$\alpha=0,\ \beta=\frac{1}{4}\pi$

$$\mathbf{s}=\frac{s}{\sqrt{2}}(\hat{x}+\hat{z})$$

$$\mathbf{u}=\frac{1}{\sqrt{2}}(\hat{x}-\hat{z})$$

$$M=\tfrac{1}{2}\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}$$

$$f_{20}{}^{10}=\tfrac{3}{4}(0.78)(NZ/A^2)R_0^2$$

$$f_{22}{}^{10}=\tfrac{1}{4}(0.78)(NZ/A^2)R_0^2$$

$$f_{2\mu}{}^{10}=0,\quad \mu\neq 0,\ 2$$

$\alpha=\frac{1}{4}\pi,\ \beta=\frac{1}{4}\pi$

$$\mathbf{s}=\tfrac{1}{2}s(\hat{x}+\hat{y}+\sqrt{2}\hat{z})$$

$$\mathbf{u}=\tfrac{1}{2}(\hat{x}+\hat{y}-\sqrt{2}\hat{z})$$

$$M=\begin{bmatrix}1 & 1 & -\sqrt{2}\\ 1 & 1 & -\sqrt{2}\\ \sqrt{2} & \sqrt{2} & -2\end{bmatrix}$$

$$f_{20}{}^{10}=\tfrac{3}{4}(0.78)(NZ/A^2)R_0^2$$

$$f_{2-2}{}^{10}=\tfrac{1}{4}(0.78)(NZ/A^2)R_0^2$$

$$f_{2\mu}{}^{10}=0,\quad \mu\neq 0,\quad -2.$$

PHYSICAL REVIEW VOLUME 151, NUMBER 3 18 NOVEMBER 1966

Convenient Analytic Form for the Deuteron Wave Function*

IAN J. MCGEE
Department of Physics, University of Wisconsin, Madison, Wisconsin
(Received 6 June 1966)

A convenient analytical form for the deuteron wave function is presented which reproduces the static properties of the deuteron, and is in accord with numerical wave functions which fit neutron-proton scattering data up to 300 MeV.

DEUTERON wave functions in numerical form have been obtained by several groups[1] from nucleon-nucleon potentials which reproduce the phenomenological phase parameter fits to the two-nucleon scattering data up to several hundred MeV. The object of this note is to present approximate expressions for one of these numerical wave functions which is convenient and accurate for most cases of interest.

The motivation for the analytic form used in this paper comes from several sources. However, it is sufficient for purposes of illustration to indicate the approach from the work of Bertocchi *et al.*[2] Their work shows that for neutron-proton potentials describable as a sum of Yukawa forms, the S and D states of the deuteron wave function can be written in coordinate space in the form:

$$u(r)=N\int_0^\infty \sigma_s(z)e^{-zr}dz,$$

$$w(r)=\rho N\int_0^\infty \sigma_d(z)e^{-zr}\left(1+\frac{3}{zr}+\frac{3}{(zr)^2}\right)dz; \qquad (1)$$

and in momentum space as

$$\tilde{u}(p)=N\int_0^\infty\frac{\sigma_s(z)dz}{p^2+z^2}=\frac{N}{p^2+\alpha^2}+N\int_{\alpha+\lambda}^\infty\frac{\hat{\sigma}_s(z)dz}{p^2+z^2},$$

$$-\tilde{w}(p)=\rho N\int_0^\infty\frac{\sigma_d(z)dz}{p^2+z^2}=\frac{\rho N}{p^2+\alpha^2}+\rho N\int_{\alpha+\lambda}^\infty\frac{\hat{\sigma}_d(z)dz}{p^2+z^2}, \qquad (2)$$

where

$$\sigma_i(z)=\delta(z-\alpha)+\hat{\sigma}_i(z)\theta(z-\alpha-\lambda),\quad i=s,d,$$

the function θ being the unit step function. Here λ is the minimum decay constant which appears in the potential, α is given by the deuteron binding energy ϵ, $\alpha=(m\epsilon)^{1/2}$, ρ is the asymptotic D to S ratio, and N is the wave-function normalization, given in terms of the deuteron effective range $\rho(-\epsilon,-\epsilon)$ by

$$N^2=2\alpha/[1-\alpha\rho(-\epsilon,-\epsilon)]. \qquad (3)$$

The weight functions $\sigma_i(z)$ in Eqs. (1) and (2) are subject to the subsidiary conditions[3]

$$\int_0^\infty\sigma_s(z)dz=0, \qquad (4)$$

$$\int_0^\infty\sigma_d(z)z^m dz=0,\quad m=-2,\ 0,\ 2. \qquad (5)$$

These sum rules are required to guarantee that the wave functions be finite at the origin and have the correct indicial behavior.

* Work supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation and in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, No. COO-881-69.

[1] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962); N. K. Glendenning and G. Kramer, Phys. Rev. **126**, 2159 (1962); S. Gartenhaus, *ibid* **100**, 900 (1956); K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, *ibid.* **126**, 881 (1962); H. Feshbach and E. L. Lomon, Ann. Phys. (N.Y.) **29**, 19 (1964).

[2] L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo Cimento **18**, 770 (1960). Although these authors only considered the S-state wave function, the formalism is easily extended to include all angular momentum wave functions. The author would like to thank Professor L. Durand, III, for calling his attention to this. See also Ref. 3.

In the present work we have approximated the weight functions $\sigma_i(z)$ in Eq. (2) by a series of delta functions, thus expressing the momentum space wave functions by a series of pole terms, and the coordinate space wave functions by a sum of exponentials or Hankel functions. The wave function can then be viewed as an extension of the familiar Hulthén wave function for the deuteron S state. Specifically, if the pole positions and their residues are denoted by ϵ_j, C_j, the S-and D-state wave functions in coordinate space have the form

$$u(r)=N(e^{-\alpha r}+\sum_{j=1}^{n} C_j e^{-\epsilon_j r}),$$
$$w(r)=\rho N(\alpha r h_2(i\alpha r)+\sum_{j=1}^{n} C_j' \epsilon_j' r h_2(i\epsilon_j' r)), \tag{6}$$

where $h_2(ix)$ is the spherical Hankel function defined by

$$xh_2(ix)=e^{-x}[1+3/x+3/x^2].$$

As noted in Eqs. (2) and (6), the asymptotic pole is fixed by the binding energy and normalization condition. The remaining poles and their residues were adjusted to fit the numerical wave function in the intermediate and short-range region and at the same time satisfy the subsidiary conditions. Several advantages are apparent with these wave functions as compared with analytical expressions given by previous authors.[4,5] The asymptotic form appears exactly; the higher order poles are simply approximations to the spectral weight function of dispersion theory.[6] The analytic expressions are simple, both in coordinate space and momentum space, hence one can do almost all the integrals encountered with these wave functions. Finally the analytic form is quite flexible since further refinement in the wave function is obtainable by merely adding more poles.

The numerical wave function which is obtained from the Hamada-Johnston potential[1] has recently been reevaluated by Partovi.[7] We have used this wave function to illustrate the method, the resulting parameters for the fitted analytic function being given in Table I. It was found that five poles (including the asymptotic pole) gave a reasonable fit for the S-state wave function, but at least six were required for a good fit to the D state.[8]

TABLE I. Fitted pole positions and residues of the S and D states for the analytic deuteron wave function, Eq. (6). $\alpha=0.2338$ F^{-1}, $N=0.8896$ $F^{-1/2}$, $\rho=0.0269$.

	S state		D state	
j	ϵ_j	C_j	ϵ_j'	C_j'
1	5.733α	−0.63608	4.833α	−20.34
2	12.844α	−6.6150	10.447α	−36.60
3	17.331α	15.2162	14.506α	−123.02
4	19.643α	−8.9651	16.868α	305.11
5			21.154α	−126.16

The procedure for obtaining these parameters was to fix the asymptotic pole and residue by the binding energy and normalization condition, and then fit the intermediate region at smaller and smaller values of r by successively adding more poles to the sum in Eq. (6). This was accomplished with a standard least-squares computer code.

Finally, the higher order poles corresponding to very small distances in coordinate space were determined by the sum rules, Eqs. (4) and (5). The hard core present in the Partovi wave function (and in most other numerical wave functions[1]) is smoothed over in this procedure to give a reasonable shape to the wave function in the region $0<r\lesssim 0.5$ F.[9] Results of this fitting procedure are sketched in Fig. 1 which compares the Partovi wave function and the analytic wave function using the parameters given in Table I. The often-used Hulthén wave function is also included for comparison but is seen to peak too low and tends to put too much of the wave function at small values of r. For $r\gtrsim 2.0$ F, the fitted wave functions are indistinguishable from the numerical wave functions, the actual difference for any value of r being less than 0.6% for the S-state and

[3] In general, the radial wave function for orbital angular momentum l is expressible as

$$u_l(r)=N_l\int_0^\infty \sigma_l(z) z r h_l(izr)dz=\sum_{s=0}^{l} a_{l,s} r^{-l+2s}\int_0^\infty \sigma_l(z) z^{-l+2s}dz + \sum_{s=0}^{\infty} b_{l,s} r^{l+1+s}\int_0^\infty \sigma_l(z) z^{l+1+s}dz$$

where

$$a_{l,s}=-(N_l/\sqrt{\pi})(i/2)^{2l-s}\Gamma(l-s+\tfrac{1}{2})/\Gamma(s+1)$$

and

$$b_{l,s}=N_l(\sqrt{\pi})2^{-s-l-1}i^{2s+l}/[\Gamma(1+\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+l+\tfrac{3}{2})].$$

The origin of the subsidiary conditions is now clear. In order that $u_l(r)$ be finite at the origin and behave initially as r^{l+1}, the first sum in the above relation must vanish; hence there must be $l+1$ sum rules of the type given in Eq. (5) with $m=-l, -l+2, \cdots, l$. For the nucleon-nucleon case, the tensor potential couples the S and D states leading to the occurrence of expressions like $r\ln r$ in $u(r)$, $r^3\ln r$ in $w(r)$, so that strictly speaking, neither $u(r)$ nor $w(r)$ is a pure power series in r. Thus the final sum rule for $w(r)$ need not hold (see Hulthén and Sugawara, Ref. 4, Secs. 28 and 33) although we retain it in the present work to improve the fit to the numerical wave function at small r.

[4] H. Kottler and K. L. Kowalski, Nucl. Phys. **53**, 334 (1964); M. J. Moravcsik, *ibid.* **7**, 113 (1958); G. Ernst and S. Flügge, Z. Physik **162**, 448 (1961); L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, Sec. 33.

[5] An analytical wave function quite similar in form to the present work has been developed by M. Gourdin, M. Le Bellac, F. M. Renard, and J. Tran Thanh Van, Nuovo Cimento **37**, 524 (1965). They consider four poles for both the S and D states and adjusted the free parameters to fit low-energy deuteron photodisintegration data.

[6] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

[7] F. Partovi, Ann. Phys. (N.Y.) **27**, 79 (1964).

[8] In order to satisfy the subsidiary conditions Eqs. (4) and (5), it is evident that a minimum of two poles is required for the S state (the Hulthén wave function is an example), and four poles are found necessary for the D state. This minimum requirement however leaves little flexibility for adjusting the shape of the wave functions. Our analysis shows therefore that at least two more poles than the minimum are required to improve significantly the shape of the wave function.

[9] The smooth shape in the region $0<r\lesssim 0.5$ F was obtained for the S state by requiring that the parameters in Eq. (6) satisfy three sum rules instead of one. [Thus its indicial behavior was artificially set at $u(r)\sim r^3$.]

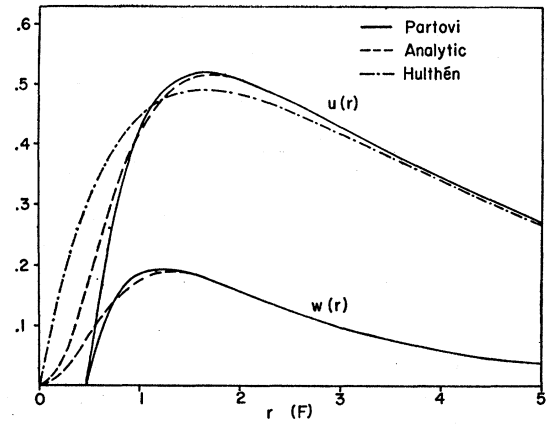

FIG. 1. S and D states of the deuteron for the numerical wave function obtained from the Hamada-Johnston potential and the analytic wave function with the parameters of Table I. Wave functions are normalized as $\int_0^\infty [u^2(r)+w^2(r)]dr=1$. The parameters for the Hulthén S-state wave function are $N^2=0.765$ F^{-1}, $\beta=1.26$ F, corresponding to a normalization $\int_0^\infty u_H^2(r)dr=0.93$.

0.7% for the D state. This error is largest for $r\lesssim 2.5$ F and decreases very rapidly as the wave functions approach their asymptotic values. Near the origin the abrupt character of the hard-core wave functions is replaced by the smooth behavior of the analytic wave functions.[9] However there is a question of how well one ought to fit the core region since its form reflects our inability to specify the wave function in this region from experimental data. Only in very high energy scattering could one hope to detect differences in the two types of core behavior. The important feature of the present analytic wave function is that it does very well in the region where all numerical wave functions agree on the basis of scattering data up to $\sim$300 MeV. Furthermore, refinement in the fit, without loss in simplicity of form, can easily be obtained by adding more poles and sum rules to Eq. (6).

The parameters given in Table I give a deuteron effective range $\rho(-\epsilon, -\epsilon)=1.749$ F, ($N=0.8896$ F$^{-1/2}$), a quadrupole moment, $Q=0.282$ F^2, an asymptotic D to S ratio, $\rho=0.0269$, and a percentage D state of 7%, all in agreement with the experimental data.[10]

The author is indebted to Professor L. Durand, III for calling attention to this problem, and to Dr. B. M. Casper for supplying several numerical wave functions.

[10] M. J. Moravcsik, *The Two Nucleon Interaction* (Clarendon Press, Oxford, England, 1963).

PHYSICAL REVIEW VOLUME 151, NUMBER 3 18 NOVEMBER 1966

Comprehensive Formalism for Nuclear Reaction Problems. I. Derivation of Existing Reaction Theories*

A. M. LANE
Atomic Energy Research Establishment, Harwell, Berkshire, England

AND

D. ROBSON
Department of Physics, Florida State University, Tallahassee, Florida

(Received 24 March 1966)

An attempt is made to reduce the confusion arising from the existence of many different theories of resonance reactions. The method is to set up a formalism which is sufficiently general so that individual theories may be obtained by appropriate specialization. Such a formalism is obtained by suitably generalizing that devised by Bloch for a smaller range of theories. The formalism facilitates comparison of the structure of the various theories. Also, as we shall show in a subsequent paper, it gives a neat and systematic framework for study of the line-broadening problem.

I. INTRODUCTION

THERE now exist a multitude of theories of nuclear reactions, expressed in different formalisms and designed for various objectives. In this situation, it is useful to have a formal framework such that any particular theory can be derived by appropriate specialization. This facilitates comparison of different theories, and creates order out of chaos. Such a framework was given by Bloch[1] several years ago, and he showed how it could be specialized to the theories then available. The number of theories has since doubled, and it is again desirable to systematize the field in this manner. It turns out that the method

* Supported in part by the U. S. Air Force Office of Scientific Research.

[1] C. Bloch, Nucl. Phys. 4, 503 (1957).