phys. Rev. **106**, 620 (1957).

Here $\alpha=(2\hbar\beta)^{-1/2}(q_0+i\beta p_0)$, and in the last equation for ρ the symbol $\mathfrak{N}$ means[4] that, in the expression which follows, the a or "annihilation" operators all operate to the right of all $a^\dagger$ or "creation" operators.

Several important conclusions may be drawn from these results.

1. From the derivation of ρ in Appendix A, we see that the requirement that the density matrix be positive-definite demands

$$\Delta p\Delta q\geq\tfrac{1}{2}\hbar.$$

This is the uncertainty principle, but with a subtle difference. The familiar derivation of the uncertainty principle determines the minimum product in the uncertainties of two separate single measurements, one of momentum and the other of position, made on separate elements of an ensemble composed of identical systems each in the same state. The present statement should be interpreted differently. It asserts that the most accurate knowledge of the state of the system after a simultaneous measurement is that for which $\Delta p\Delta q=\frac{1}{2}\hbar$. It does not refer to the accuracy of the measurement itself, for that depends upon the initial state of the system, but rather to the accuracy of our knowledge of the state of the system subsequent to the simultaneous measurement. The details of the measurement apparatus set the ratio $\Delta q/\Delta p$, that is, how accurately the state is known in the position variable relative to the momentum variable, and they also set the value of the uncertainty product $\Delta p\Delta q$ consistent with $\Delta p\Delta q\geq\frac{1}{2}\hbar$. Hereafter, we shall refer to the measurement which yields the minimum uncertainty in knowledge of the subsequent state, i.e., $\Delta p\Delta q=\frac{1}{2}\hbar$, as an *ideal simultaneous measurement.*

2. Immediately after the ideal simultaneous measurement, the system is in a pure coherent state.[5] As shown in Appendix B, when $\Delta p\Delta q=\frac{1}{2}\hbar$

$$\rho=|\alpha\rangle\langle\alpha|\,,$$

or

$$|\psi_\alpha\rangle=|\alpha\rangle,$$

where

$$a|\alpha\rangle=\alpha|\alpha\rangle.$$

The coherent state $|\alpha\rangle$ is the familiar minimum-uncertainty wave packet,

$$\psi_\alpha(q')=[2\pi(\Delta q)^2]^{-1/4}\times\exp[-(q'-q_0)^2/4(\Delta q)^2+iq'p_0/\hbar].$$

This result is that arrived at by Arthurs and Kelly[2] by quite different means.

3. If the simultaneous measurement is carried out with less than ideal accuracy, that is, if $\Delta p\Delta q>\frac{1}{2}\hbar$, then the state of the system after the measurement is a mixed state and must be described in a statistical manner, as by the density operators given above. The entropy S of the system right after a simultaneous measurement is then

$$S=k\left\{\frac{\lambda}{e^\lambda-1}-\ln(1-e^{-\lambda})\right\};\quad \lambda=\ln\frac{\Delta p\Delta q+\frac{1}{2}\hbar}{\Delta p\Delta q-\frac{1}{2}\hbar}.$$

In the case of an ideal measurement, i.e., $\Delta p\Delta q=\frac{1}{2}\hbar$, the state after the measurement is a pure coherent state, and the entropy after the measurement is zero, which is the smallest value an entropy can have. The application of the variational procedure for an ideal simultaneous measurement is clearly a singular one. The pure coherent state obtained from this procedure for the ideal case, i.e., $\Delta p\Delta q=\frac{1}{2}\hbar$, can, however, be regarded as a limit approached from the results when $\Delta p\Delta q$ exceeds $\frac{1}{2}\hbar$. The real reason for this singularity comes from the fact that the entropy for any pure state is zero; thus the maximization is unnecessary and the constraints alone determine the state after the measurement. In fact, if one limits himself to the systems describable by pure states, then the state that the system is left in after an ideal simultaneous measurement—just as in the case of the precise measurement—can be obtained from the necessary and sufficient condition that it satisfy the constraints q_0, p_0, and $\Delta p\Delta q=\frac{1}{2}\hbar$. It can be shown that this condition also leads to the same coherent state.

THE IDEAL SIMULTANEOUS MEASUREMENT

From the results above, we can develop a formalism to describe the simultaneous measurement which parallels in most respects that of the single measurement. Since the state of the system after a simultaneous measurement of p and q is the eigenstate $|\alpha\rangle$ of the operator $a=(2\hbar\beta)^{-1/2}(q+i\beta p)$ having an eigenvalue $\alpha=(2\hbar\beta)^{-1/2}\times(q_0+i\beta p_0)$, we can look upon the operator a as describing the simultaneous ideal measurement of q and p by an experimental apparatus which distributes the relative accuracies of the two variables given by $\beta=\Delta q/\Delta p$. Then, after a simultaneous measurement of q and p resulting in values q_0 and p_0, the system jumps to an eigenstate of the variable a belonging to an eigenvalue α equal to the measured values. This result parallels the formalism of the single measurement except that "observables" for simultaneous measurement are no longer represented by Hermitian operators.

In accordance with the fundamental probability interpretation of quantum mechanics, we postulate that the probability $P(q',p')$ of an ideal simultaneous measurement which yields the values q' and p' is proportional to $\langle q',p'|\rho|q',p'\rangle$,wehre we use the notation $|q',p'\rangle$ to represent the coherent minimum-uncertainty wave packet previously symbolized by $|\alpha\rangle$, for which

$$a|q',p'\rangle=(2\hbar\beta)^{-1/2}(q+i\beta p)|q',p'\rangle=(2\hbar\beta)^{-1/2}(q'+i\beta p')|q',p'\rangle.$$

By demanding

$$\iint P(q',p')dq'dp'=1,$$

[4] H. Heffner and W. H. Louisell, J. Math Phys. 6, 474 (1965).
[5] R. J. Glauber, Phys. Rev. 131, 2766 (1963).

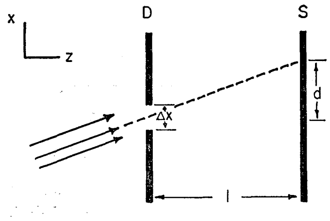

FIG. 1. A schematic for the simultaneous measurement of the x-component electron position and momentum.

we find

$$P(q',p')=h^{-1}\langle q',p'|\rho|q',p'\rangle,$$

or, for a system in a pure state $|\psi\rangle$,

$$P(q',p')=h^{-1}|\langle q',p'|\psi\rangle|^2.$$

We may now determine the probability that, having once made an ideal simultaneous measurement which yielded the result (q_0,p_0), an immediately repeated ideal measurement gives the simultaneously measured values (q',p');

$$P(q',p')=h^{-1}\langle q',p'|\mathfrak{N}\{\exp[-(a^\dagger-\alpha^*)(a-\alpha]\}|q'p'\rangle$$
$$=h^{-1}\exp\{-[(q'-q_0)^2/2\hbar\beta+(p'-p_0)^2/2(\hbar/\beta)]\}.$$

This joint probability distribution has certain familiar and certain unexpected features. First, as expected, it shows that the probability distribution is a joint Guassian distribution, with the measured results being most probably q_0 and p_0, as just previously measured. The fact that the outcome of the second measurement is not precisely determined differs from the case of a single measurement, where physical continuity requires that immediately repeated measurements yield the same results. This difference is not surprising, since complete accuracy in determining two conjugate variables in a measurement is impossible.

The variances of this probability distribution for the results of an immediately repeated ideal simultaneous measurement are such that

$$(\Delta p')^2(\Delta q')^2=\int\int(p'-p_0)^2(q'-q_0)^2P(q',p')dp'dq'=\hbar^2.$$

This is, as expected (See Appendix C), the minimum possible value, since the state immediately before the repeated measurement is a minimum wave packet. What is perhaps unexpected is that the minimum of $\Delta p'\Delta q'=\hbar$ differs from, and in fact is exactly twice, the minimum of $\Delta p\Delta q=\frac{1}{2}\hbar$. To understand this difference, we emphasize that $\Delta p\Delta q$ and $\Delta p'\Delta q'$ have quite different meanings. The former refers to the accuracy to which a quantum state may be specified, and the latter is a statistical measure for the outcomes of the ideal simultaneous measurements on an ensemble of identical systems. $\Delta p'\Delta q'$ should take on larger values than $\Delta p\Delta q$, since there are two sources of uncertainty associated with the statistics of a measurement. The first of these is the uncertainty inherent in the specification of the state under investigation. The second is that the measurement process itself produces unavoidable inaccuracies, by forcing the system into an eigenstate of the operator under observation. In the case of the ideal simultaneous measurement of position and momentum, these two independent uncertainties contribute to give a net minimum of $\Delta p'\Delta q'=\hbar$.

A THOUGHT EXPERIMENT

Since any measurement (including a so-called "precise measurement") is in practice a simultaneous measurement, all the thought experiments described in elementary quantum-mechanics texts can be modified to measure both p and q simultaneously. Such modifications are of course very elementary and perhaps trivial, once pointed out. A brief example may nevertheless be justified here to help clarify some points in this paper. Suppose a beam of electrons is moving in the x-z plane with its z-component momentum p_z precisely given. The x-component electron position q_x and momentum p_x are to be measured by the apparatus shown in Fig. 1. The apparatus consists of a diaphragm D with a slit Δx at $z=0$ and an oscilloscope screen S at $z=l$. Both the diaphragm and the screen are perpendicular to the z axis. It is obvious that the x-component electron position q_x can be determined by the position of the slit with uncertainty $\Delta q_x\simeq\Delta x$. The x-component electron momentum will be determined by the location of the bright spot on the screen where the electron impinges, i.e.,

$$d=\frac{p_x}{m}\frac{l}{p_z/m}=\frac{p_x}{p_z}l,\quad\text{or}\quad p_x=\frac{p_z}{l}d.$$

The uncertainty of the x-component electron momentum Δp_x comes from two sources: (1) the wave nature of the electron, which gives approximately $\hbar/\Delta x$ due to the scattering from the slit, and (2) the additional uncertainty due to the response of the screen, Δs. Therefore, $\Delta q_x\simeq\Delta x$, $\Delta p_x\simeq\hbar/\Delta x+\Delta s$. The apparatus then measures both q_x and p_x simultaneously with uncertainties Δq_x and Δp_x specified. In principle, we can make $\Delta s=0$ and thus have an *ideal* simultaneous measurement with $\Delta p\Delta q\simeq\hbar$. The relative uncertainty β in this case can be controlled by the size of the slit, i.e.,

$$\beta=\Delta q_x/\Delta p_x\simeq\Delta x/\hbar/\Delta x=(\Delta x)^2/\hbar.$$

If Δx is made zero, β is zero, and we approach the precise position measurement. If $\hbar$ is also zero, the measurement is then classical.

MEASUREMENT PROBABILITY

Suppose a quantum-mechanical system is subjected to an ideal simultaneous measurement at an initial time $t=0$ with known results q_0 and p_0, and thereafter remains undisturbed until a time t when another ideal simultaneous measurement of p and q is made. What are the probable outcomes of such a measurement? We may answer that question using the results just developed.

Immediately after the initial measurement, the system is in the coherent state $|q_0,p_0\rangle$ and is described by a density operator $\rho(0)$ which may be written

$$\rho(0)=|q_0,p_0\rangle\langle q_0,p_0|=\mathfrak{N}\{\exp[-(a^\dagger-\alpha_0^*)(a-\alpha_0)]\}.$$

Subsequently, the system evolves according to

$$i\hbar(d\rho/dt)=[\mathcal{H},\rho],$$

and at time t is described by $\rho(t)$. The joint probability density describing the probability of a second measurement at time t resulting in values (q',p') is

$$P(q',p',t)=h^{-1}\langle q',p'|\rho(t)|q',p'\rangle.$$

This probability density parallels the familiar classical probability density. Thus, the concepts of phase space and indeed the other classical statistical-mechanical concepts may be employed. The single distinction is that the quantum system is unperturbed, i.e., unobserved, between measurements. The measurement at time t, the probable results of which are described by $P(q',p',t)$, when actually carried out will disturb the system so that a new probability density must be calculated to determine the probable results of yet another measurement. Thus, given the state of the system at an initial time, the results of a measurement at a subsequent time depend upon whether or not an intervening measurement has been made. The probability distribution represents a simple Markov process, however, depending only upon the results of the immediately preceding measurement.

An alternative description of the joint probability density of measured results is the characteristic function defined by

$$C(\xi,\eta,t)=\int P(q',p',t)\exp[i(\xi q'+\eta p')]dq'dp',$$

or alternatively,

$$P(q',p',t)=\frac{1}{(2\pi)^2}\int C(\xi,\eta,t)\exp[-i(\xi q'+\eta p')]d\xi d\eta.$$

An alternative expression may be developed[6] for $C(\xi,\eta,t)$ by recognizing that

$$(2\hbar\beta)^{-1/2}(q+i\beta p)|q',p'\rangle=|q',p'\rangle(2\hbar\beta)^{-1/2}(q'+i\beta p').$$

Thus

$$C(\xi,\eta,t)=\frac{1}{2\pi\hbar}\int\langle q',p'|e^{-\gamma^*a^\dagger}\rho(t)e^{\gamma a}|q',p'\rangle dq'dp'$$

$$=\mathrm{Tr}e^{\gamma a}e^{-\gamma^*a^\dagger}\rho(t),$$

where

$$\gamma=(\hbar/2\beta)^{1/2}(\eta+i\beta\xi),$$

and where a and $a^\dagger$ are defined as before. The calculation of the characteristic function is simplified if it is carried out in the Heisenberg picture, to give

$$C(\xi,\eta,t)=\mathrm{Tr}e^{\gamma a(t)}e^{-\gamma^*a^\dagger(t)}\rho(0),$$

which, when the initial state represents the result of an ideal simultaneous measurement, further simplifies to

$$\begin{aligned}C(\xi,\eta,t)&=\langle q_0,p_0|e^{\gamma a(t)}e^{-\gamma^*a^\dagger(t)}|q_0p_0\rangle\\&=\langle q_0,p_0|e^{i\xi q(t)+i\eta p(t)}|q_0,p_0\rangle\\&\qquad\times\exp[-\tfrac{1}{4}\hbar(\eta^2/\beta+\beta\xi^2)].\end{aligned}$$

AN EXAMPLE

It is instructive to consider a simple example of the foregoing results. Consider a simple harmonic oscillator of unit mass described by the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}(p^2+\omega^2q^2).$$

Suppose that at $t=0$ an initial ideal simultaneous measurement was performed yielding values q_0 and p_0. With no intervening measurement we wish to infer the probability distribution of the results of a subsequent ideal simultaneous measurement to be carried out at time t. The Heisenberg equations of motion yield the familiar relations

$$\begin{aligned}q(t)&=q\cos\omega t+(p/\omega)\sin\omega t,\\p(t)&=p\cos\omega t-\omega q\sin\omega t.\end{aligned}$$

Upon substituting these values in the previous expression for $C(\xi,\eta,t)$ and using the eigenvalue property of the coherent states $|q_0,p_0\rangle$, we find that the characteristic function is of the form

$$C(\xi,\eta,t)=\exp[i(m_q\xi+m_p\eta)-\tfrac{1}{2}(\sigma_q^2\xi^2+\lambda_{qp}\xi\eta+\sigma_p^2\eta^2)],$$

where

$$\begin{aligned}m_q&=q_0\cos\omega t+(p_0/\omega)\sin\omega t,\\m_p&=p_0\cos\omega t-\omega q_0\sin\omega t,\\\sigma_q^2&=\tfrac{1}{2}\hbar\beta[1+\cos^2\omega t+(1/\beta^2\omega^2)\sin^2\omega t],\\\lambda_{qp}&=\hbar[(1/\beta\omega)-\beta\omega]\cos\omega t\sin\omega t,\\\sigma_p^2&=(\hbar/2\beta)[1+\cos^2\omega t+\beta^2\omega^2\sin^2\omega t].\end{aligned}$$

This form of the characteristic function immediately shows that the probability of measuring q' and p' at time t is a joint Gaussian distribution with a mean value of q' equal to m_q and of p' equal to m_p, and with corresponding variances of σ_q^2 and σ_p^2 and covariance λ_{qp}.

Several points are worth noting. First, the mean values of the measurement are identical to the classical values. In addition, the probability of measurement reproduces itself each period. Thus when the second measurement occurs an integral number of periods after the initial measurement, the characteristic function reduces to

$$C(\xi,\eta,t)=\exp[i(\xi q_0+\eta p_0)-\xi^2(\hbar\beta)-\eta^2(\hbar/\beta)],$$

[6] C. Y. She, Stanford Electronics Laboratory Report No. 64-074, Stanford University, 1964 (unpublished).

appropriate to a probability density of

$$P(q',p',t)=h^{-1}\{-[(q'-q_0)^2/2\beta\hbar+(p'-p_0)^2/2(\hbar/\beta)]\}.$$

This is precisely the expression previously derived for the case of a second measurement immediately following the first.

One also notes that when the second measurement follows the first by a half-period (or by an odd-integral multiple of a half-period) the result is identical except that the mean values are reversed in sign. One further notes that whenever the second measurement follows the initial measurement by an integral number of quarter periods, the covariance becomes zero and the results of the second measurement are uncorrelated with one another.

Finally, it is interesting to observe the result of demanding that the ratio of the measurement uncertainties be such that $\beta=1/\omega$. Under this condition, the following apply: (a) The covariance is zero and the measurement results are uncorrelated; (b) The total uncertainties of both q and p are independent of time and become simply $\hbar/\omega$ and $\hbar\omega$, respectively; (c) The measurement operators a and $a^\dagger$ become identical to the usually defined annihilation and creation operators; and (d) The uncertainty in the oscillator energy splits equally between the variables q and p, and it takes its minimum value of one quantum, that is,

$$\Delta E=\tfrac{1}{2}[(\langle p^2\rangle-\langle p\rangle^2)+\omega^2(\langle q^2\rangle-\langle q\rangle^2)]=\hbar\omega.$$

PHASE SPACE

We have defined, for the case of two canonically conjugate variables q and p, a probability density $P(q',p',t)$ which determines the probability of a simultaneous measurement yielding values q' and p' within a range dq' and dp'. This function is completely analogous to the classical probability density and, in fact, reduces to exactly the classical form as Planck's constant is made to approach zero. Nonclassical conjugate variables such as spin may also be treated in this same fashion. That is, given two conjugate variables (ξ,η), the state of the system $|\xi',\eta'\rangle$ after a simultaneous measurement may be found by the maximization-of-entropy approach and the resulting probability density function is given by

$$P(\xi',\eta')=K\langle\xi',\eta'|\rho|\xi',\eta'\rangle,$$

where K is the constant of normalization. Gordon[7] has derived the simultaneous-measurement state for the case of spin variables by a different method and finds it similar to the minimum-uncertainty wave packet or coherent state obtained for position and momentum measurement. He also arrives at an expression for the probability density of simultaneous measurement equivalent to that presented here.

The fact that a joint probability density function representing the measurement of conjugate variables can be defined means that the concept of phase space can be carried over directly into quantum mechanics with the qualification that a point in phase space represents the result of a measurement, but does not represent the state of the system.

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APPENDIX A: THE STATE OF THE SYSTEM AFTER A SIMULTANEOUS MEASUREMENT

We have seen that maximizing the entropy subject to the constraints

$$\langle p\rangle=p_0,$$
$$\langle q\rangle=q_0,$$
$$\langle p^2\rangle=p_0^2+(\Delta p)^2,$$
$$\langle q^2\rangle=q_0^2+(\Delta q)^2$$

demands a density matrix of the form

$$\rho=\exp[\lambda_0+\lambda_1q^2+\lambda_2q+\lambda_3p^2+\lambda_4p].$$

By defining

$$a=(2\hbar\beta)^{-1/2}(q+i\beta p),$$
$$a^\dagger=(2\hbar\beta)^{-1/2}(q-i\beta p),$$

and recognizing

$$[a,a^\dagger]=1,$$

the density matrix may be placed in the form

$$\rho=\exp[\mu_0-\mu_1(a^\dagger-\alpha^*)(a-\alpha)],$$

where now the five constants μ_0, μ_1, β, Reα, and Imα must be determined from the constraints. This may be done most easily by placing ρ in normal form,[4] so that all a operators operate to the right and all $a^\dagger$ operators operate to the left. The trace may then easily be taken using a complete set of coherent states[5] $|\alpha'\rangle$ and the completeness relation

$$\frac{1}{\pi}\int\int|\alpha'\rangle\langle\alpha'|d^2\alpha'=1.$$

Thus in normal form, ρ becomes

$$\rho^{(n)}=\mathfrak{N}\{\exp[\mu_0+(e^{-\mu_1}-1)(a^\dagger-\alpha^*)(a-\alpha)]\},$$

and the trace of ρ which must be unity is

$$\mathrm{Tr}\rho=\frac{1}{\pi}\int\int\langle\alpha'|\rho^{(n)}|\alpha'\rangle d^2\alpha'$$
$$=\frac{1}{\pi}\int\int\exp[\mu_0+(e^{-\mu_1}-1)(\alpha'^*-\alpha^*)\times(\alpha'-\alpha)]d^2\alpha'=e^{\mu_0}/(1-e^{-\mu_1})=1.$$

[7] J. P. Gordon (private communication).

Thus,

$$e^{\mu_0}=(1-e^{-\mu_1}).$$

We next demand

$$(2\hbar\beta)^{-1/2}(\langle q\rangle+i\beta\langle p\rangle)=(2\hbar\beta)^{-1/2}(q_0+i\beta p_0),$$

or

$$\mathrm{Tr}\rho a=(2\hbar\beta)^{-1/2}(q_0+i\beta p_0).$$

Again the trace is easily determined, as before:

$$\mathrm{Tr}\rho a=\frac{1}{\pi}\int\int\langle\alpha'|\rho^{(n)}a|\alpha'\rangle d^2\alpha$$

$$=\frac{1}{\pi}(1-e^{-\mu_1})\int\int\alpha'\exp[(e^{-\mu_1}-1)(\alpha'^*-\alpha^*)\times(\alpha'-\alpha)]d^2\alpha'$$

$$=\alpha.$$

Thus,

$$\alpha=(2\hbar\beta)^{-1/2}(q_0+i\beta p).$$

There remain the constants β and μ_1 to be determined from

$$\langle(q-q_0)^2\rangle=\mathrm{Tr}[\tfrac{1}{2}\hbar\beta(a^\dagger-\alpha^*+a-\alpha)^2\rho]=(\Delta q)^2,$$

and

$$\langle(p-p_0)^2\rangle=\mathrm{Tr}[-(\hbar/2\beta)(a^\dagger-\alpha^*-a+\alpha)^2\rho]=(\Delta p)^2.$$

The calculation of these traces can be somewhat simplified (as can the preceding examples) by defining two new displaced operators

$$c=a-\alpha,$$
$$c^\dagger=a^\dagger-\alpha^*,$$

where $[c,c^\dagger]=1$, and again using the complete set of coherent eigenstates of the operator c. Thus

$$(\Delta q)^2=\frac{1}{\pi}\int\int\tfrac{1}{2}\hbar\beta\langle\alpha|c^{\dagger 2}\rho+\rho c^2+2c^\dagger c\rho+\rho|\alpha\rangle d^2\alpha$$

$$=\frac{\hbar\beta}{2\pi}\int\int\langle\alpha|c^{\dagger 2}\rho^{(n)}+\rho^{(n)}c^2+2c^\dagger\rho^{(n)}c+2c^\dagger\partial\rho^{(n)}/\partial c^\dagger+\rho^{(n)}|\alpha\rangle d^2\alpha,$$

or, letting $\alpha=x+iy$, we find

$$(\Delta q)^2=\frac{\hbar\beta}{2\pi}(1-e^{-\mu_1})\int\int[2x^2-2y^2+2(x^2+y^2)e^{-\mu_1}+1]\times\exp[(e^{-\mu_1}-1)(x^2+y^2)]dxdy$$

$$=\tfrac{1}{2}\hbar\beta(1+e^{-\mu_1})/(1-e^{-\mu_1}).$$

By the same method we obtain

$$(\Delta p)^2=(\hbar/2\beta)(1+e^{-\mu_1})/(1-e^{-\mu_1}).$$

In addition, the requirement that the density matrix be positive-definite must be met, and it demands[5] that

$$1-e^{-\mu_1}\geq 0.$$

Solving for β and μ_1 in compliance with the above requirements, we find

$$\beta=\Delta q/\Delta p\geq 0,\quad \frac{1+e^{-\mu_1}}{1-e^{-\mu_1}}=\frac{2\beta(\Delta p)^2}{\hbar}=\frac{\Delta p\Delta q}{\frac{1}{2}\hbar}\geq 1,$$

and

$$\mu_1=\ln[(\Delta p\Delta q+\tfrac{1}{2}\hbar)/(\Delta p\Delta q-\tfrac{1}{2}\hbar)].$$

With these values for the constants and using the definitions of a and $a^\dagger$, we may write the density matrix in any of the several forms in the text.

APPENDIX B: THE STATE OF THE SYSTEM AFTER AN IDEAL SIMULTANEOUS MEASUREMENT

For an ideal simultaneous measurement, $\Delta p\Delta q=\frac{1}{2}\hbar$, and the density operator expressing the state immediately following the measurement is

$$\rho=\mathfrak{N}\{\exp[-(a^\dagger-\alpha^*)(a-\alpha)]\}.$$

We wish to show that this is equivalent to the pure coherent state

$$\rho=|\alpha\rangle\langle\alpha|,$$

where

$$a|\alpha\rangle=\alpha|\alpha\rangle.$$

We do this by showing

$$\langle\gamma|\rho^{(n)}|\delta\rangle=\langle\gamma|\alpha\rangle\langle\alpha|\delta\rangle$$

for all γ and δ.

The coherent states have the property[5]

$$\langle\gamma|\alpha\rangle=\exp[-\tfrac{1}{2}|\gamma|^2-\tfrac{1}{2}|\alpha|^2+\gamma^*\alpha].$$

Thus we find

$$\langle\gamma|\rho|\delta\rangle=\langle\gamma|\mathfrak{N}\{\exp[-(a^\dagger-\alpha^*)(a-\alpha)]\}|\delta\rangle$$
$$=\exp\{-\tfrac{1}{2}|\gamma|^2-\tfrac{1}{2}|\delta^2|-|\alpha|^2+\gamma^*\alpha+\alpha^*\delta\},$$

and

$$\langle\gamma|\alpha\rangle\langle\alpha|\delta\rangle=\exp\{-\tfrac{1}{2}|\gamma|^2-\tfrac{1}{2}|\delta|^2-|\alpha|^2+\gamma^*\alpha+\alpha^*\delta\}.$$

Since the two expressions are equal for all γ and δ, we conclude that

$$\rho=\mathfrak{N}\{\exp[-(a^\dagger-\alpha^*)(a-\alpha)]\}=|\alpha\rangle\langle\alpha|.$$

APPENDIX C: THE MINIMUM VALUE OF $\Delta p'\Delta q'$ FOR THE IDEAL SIMULTANEOUS MEASUREMENT

To prove that $\hbar^2$ is the minimum possible value of

$$(\Delta p')^2(\Delta q')^2=\int\int(p'-p_0)^2(q'-q_0)^2P(q',p')dq'dp',$$

where

$$P(q',p')=h^{-1}\langle q',p'|\rho|q',p'\rangle=h^{-1}\langle\alpha|\rho|\alpha\rangle,$$

we use Glauber's P representation[5] for the density operator:

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha=\frac{\pi}{h}\int\int P(\alpha)|\alpha\rangle\langle\alpha|dp'dq'.$$

With the P representation and a few steps of manipulation, we can show that

$$(\Delta p')^2(\Delta q')^2=\hbar^2+\frac{\pi}{h}\int\int(q'-q_0)^2(p'-p_0)^2P(\alpha)dp'dq'.$$

Now, we compute $f(\alpha^*)$ by Eq. (4.11) of Glaubers'[5] with the choice

$$|f\rangle=\tfrac{1}{2}i\hbar[(a^\dagger-\alpha_0^*)^2-(a-\alpha_0)^2]|\alpha\rangle.$$

Substituting the result for $|f(\alpha^*)|^2$ just obtained into Eq. (7.9) of Glauber's, we obtain

$$\frac{\pi}{h}\int\int(q'-q_0)^2(p'-p_0)^2P(\alpha)dp'dq'\geq 0,$$

where the equality holds if $P(\alpha)=\delta^2(\alpha-\alpha_0)=(h/\pi)\times\delta(q'-q_0)\delta(p'-p_0)$. Therefore, $(\Delta p')^2(\Delta q')^2\geq\hbar^2$ and the equality (minimum) holds if the state immediately before the ideal simultaneous measurement is a coherent state, i.e., if $P(\alpha)=\delta^2(\alpha-\alpha_0)$ or $\rho=|\alpha_0\rangle\langle\alpha_0|$.

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Characteristic States of the Electromagnetic Radiation Field

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It has been argued that the positive-frequency part of the quantized electromagnetic field is the "observable" that one would most naturally associate with field measurements using quantum photodetectors. However, since it is possible in principle to make field measurements via the process of stimulated emission, the question of the possible solutions of the characteristic-value equation for the creation operator $a^\dagger$ is examined. Various proofs are given to demonstrate that the characteristic kets of $a^\dagger$ are not physically admissible states of the radiation field. The possible existence of other useful basis states besides $|n\rangle$, $|\alpha\rangle$, and states generated from these by unitary transformations is then considered. It is shown that when certain restrictions are placed on the correspondence between Hermitian combinations of the arbitrary non-normal operators b and $b^\dagger$ and the harmonic-oscillator variables x, p, and H, then the only possible basis states are the coherent states $|\alpha\rangle$ and the number states $|n\rangle$. A λ-dependent variation on the photon annihilation operator a is also considered. Its characteristic states for $-1\leq\lambda\leq 1$ are derived, and shown to form a complete set.

I. INTRODUCTION

THE recent development[1–7] of a quantum-mechanical theory of optical coherence has demonstrated the utility of the characteristic states of the non-Hermitian, non-normal boson annihilation operator a, the quasiclassical or coherent states. For a single-mode radiation field the coherent state vector $|\alpha\rangle$ satisfies the characteristic value equation,

$$a|\alpha\rangle=\alpha|\alpha\rangle, \quad (1)$$

with α as its corresponding complex characteristic value. Although the $|\alpha\rangle$ states are not orthogonal, that is,

$$|\langle\alpha|\beta\rangle|^2=\exp(-|\alpha-\beta|^2), \quad (2)$$

they can be normalized to unity, $\langle\alpha|\alpha\rangle=1$. The $|\alpha\rangle$ states also constitute a basis for the representation of arbitrary states and operators of the radiation field since the nonorthogonal projection operators $|\alpha\rangle\langle\alpha|$ satisfy a completeness relation of the form[8]

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \quad (3)$$

where $d^2\alpha=d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$ is the real element of area, and the integration extends over the entire complex plane. Because of their nonorthogonality, expansions in terms of coherent states are in general not unique unless additional restrictions are placed upon the expansion coefficients.[4] In contrast to the infinite complete sequence of occupation-number states $|n\rangle$, $n=0, 1, 2, \cdots, \infty$ which form an orthonormal basis for the field state vectors, the basis formed by the $|\alpha\rangle$ characteristic states constitutes a complete nondenumerable infinity of normalized characteristic vectors which are not

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