Coupled-Channel Schrödinger-Equation Model for Neutron-Alpha and Deuteron-Triton Scattering. I*

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We have considered a two-body model for neutron-alpha and deuteron-triton scattering in which pairs of states are coupled together by a real, central, Saxon potential. All couplings obey conservation of total angular momentum and parity. The $D_{3/2}$ excited state of He^5 (or the 107-keV "resonance" in the $d+T$ system) is thus explained by a strong coupling. This simple model adequately explains the elastic and reaction data except for the low-angle left-right asymmetry for the (mirror) reaction proton.

I. INTRODUCTION

Helium is the most commonly used analyzer for polarized neutrons and so it is desirable to understand the scattering of neutrons by α particles as well as possible. The neutron-alpha system is also simple in that (1) the large binding energy of the α particle provides a large energy region in which only elastic processes can occur, (2) it is one the simplest (nontrivial) spin systems, and (3) there are no Coulomb effects to be taken into account. Furthermore, the neutron-alpha system has a mirror process $p+\alpha$ which is quite well understood both theoretically and experimentally. Because of the simplicity of the $n+\alpha$ system (1)–(3), it is an ideal process to study in a model calculation.

The $n+\alpha$ phase shifts are usually deduced from the $p+\alpha$ phase shifts, which are more accurately known. The starting point of the model studied here is the work of Gammel and Thaler[1] who found a $p+\alpha$ potential that produced a smooth set of phase shifts. These phase shifts are not in agreement with recent experimental results[2–4] of the $n+\alpha$ system. Hoop and Barschall[5] have deduced new set of $n+\alpha$ phase shifts from a new set of $p+\alpha$ phase shifts by Weitkamp and Haeberli.[6] These phase shifts are in excellent agreement with all of the existing $n+\alpha$ elastic differential cross sections and polarizations, $d+T$ total reaction cross sections and the resonance parameters of the $D_{3/2}$ excited state of He^5.

There is no experimental data concerning the reaction polarizations of the deuteron or triton. A great deal of data exists,[7–11] however for the inverse process $T(d,n)He^4$. The total cross section for $n+\alpha$ is known[2,1] from 14- to 30-MeV laboratory neutron energy and the $d+T$ total cross section (integrated 20° to 180° c.m. angle) is known.[13,14] The reaction differential cross section for the process $T(d,n)He^4$ is well known,[7,11] as well as the elastic $d+T$ differential cross sections.[15,16] The $D_{3/2}$ excited state of He^5 was first observed in $d+T$ system by Connor *et al.*[17] The $d+T$ system, unfortunately, is not so well explored as the $n+\alpha$ system. Laskar[18] *et al.* have calculated phase shifts for elastic $d+T$ scattering within the resonating-group formalism. Their calculation produced a smooth set of phase shifts which are in reasonable agreement with the experimental data, and which showed that Serber exchange described the forces better than the Biel or symmetrical exchange.

There are many models for nuclear reactions[19–25] but the one which seemed to generalize the previous 5-body nuclear scattering[1,5,18] most naturally was the coupled Schrödinger-equation model proposed by Newton[19] for inelastic scattering. The bibliographies of Refs. 19–24 contain a detailed list of references to most aspects of reaction theory. In Newton's model, the $^2D_{3/2}$ excited state of He^5 will be a consequence of the opening of the $d+T$ reaction channel, just as Baz[26] suggested some time ago.

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[1] J. L. Gammel and R. M. Thaler, Phys. Rev. **109**, 2041 (1958).
[2] R. E. Shamu and J. G. Jenkin, Phys. Rev. **135**, B99 (1964).
[3] T. H. May, R. L. Walter, and H. H. Barschall, Nucl. Phys. **45**, 17 (1963).
[4] R. B. Perkins and C. Glashausser, Nucl. Phys. **60**, 433 (1964).
[5] B. Hopp, Jr., and H. H. Barschall, Nucl. Phys. **83**, 65 (1966).
[6] W. G. Weitkamp and W. Haeberli, Nucl. Phys. **83**, 46 (1966).
[7] S. J. Bame, Jr., and J. E. Perry, Phys. Rev. **107**, 1616 (1957).
[8] J. Christiansen, F. W. Büsser, F. Niebergall, and G. Sohngen (private communication to J. L. G.).
[9] R. I. Brown and W. Haeberli, Phys. Rev. **130**, 1163 (1963).
[10] S. D. Baker, G. Roy, G. C. Phillips, and G. K. Walters, Phys. Rev. Letters **15**, 115 (1965).
[11] L. Stewart, J. E. Brolley, Jr., and L. Rosen, Phys. Rev. **119**, 1649 (1960).
[12] Los Alamos Physics and Cryogenics Groups, Nucl. Phys. **12**, 291 (1959).
[13] H. V. Argo, R. G. Taschek, H. M. Agnew, A. Hemmendinger, and W. T. Leland, Phys. Rev. **87**, 612 (1952).
[14] E. J. Stovall, Jr., W. R. Arnold, J. A. Phillips, G. A. Sawyers, and J. L. Tuck, Phys. Rev. **87**, 159 (1952).
[15] W. R. Stratton, G. D. Freier, G. R. Keepin, D. Rankin, and T. F. Stratton, Phys. Rev. **88**, 257 (1952).
[16] J. C. Allred, A. H. Armstrong, A. M. Hudson, R. M. Potter, E. S. Robinson, L. Rosen, and E. J. Stovall, Jr., Phys. Rev. **88**, 425 (1952).
[17] T. P. Connor, T. W. Bonner, and T. R. Smith, Phys. Rev. **88**, 473 (1952).
[18] W. Laskar, C. Tate, and P. G. Burke, Proc. Phys. Soc. (London) **77**, 1014 (1961).
[19] R. G. Newton, Ann. Phys. (N.Y.) **4**, 29 (1958).
[20] L. Fonda and R. G. Newton, Ann. Phys. (N.Y.) **10**, 490 (1960).
[21] H. Feshbach, Ann. Phys. (N.Y.) **5**, 357 (1958).
[22] H. Feshbach, Ann. Phys. (N.Y.) **19**, 287 (1962).
[23] R. Lipperheide, Ann. Phys. (N.Y.) **17**, 114 (1962).
[24] G. R. Satchler, Nucl. Phys. **77**, 481 (1966).
[25] L. Wolfenstein, Ann. Rev. Nucl. Sci., **6**, 43 (1956).
[26] A. I. Baz, Zh. Eksperim. i Teor Fiz. **33**, 923 (1957) [English transl.: Soviet Phys.—JETP **6**, 709 (1958)].

TABLE I. The $n+\alpha$ and $d+$T total spin states.

A. $n+\alpha$ $S=\frac{1}{2}$	
Even parity	Odd parity
$^2S_{1/2}$	$^2P_{1/2}$ $^2P_{3/2}$
$^2D_{3/2}$ $^2D_{5/2}$	
B. $d+T$ $S=\frac{3}{2}$, $S=\frac{1}{2}$	
Even parity	Odd parity
$^4S_{3/2}$ $^2S_{1/2}$	$^4P_{1/2}$ $^4P_{3/2}$ $^4P_{5/2}$
$^4D_{1/2}$ $^4D_{3/2}$ $^4D_{5/2}$ $^4D_{7/2}$	$^2P_{1/2}$ $^2P_{3/2}$
$^2D_{3/2}$ $^2D_{5/2}$	

Recently, the measurement of the left-right scattering asymmetry[10] and its comparison to the spin polarization[9] have attracted interest.[27–30] The qualitative features of these experiments[9,10] can be understood with nondynamical studies[28,30] but Tanifuji[29] has shown that the results are incompatible with ordinary direct reaction theory with simple potentials, i.e. central, tensor, and $\mathbf{l}\cdot\mathbf{s}$ potential in *uncoupled* Schrödinger equations.

In Sec. II the formalism of the model is presented and discussed. In Sec. III, the results and conclusions are presented and discussed.

II. MODEL AND CALCULATIONS

The model chosen for our calculations was the two-body model proposed by Newton[19] and Fonda and Newton.[20] Wills *et al.*[31] recently used a similar model which is different from ours in three respects: (1) it consists of a relativistic set of coupled Schrödinger equations whereas our model consists of nonrelativistic coupled Schrödinger equations; (2) Wills *et al.* were interested in peripheral interactions so that their entire coupling matrix was determined from the possible one-particle exchanges, whereas we take the coupling potentials as parameters in order to fit all the reaction data; (3) Wills *et al.* neglected spin effects whereas we were interested in such observables as spin polarizations and reaction left-right asymmetry.

The "channels" in our model will be the $n+\alpha$ channel and the $d+$T channel (listed in Table I) which are pairwise coupled subject to conservation of total angular momentum J and parity P. Pairs of states are coupled in the equation

$$\left\{\begin{pmatrix} T_1 & 0 \\ 0 & T_2\end{pmatrix}+\begin{pmatrix} V_{11}(J,P,r) & V_{12}(J,P,r) \\ V_{21}(J,P,r) & V_{22}(J,P,r)\end{pmatrix}-\begin{pmatrix} E & 0 \\ 0 & E-Q\end{pmatrix}\right\} \times\begin{pmatrix} u(J,P,l_1,r) \\ w(J,P,l_2,r)\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}, \quad (1)$$

[27] M. Tanifuji, Phys. Rev. Letters **15**, 113 (1965).
[28] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Phys. Rev. **143**, 1324 (1966).
[29] M. Tanifuji, Nucl. Phys. (to be published).
[30] I. Duck, Nucl. Phys. **80**, 617 (1966).
[31] J. G. Wills, D. Ellis, and D. B. Lichtenberg, Phys. Rev. **143**, 1375 (1966).

where T_1 is the kinetic energy operator for the $n+\alpha$ channel and T_2 is the kinetic energy operator for the $d+$T channel. Hence,

$$T_1=-\frac{\hbar^2}{2\mu_1}\left(\frac{d^2}{dr^2}-\frac{l_1(l_1+1)}{r^2}\right), \quad (2)$$

and

$$T_2=-\frac{\hbar^2}{2\mu_2}\left(\frac{d^2}{dr^2}-\frac{l_2(l_2+1)}{r^2}\right), \quad (3)$$

where μ_1 and μ_2 are the $n+\alpha$ and $d+$T reduced masses, l_1, l_2 are the channel orbital angular momenta, and r is the relative coordinate in each channel. In Eq. (1), the two diagonal potentials[32] $V_{11}(J,P,r)$ and $V_{22}(J,P,r)$ are chosen as the Gammel-Thaler[1] $p+\alpha$ potential and a soft potential which reproduces the observed $d+$T angular distributions. The coupling potential $V_{12}(J,P,r)$ is adjusted in depth to provide the experimental reaction cross section. The parameters in all potentials are listed in Table II. E is the center-of-mass neutron energy in MeV and $Q=17.67$ MeV for the $d+$T threshold. The functions $u(J,P,l_1,r)$ and $w(J,P,l_2,r)$ are the channel eigenfunctions for $n+\alpha$ and $d+$T, respectively. The same relative coordinate r is used in both channels because in this model we assume that the deuteron and triton can be "effectively" treated as having no internal structure as in Ref. 19. Thus the polarization of deuteron and triton wave functions is a possible correction to this model.

All of the $n+\alpha$ states of Table I are coupled to $d+$T states according to the following scheme: $^2D_{3/2}(n+\alpha)$ couples to $^4S_{3/2}(d+T)$, $^2S_{1/2}(n+\alpha)$ couples to $^2S_{1/2}(d+\mathrm{T})$, $^2P_{3/2}(n+\alpha)$ couples to $^2P_{3/2}(d+\mathrm{T})$, $^2P_{1/2}(n+\alpha)$ couples to $^2P_{1/2}(d+\mathrm{T})$, and $^2D_{5/2}(n+\alpha)$ couples to $^2D_{5/2}(d+\mathrm{T})$. All other couplings are neglected at present. An inspection of Table I indicates that this neglect of other couplings is a serious limitation on the model. For instance the $^2D_{3/2}(n+\alpha)$ state could be coupled to $^4S_{3/2}(d+\mathrm{T})$, $^2D_{3/2}(d+\mathrm{T})$, and $^4D_{3/2}(d+\mathrm{T})$ while conserving J and P. It is of interest to see how important the pairwise couplings are, in the spirit of a model calculation. If agreement is found, it will provide an "*a posteriori*" justification for the neglect of additional couplings. At present, our computer codes are being adapted to include all possible couplings which conserve J and P.

For a given value of J and P, Eq. (1) is solved numerically to obtain the scattering solution (in the asymptotic region) which, for an incident-channel α scattering into a final-channel β can be written as

$$\Phi_{\alpha\beta}(r)\sim\delta_{\alpha\beta}e^{i\mathbf{k}_\alpha\cdot\mathbf{r}}\chi_s^{m_s}+\sum_{s',m_{s'}}M(\beta\leftarrow\alpha,s'm_{s'},sm_s) \times(\chi_{s'}^{m_{s'}})(e^{ik_\beta r}/r), \quad (4)$$

where k_α and k_β are the magnitudes of center-of-mass

[32] In Ref. 23, R. Lipperheide has shown explicitly that hard cores in the potentials can lead to no difficulties so long as the product of the wave function and hard core vanishes.

TABLE II. The elements of the potential matrix.

A.

$$V_{11}(r)=\infty \qquad (r\leq r_c)$$
$$\quad =V_c(r)+(\mathbf{L}\cdot\mathbf{S})V_{\mathrm{L\cdot S}}(r) \qquad (r>r_c)$$

$$V_c(r)=\frac{V_c}{1+[r/D-1]\exp[(r-R)/D]}$$

$$V_{\mathrm{L\cdot S}}(r)=-\frac{D^2}{r}\frac{V_{\mathrm{L\cdot S}}}{V_c}\left[\frac{d}{dr}V_c(r)\right]$$

$r_c^+=r_c^-=0.183\times10^{-13}$ cm
$R^+=1.70\times10^{-13}$ cm
$R^-=1.85\times10^{-13}$ cm
$D^+=0.850\times10^{-13}$ cm
$D^-=0.925\times10^{-13}$ cm
$V_c^+=-45.6$ MeV
$V_c^-=-40.0$ MeV
$V_{\mathrm{L\cdot S}}^+=-15.0$ MeV
$V_{\mathrm{L\cdot S}}^-=-30.0$ MeV

B.

$$V_{22}(r)=V_0/\{1+(R_0/a_0-1)\exp[(r-R_0)/a_0]\}$$

$R_0=1.250\times10^{-13}$ cm, $a_0=0.850\times10^{-13}$ cm } all states.

$d+$T state	V_0 depth in MeV
^4S	-52.0
^4P	-23.0
^4D	-50.0
^2S	-54.0
^2P	$+1.0$
^2D	-59.0

C.

$$V_{12}(r)=V_1/\{1+(r/a_1-R_1)\exp[(r-R_1)/a_1]\}$$

$R_1=1.000\times10^{-13}$ cm, $a_1=0.850\times10^{-13}$ cm } all states.

$n+\alpha$ state	$d+T$ state	V_1 depth in MeV
$^2D_{3/2}$	$^4S_{3/2}$	-74.0
$^2S_{1/2}$	$^2S_{1/2}$	-9.0
$^2P_{3/2}$	$^2P_{3/2}$	-27.0
$^2P_{1/2}$	$^2P_{1/2}$	-24.0
$^2D_{5/2}$	$^2D_{5/2}$	-6.0

wave number of the initial and final channels in the coordinate system of Fig. 1. $\chi_s^{m_s}$ and $\chi_{s'}^{m_{s'}}$ are the initial and final spin functions, and $M(\beta\leftarrow\alpha,\, s'm_{s'},\, sm_s)$ is the scattering matrix in total spin representation except for the factor $(i)^{l-l'}$. Explicitly,

$$M(\beta\leftarrow\alpha,\, s'm_{s'},\, sm_s)$$
$$=\frac{(4\pi)^{1/2}}{k_\alpha}\sum_{J,l,l'}\{(2l+1)^{1/2}C(Jm_J,l0,sm_s)$$
$$\times t(J,P,l,l')_{\beta\alpha}(i)^{l-l'}$$
$$\times C(Jm_J,l'm_l',s'm_s')Y_{l'}^{m_l'}(\theta,\phi)\}\,, \quad (5)$$

where k_α is the magnitude of the wave number in the incoming channel, $C(Jm_J,l0,sm_s)$ and $C(Jm_J,l'm_l',s'm_s')$ are the initial- and final-state Clebsch-Gordan coefficients, $Y_{l'}^{m_l'}(\theta,\phi)$ is the spherical harmonic of the final

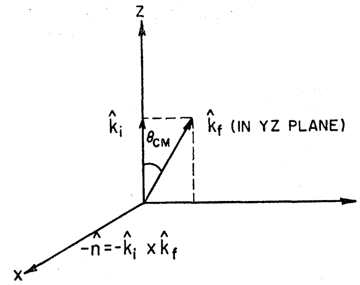

FIG. 1. The coordinate system used for the calculation. The initial beam is in the direction of the unit vector $\hat{k}_i$ and the scattered wave is in the direction of the unit vector $\hat{k}_f$ so that cross sections are measured in the y-z (scattering) plane.

state, and $t(J,P,l,l')_{\beta\alpha}$ is the transition matrix element whose relationship to the S matrix is

$$t(J,P,l,l')_{\beta\alpha}=(1/2i)[\delta_{\alpha\beta}-S(J,P,l,l')_{\beta\alpha}]. \quad (6)$$

In general elastic-scattering differential cross section for an unpolarized scattering experiment is given by

$$[d\sigma(\theta)/d\Omega]_{\alpha\alpha}=[1/(2I+1)(2i+1)]\,\mathrm{Tr}\{MM^\dagger\} \quad (7)$$

where I and i are the target and projectile spins. Here α may denote either a $n+\alpha$ or $d+$T system. This reduces to the usual formulas for elastic scattering found in Refs. 18 and 33. The unpolarized reaction differential cross sections for scattering from the α channel into the β channel is given by

$$[d\sigma(\theta)/d\Omega]_{\beta\alpha}=[1/(2i+1)(2I+1)]\,\mathrm{Tr}[M_{\mathrm{re}}M_{\mathrm{re}}^\dagger]\,, \quad (8)$$

where M_{re} is the reaction scattering matrix and I and i are the spins of the target and projectile in channel α. The polarization of the reaction neutron from the initially unpolarized process $\mathrm{T}(d,n)\mathrm{He}^4$ is given by

$$\mathbf{P}=\langle\boldsymbol{\sigma}\rangle_f=\mathrm{Tr}(M_{\mathrm{re}}M_{\mathrm{re}}^\dagger\boldsymbol{\sigma})/\mathrm{Tr}(M_{\mathrm{re}}M_{\mathrm{re}}^\dagger)\,, \quad (9)$$

where M_{re} is the 2×6 reaction scattering matrix and $\boldsymbol{\sigma}$ is the 2×2 Pauli spin operator. In the case of scattering with polarized targets, to get the left-right asymmetry the scattering matrix M_{re} must be rotated into individual particle representation using the usual Clebsch-Gordan matrix. Let N_{re} denote the scattering matrix thus obtained.

$$A(\theta)=\mathrm{Tr}[N_{\mathrm{re}}(\mathbf{1}\otimes\boldsymbol{\sigma})N_{\mathrm{re}}^\dagger]/\mathrm{Tr}[N_{\mathrm{re}}N_{\mathrm{re}}^\dagger]\,, \quad (10)$$

where $\mathbf{1}\otimes\boldsymbol{\sigma}$ is a 6×6 direct product of the 3×3 unit operator for the unpolarized deuteron and the Pauli spin operator $\boldsymbol{\sigma}$ for the initially polarized He^3 target. One notes that the denominator of Eq. (9) is identical to the denominator of Eq. (10) because the trace is an invariant quantity.

The relations between the phase shifts of Tables III and IV and the differential cross sections for the $n+\alpha$ and $d+$T systems are listed in Refs. 33 and 18, respectively (except that complex phase shifts were used when above the lowest threshold). The neutron polarization for elastic scattering is given by Burke.[33] The explicit formulas for the inelastic scattering, polariza-

[33] P. G. Burke, *Nuclear Forces and the Few Nucleon Problem* (Pergamon Press, Inc., New York, 1960), Vol. II, p. 481.

TABLE III. Real parts of the neutron-alpha phase shifts and inelastic parameters. The inelastic parameter is $\gamma_l^{J,P} \equiv (1-|\eta_l^{J,P}|^2) \equiv (1-e^{-4IM\delta_l})$, where $\eta_l^{J,P}=e^{2i\delta_l J,P}$ and $\delta_l^{J,P}$ is the complex phase shift.

E(lab, n) (MeV)	$\delta(S_{1/2})$ (rad)	$\delta(P_{3/2})$ (rad)	$\delta(P_{1/2})$ (rad)	$\delta(D_{5/2})$ (rad)	$\delta(D_{3/2})$ (rad)
16.00	−1.465	−1.421	1.001	0.180	0.131
18.00	−1.514	−1.448	1.003	0.202	0.145
20.00	−1.556	−1.475	1.051	0.225	0.160
21.00	−1.567	−1.487	1.061	0.260	0.168
22.00	1.566	−1.499	1.069	0.277	0.179
22.15	1.566 (0.0189)	−1.499 (0.0001)	1.069 (0.0001)	0.283 (0.002)	0.9093 (0.9491)
22.20	1.559 (0.0485)	−1.499 (0.0002)	1.069 (0.0002)	0.284 (0.005)	0.1652 (0.551)
22.45	1.551 (0.0780)	−1.500 (0.0003)	1.019 (0.0003)	0.295 (0.008)	−0.336 (0.060)
22.60	1.500 (0.0883)	−1.540 (0.0005)	1.012 (0.0075)	0.299 (0.082)	−0.001 (0.042)
23.60	1.450 (0.1050)	−1.565 (0.0009)	0.909 (0.0171)	0.307 (0.096)	0.142 (0.030)
24.00	1.411 (0.0912)	1.552 (0.0023)	0.907 (0.0196)	0.315 (0.084)	0.148 (0.052)
26.00	1.372 (0.0820)	1.560 (0.0125)	0.905 (0.0200)	0.327 (0.075)	0.154 (0.066)
28.00	1.327 (0.0760)	1.531 (0.0300)	0.903 (0.0250)	0.358 (0.052)	0.158 (0.079)
30.00	1.281 (0.0650)	1.455 (0.0381)	0.901 (0.0301)	0.391 (0.030)	0.161 (0.102)
32.00	1.150 (0.0541)	1.380 (0.0473)	0.870 (0.0360)	0.416 (0.021)	0.165 (0.120)
34.00	1.010 (0.0421)	1.311 (0.0620)	0.782 (0.0412)	0.424 (0.015)	0.173 (0.150)
37.00	0.990 (0.0311)	1.21 (0.0814)	0.751 (0.039)	0.456 (0.012)	0.191 (0.173)
40.00	0.892 (0.028)	1.03 (0.0934)	0.700 (0.031)	0.473 (0.008)	0.211 (0.199)
42.00	0.850 (0.022)	0.985 (0.1261)	0.652 (0.021)	0.488 (0.001)	0.224 (0.211)

tions, and asymmetries are given below because Huby[34] has shown that the older papers which list these formulas contain an inconsistency in the phasing of the matrix elements. The differential cross section of the reaction neutron is found from Eq. (8) to be

$$[d\sigma(\theta)/d\Omega]_{n,\mathrm{re}} = (1/6k^2)D(\theta)\,, \tag{11}$$

where k is the magnitude of the initial-state center-of-mass wave number and $D(\theta)$ is given by

$$\begin{aligned} D(\theta) = {} & 2|t_{0,0}^{1/2}|^2+2|t_{1,1}^{1/2}|^2+2(3\cos^2\theta+1)|t_{1,1}^{3/2}|^2+2|t_{0,2}^{3/2}|^2\{6\sin^2\theta\cos^2\theta+\tfrac{1}{2}(3\cos^2\theta-1)^2+\tfrac{3}{2}\sin^4\theta\} \\ & +2|t_{2,2}^{5/2}|^2\{(9/4)(3\cos^2\theta-1)^2+9\sin^2\theta\cos^2\theta\}+4\cos\theta[\mathrm{Re}(t_{0,0}^{1/2}t_{1,1}^{1/2*})]+8\cos\theta[\mathrm{Re}(t_{0.0}^{1/2}t_{1,1}^{3/2*})] \\ & +4(3\cos^2\theta-1)[\mathrm{Re}(t_{1,1}^{1/2}t_{1,1}^{3/2*})]+6(3\cos^2\theta-1)[\mathrm{Re}(t_{2,2}^{5/2}t_{0,0}^{1/2*})] \\ & +6\cos\theta(5\cos^2\theta-3)[\mathrm{Re}(t_{2,2}^{5/2}t_{1,1}^{1/2*})]+24\cos^3\theta[\mathrm{Re}(t_{2,2}^{5/2}t_{1,1}^{3/2*})]\,. \end{aligned} \tag{12}$$

In the above expression $t_{ll'}^J$ denote the quantities defined in Eq. (6). The reaction-neutron polarization from an initially unpolarized beam can be evaluated from Eq. (9) and is given by $P_{\mathrm{re},n}(\theta)$ where

$$P_{\mathrm{re},n}(\theta)=4(p_1\sin\theta+p_2\sin\theta\cos\theta+p_3\sin\theta\cos^2\theta)/D(\theta)\,, \tag{13}$$

where

$$p_1=\mathrm{Im}[t_{0,0}^{1/2}(t_{1,1}^{3/2*}-t_{1,1}^{1/2*})]+\tfrac{3}{2}\,\mathrm{Im}[t_{2,2}^{5/2}(t_{1,1}^{1/2*}-t_{1,1}^{3/2*})]\,, \tag{14}$$

$$p_2=3\,\mathrm{Im}[t_{0,0}^{1/2}t_{2,2}^{5/2*}]+3\,\mathrm{Im}[t_{1,1}^{1/2}t_{1,1}^{3/2*}]\,, \tag{15}$$

$$p_3=(15/2)\,\mathrm{Im}[t_{1,1}^{1/2}t_{2,2}^{5/2*}]+\tfrac{3}{2}\,\mathrm{Im}[t_{1,1}^{3/2}t_{2,2}^{5/2*}]\,. \tag{16}$$

Also, the left-right asymmetry $A(\theta)$ from an initially polarized target is found from Eq. (10) to be

$$A(\theta)=4(a_1\sin\theta+a_2\sin\theta\cos\theta+a_3\sin\cos^3\theta+a_4\sin^3\theta)/D(\theta)\,, \tag{17}$$

where

$$a_1=2\,\mathrm{Im}[t_{2,2}^{5/2}t_{1,1}^{1/2*}]+\mathrm{Im}[t_{2,2}^{5/2}t_{1,1}^{3/2*}]+\tfrac{1}{3}\,\mathrm{Im}[t_{0,0}^{1/2}(t_{1,1}^{1/2*}-t_{1,1}^{3/2*})]+\tfrac{2}{3}\,\mathrm{Im}[t_{0,2}^{3/2}t_{1,1}^{3/2*}]+\tfrac{2}{3}\,\mathrm{Im}[t_{1,1}^{1/2}t_{0,2}^{3/2*}]\,, \tag{18}$$

$$a_2=2\,\mathrm{Im}[t_{0,2}^{3/2}t_{2,2}^{5/2*}]+\mathrm{Im}[t_{1,1}^{3/2}t_{1,1}^{1/2*}]+\mathrm{Im}[t_{2,2}^{5/2}t_{0,0}^{1/2*}]\,, \tag{19}$$

$$a_3=(9/4)\,\mathrm{Im}[t_{0,2}^{3/2}t_{2,2}^{5/2*}] \tag{20}$$

$$a_4=\tfrac{5}{2}\,\mathrm{Im}[t_{1,1}^{1/2}t_{2,2}^{5/2*}]+\tfrac{1}{2}\,\mathrm{Im}[t_{1,1}^{3/2}t_{2,2}^{5/2*}]\,. \tag{21}$$

The computer program which solved Eq. (1) numerically was checked as follows: In addition to extensive hand calculations, the zero-coupling case was used to check the diagonal part of the program, and the off-diagonal K-matrix elements were found to vary as the ratio of the channel reduced masses, as they should.

[34] R. Huby, Proc. Phys. Soc. (London) **A67**, 1103 (1954).

TABLE IV. Real parts of the deuteron-triton phase shifts and inelastic parameters. The inelastic parameters $\gamma_l^{J,P}$ is defined as $\gamma_l^{J,P} \equiv (1-|\eta_l^{J,P}|^2) = (1-e^{-4\,\mathrm{Im}\,\delta_l J,P})$ where $\eta_l^{J,P} \equiv e^{2i\,{}^{2s+1}\delta_l}$ and ${}^{(2s+1)}\delta_l$ is the complex phase shift in the state J, P, l, S.

A. Real parts of the $D+T$ phase shifts.

E(lab,d) (MeV)	Re($^4\delta_0$) (rad)	Re($^4\delta_1$) (rad)	Re($^4\delta_2$) (rad)	Re($^2\delta_0$) (rad)	Re($^2\delta_1$) (rad)	Re($^2\delta_2$) (rad)
0.100	0.0432	0.0001	0.	0.0361	0.	0.
0.150	0.0460	0.0001	0.	0.0522	0.	0.
0.250	0.0610	0.0050	0.0010	0.201	−0.00010	0.0001
1.20	0.257	−0.0400	0.0436	0.451	−0.00025	0.0036
4.00	0.0851	−0.219	0.214	0.184	−0.00260	0.0892
6.00	−0.0430	−0.451	0.521	−0.075	−0.00910	0.214
8.00	−0.162	−0.573	0.842	−0.300	−0.0834	0.436
10.00	−0.259	−0.618	1.102	−0.376	−0.1813	0.725
12.00	−0.331	−0.682	1.390	−0.452	−0.211	0.834
14.00	−0.407	−0.771	1.250	−0.580	−0.243	0.910

B. Inelastic parameters for those states in which $\gamma_l^{J,P}$ does not vanish.

E(lab,d) (MeV)	$\gamma_0^{1/2,+}$	$\gamma_0^{3/2,+}$	$\gamma_1^{1/2,-}$	$\gamma_1^{3/2,-}$	$\gamma_2^{5/2,+}$
0.100	0.0189	0.9491	0.0001	0.0001	0.002
0.150	0.0440	0.583	0.0002	0.0002	0.004
0.250	0.0513	0.124	0.0003	0.0002	0.006
1.20	0.0792	0.064	0.0009	0.0004	0.013
4.00	0.0941	0.036	0.0089	0.0007	0.088
6.00	0.1022	0.047	0.0158	0.0008	0.092
8.00	0.087	0.059	0.0198	0.0079	0.079
10.00	0.0810	0.067	0.0211	0.0141	0.069
12.00	0.0755	0.083	0.0281	0.0300	0.045
14.00	0.0556	0.110	0.0310	0.0401	0.023

TABLE V. The reaction t-elements $t(J,P,l,l')$ for the process $T(d,n)He^4$ calculated from the potentials of Table II.

E(lab,d) (MeV)	Re($t_{0,0}^{1/2}$)	Im($t_{0,0}^{1/2}$)	Re($t_{1,1}^{1/2}$)	Im($t_{1,1}^{1/2}$)	Re($t_{1,1}^{3/2}$)	Im($t_{1,1}^{3/2}$)	Re($t_{0,2}^{3/2}$)	Im($t_{0,2}^{3/2}$)	Re($t_{2,2}^{5/2}$)	Im($t_{2,2}^{5/2}$)
1.00 (not used)	0.0585	−0.0943	0.00343	0.00574	−0.0002	0.0025	0.2014	0.3069	−0.096	−0.165
2.10	0.0310	−0.070	0.0022	0.00213	−0.00018	0.0023	−0.0697	−0.00672	−0.1456	0.0288
2.90	0.041	−0.077	0.0032	0.0034	−0.00018	0.0025	−0.071	−0.006	−0.132	0.032
6.00	0.173	−0.1255	0.0271	0.0227	0.00309	0.0506	0.1764	0.1515	−0.0876	−0.0848
10.00	0.016	−0.113	0.0391	0.0284	0.00364	0.0687	0.1759	0.0857	−0.0819	−0.059

Another check on the computer program was made by using a potential matrix of equal-range square wells for all three potentials $V_{11}(r)$, $V_{22}(r)$, and $V_{12}(r)$ in Eq. (1). The coupled-channel tangent matrix was then checked against a single-channel code and satisfactory agreement was obtained.[35]

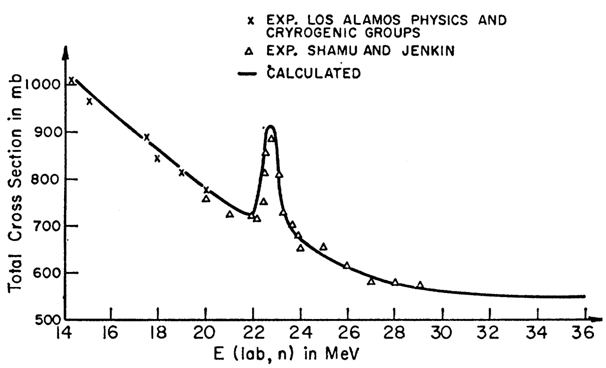

FIG. 2. Total neutron-alpha cross section at energies from 14- to 30-MeV laboratory neutron energy. × and △ denote measured values from Refs. 12 and 2, respectively. The solid curve is calculated with the phase shifts and inelastic parameters from Table III.

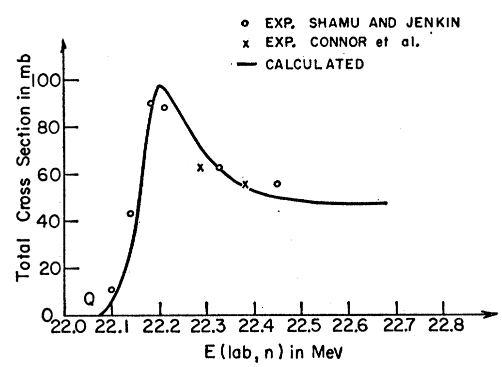

FIG. 3. Total deuteron reaction cross section at energies near the 22.05-MeV $^2D_{3/2}$ resonance. Experimental points are from Refs. 2 and 17 and the solid curve is calculated with the inelastic parameters in Table III.

[35] R. G. Newton (private communication to B.D.F.).

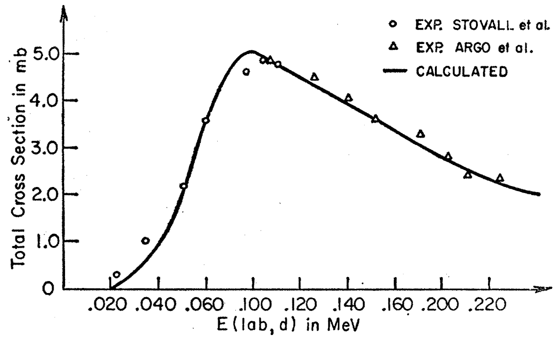

FIG. 4. Total cross section for $d+\mathrm{T}$ versus energy near the 107 keV "resonance." Experimental points are from Refs. 13 and 14 and the solid curve is calculated with the inelastic parameters from Table IV.

In the next section, the results of the "model" calculations are presented and compared with experimental data.

III. RESULTS AND DISCUSSION

The potentials of Table II were used in Eq. (1) to obtain the phase shifts and inelastic parameters of Tables III, IV, and V. In Figs. 2, 3–4, comparison with the experimental values of total cross sections of $n+\alpha$, $\alpha(n,d)\mathrm{T}$, and $d+\mathrm{T}$ is shown and the agreement is excellent in every case. In Figs. 5–7 the differential cross sections in the center of mass frame are plotted for elastic $\alpha(n,n)\alpha$, elastic $\mathrm{T}(d,d)\mathrm{T}$, and reaction $\mathrm{T}(d,n)\alpha$. While the elastic differential cross sections fit quite well, the reaction differential cross section is only in qualitative agreement, so that the absorption matrix of Tables III and IV is not exact. Figure 8 shows a fit to

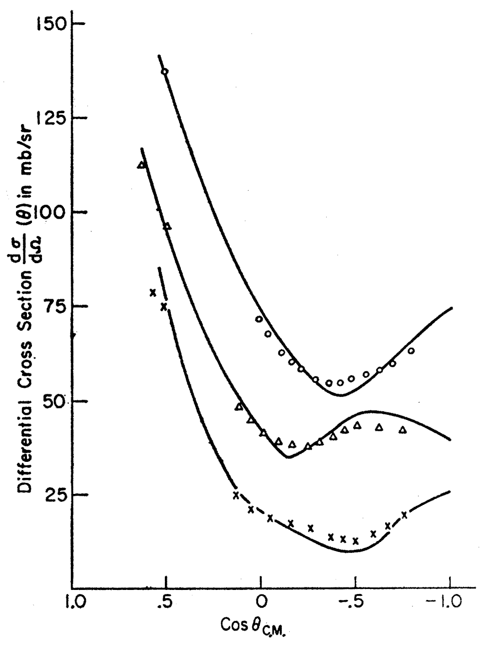

FIG. 5. Elastic neutron-alpha differential cross section versus cosine of center-of-mass scattering angle. Experimental points are from Ref. 5 and the solid curves are calculated with the phase shifts of Table III. ○ $E_N=20.91$ MeV (subtract 50 from ordinate). △ $E_N=22.20$ (subtract 25 from ordinate). ✕ $E_N=22.6$. — Calculated.

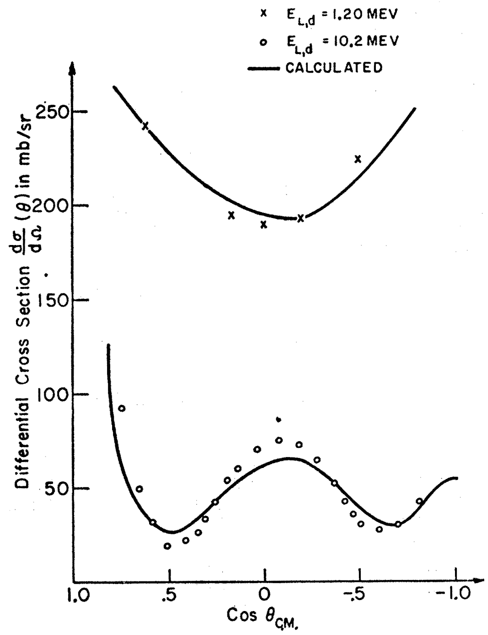

FIG. 6. Elastic deuteron-triton cross section versus cosine of center of mass scattering angle. Experimental points are from Refs. 15 and 16 and the solid lines were calculated with the phase shifts from Table IV.

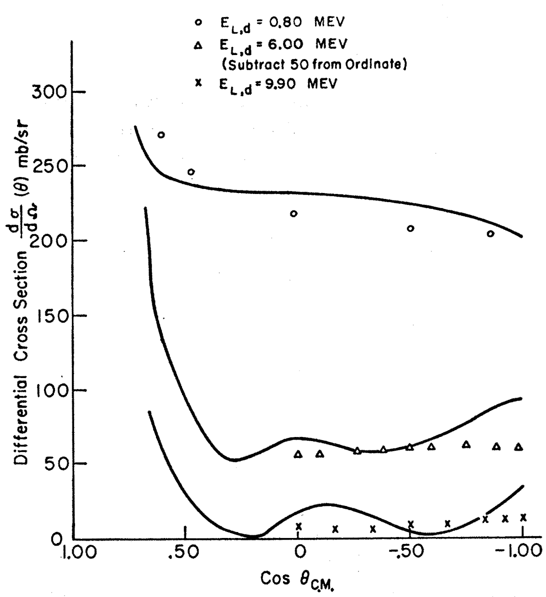

FIG. 7. Reaction neutron differential cross section versus cosine of center-of-mass scattering angle. Experimental points are from Refs. 7 and 11 and the solid curves were calculated with the t elements of Table V.

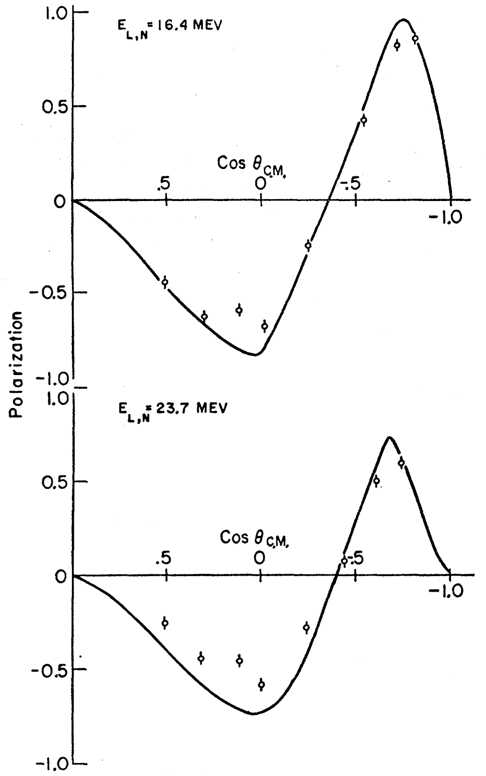

FIG. 8. Elastic neutron polarization versus cosine of center-of-mass scattering angle. Experimental points are from Ref. 3 and the solid curves are calculated with the phase shifts of Table III.

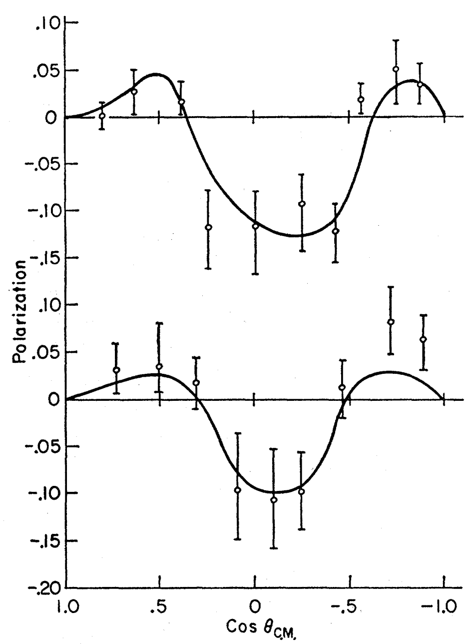

FIG. 9. Reaction neutron-spin polarization for the reaction neutron versus cosine of center-of-mass scattering angle. The experimental points are from Ref. 8 and the solid curves were calculated with the t elements of Table V.

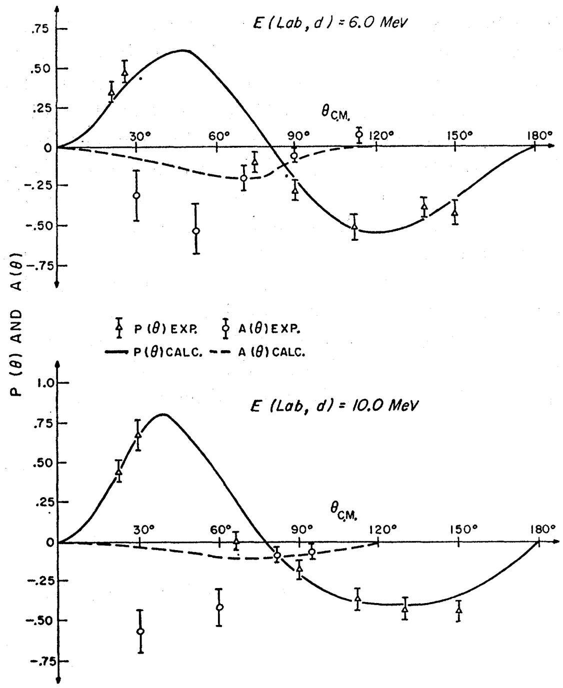

FIG. 10. Reaction polarization and asymmetry versus center-of-mass scattering angle from Refs. 9 and 10. These data are from the mirror process $He^3(d,p)He^4$ in Refs. 9 and 10. The calculated curves are calculated with Eqs. (13), (17) using the t elements of Table V, except that the energy is shifted by the (center-of-mass) energy difference between the excited states of Li^5 and He^5.

$P(\theta)$ and Fig. 9 shows the fit to $P_{re,n}(\theta)$. The fit to $P_{re,n}(\theta)$ is good considering the qualitative nature of the agreement of the reaction differential cross section. Upon *assuming* charge symmetry and shifting the energy axis by the He^5-Li^5 center-of-mass energy difference, reaction polarization and asymmetry $A(\theta)$ for the proton from the mirror process $He^3(d,p)He^4$ are calculated from Table V and the results presented in Fig. 10. The reaction-proton polarization agrees with the experimental data but $A(\theta)$ only agrees with the data at angles near 90°.

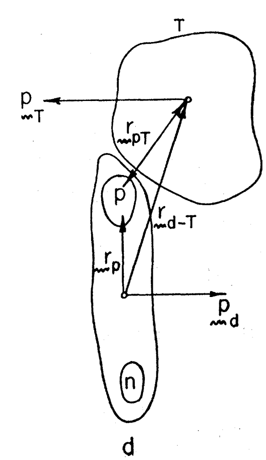

FIG. 11. The geometry of stripping in the reaction $T(d,n)He^4$ in the center-of-mass frame. The momenta of the deuteron and the triton are along $\mathbf{p}_d$ and $\mathbf{p}_T$, respectively. The vectors $\mathbf{r}_{pT}$ and $\mathbf{r}_{dT}$ are the vectors along which the triton "effectively interacts" with the proton and the deuteron and $\mathbf{r}_p$ locates the proton with respect to the center of the deuteron.

The lack of detailed agreement between calculated and experimental values of $A(\theta)$ does indicate that the scattering matrix of this model is not exact. Two possible reasons for this are, (1) the neglected couplings in the $d+\mathrm{T}$ system as shown in Table I might drastically change the structure of the scattering matrix (as previously mentioned, we are in process of correcting this flaw), and (2) the coupling potentials that were used were central potentials that depended only upon J and P, and this may be too simple to fit the data. It is interesting to note that the fact that $A(\theta)$ was too small at most angles seems to be a disease that this model shares with conventional direct-reaction theories. Tanifuji[27,29] has emphasized that since the $p+\mathrm{T}$ system exhibits strong spin polarization,[36] the neglect of the stripping process as shown in Fig. 11 is to neglect a spin-dependent potential. This neglect would not affect differential and total cross sections, but could make it impossible to simultaneously fit the spin polarizations and left-right asymmetry. Still, because of the complex energy behavior of the solutions to the coupled equations, it is proposed to first couple all the $d+\mathrm{T}$ states in Table I with central coupling potentials to get the best fit to experimental data before resorting to hard cores, spin-dependent stripping potentials, and nonlocal coupling potentials.

[36] T. A. Tombrello, Phys. Rev. **138**, B40 (1965).

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Improved Solution to the Bethe-Faddeev Equations*

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An approximate analytic solution is derived for the Bethe-Faddeev three-body equations in nuclear matter. The solution is no more complicated than the original approximation proposed by Bethe, but it is more accurate and avoids the discontinuities that appear in the original solution. In a certain limiting case, the solution agrees with the one previously proposed by Moszkowski on the basis of a variational treatment.

I. INTRODUCTION

RECENTLY, Bethe[1] has shown that the contribution of three-body correlations to the energy per particle of nuclear matter is given by

$$E^{(3)}/A=\rho^2\int g(r_{23})F(r_{23})d\tau_{23}. \qquad (1)$$

In this formula, ρ is the particle density, $g(r_{23})$ gives the radial dependence of the off-energy-shell effective interaction or G matrix, and $F(r_{23})$ is defined by

$$F(r_{23})=\int \eta(r_{12})Z_1(r_{12},r_{13},r_{23})d\tau_1. \qquad (2)$$

Here, $\eta(r_{12})$ is the on-energy-shell two-body "difference function," i.e., it is the difference between the unperturbed and the correlated wave functions for two particles. The three-body function Z_1, which is called $\Phi-\Psi^{(1)}$ by Bethe, satisfies the "Bethe-Faddeev equation"

$$Z_1(r_{12},r_{13},r_{23})=\eta(r_{12})+\eta(r_{13})-(1/e_{12})g_{12}Z_3(r_{12},r_{13},r_{23}) -(1/e_{13})g_{13}Z_2(r_{12},r_{13},r_{23}), \qquad (3)$$

with two similar equations for Z_2 and Z_3. The operator g_{12}, the off-energy-shell G matrix for particles 1 and 2, obeys the equation

$$g_{12}=v_{12}-v_{12}(1/e_{12})g_{12}, \qquad (4)$$

where v_{12} is the nucleon-nucleon potential. The propagator $(1/e_{12})$ is given by

$$e_{12}=-\nabla_{12}^2+\gamma^2, \qquad (5)$$

where γ is a constant which is estimated by Bethe to be between 3.1 F^{-1} and 3.7 F^{-1}, depending on the radius of the repulsive core in the two-body potential.

These equations were derived with the aid of three approximations, as discussed by Bethe.[1] (1) The initial momentum of each of the three interacting particles has been put equal to zero. (2) Reference-spectrum approximation: The energy spectrum for intermediate states is pure kinetic energy, and the exclusion principle is neglected for these states. (3) The dependence of e_{12} on the momentum of particle 3 has been averaged out.

Simple and accurate methods are known for calculating the two-body functions $\eta(r)$ and $g(r)$.[2] The problem, therefore, is to solve (3) for the three-body function Z_1, and Bethe[1] has found a very simple approximate solu-

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[1] H. A. Bethe, Phys. Rev. **138**, B804 (1965).
[2] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963). This paper, and its authors, will be referred to as BBP.