of H_0 whereas this no longer remains true if $\omega_c\tau \gg 1$. Cyclotron effects alter the diffusion of electrons into the metal which manifests itself in a more complex diffusion constant. For the extreme cases of parallel and perpendicular fields we find that $D_\perp$ is equal to D_0 and $D_{\parallel}$ equals $D_0/(1+\omega_c^2\tau^2)$ at resonance. It was also pointed out that these results will affect the transmitted magnetization in a way that agrees with the conclusions in Ref. (2). The effective Bloch equation is valid for all orientations of the static field and should be helpful in a semiquantitative understanding of recent experiments in conduction-electron spin resonance.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge conversations with Professor T. G. Castner which stimulated this work and to thank him for many helpful discussions. I would also like to thank Professor R. S. Knox for valuable suggestions.

PHYSICAL REVIEW VOLUME 151, NUMBER 1 4 NOVEMBER 1966

Magnetic Irreversible Solution of the Ginzburg-Landau Equations*

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(Received 10 June 1966)

We present the exact magnetic irreversible solution of the Ginzburg-Landau equations for a cylinder of infinite length (whose Ginzburg-Landau parameter κ is unity and whose radius R is three coherence lengths ξ) in an axial magnetic field H_0 for all values of H_0. Solutions for other values of κ (0.3 to 3) and R/ξ (2 to 12) are also discussed. We have determined, as a function of H_0 and as a function of position, the order parameter, the vector potential, the internal magnetic field, and the current density; and also as a function of H_0, the total number of superconducting electrons per unit length of the cylinder and the magnetization per unit volume. This solution is magnetically irreversible and hysteretic because of persistent currents which flow in the sample perpendicular to the applied magnetic field. The magnetization is reversible only over intervals of H_0 over which the number of fluxoids is conserved; otherwise it is irreversible. This solution does not depend on defects and is the counterpart to Abrikosov's magnetic reversible mixed-state solution. It is dominant in thin specimens.

I. INTRODUCTION

It was predicted by Abrikosov[1] that the magnetization per unit volume $4\pi M$ of a type-II superconductor of infinite extent is magnetic reversible when in the mixed state. $4\pi M$ depends only on the value of the applied magnetic field H_0 and not on the previous history of the sample. Experimentally, however, a certain degree of irreversibility is always found, and it is large for Ginzburg-Landau[2] κ values of order unity[3] and small when $\kappa \gg 1$. Thin samples appear to be always irreversible[4] regardless of the quality of the sample preparation. For magnetic fields H_0 larger than the bulk critical field H_{c2} the surface remains superconducting up to[5] H_{c3}. The superconducting surface is able to carry persistent circulating currents[6] around the circumference of the sample. The direction of circulation of these currents depends on the direction of the change of the external magnetic field.[7] A long macroscopic cylinder, therefore, appears as a "giant vortex" ($H_{c2} \lesssim H_0 \leq H_{c3}$) whose physical size is determined by the sample dimensions. Such a surface state in the form of a giant vortex exists also for $H_0 < H_{c2}$ as has been shown experimentally[8] just below H_{c2}. This state is quite different from the mixed state since it can carry a total current whereas the ideal mixed state (without pinning centers) cannot.[9,10] Over a finite interval of the external magnetic field this current can be changed such that it conserves the number of fluxoids enclosed in the sample.[6–8,11] When a total current is flowing in a macroscopic specimen in order to conserve the number

* Work supported by Division of Research, Metallurgy and Materials Programs, U. S. Atomic Energy Commission, Contract No. AT-(11-1)-GEN-8.

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of fluxoids b, the sample is not in the lowest possible energy state. From experiments[7] it appears that the conservation of fluxoids may override the lowest energy condition over a finite interval of the external field, namely as long as $G_{SH}-G_{NH}\leq 0$ is satisfied, where the G's are the magnetic Gibbs function of the superconducting and normal state, respectively. When the applied magnetic field exceeds this magnetic reversible interval, the number of enclosed fluxoids is changed and hysteresis occurs.[7,8,11]

From the above arguments we are led to believe that apart from the Meissner and mixed state another state exists which is related to the current-carrying surface state[6,7] above H_{c2} and also to the Meissner state (which is a special case of the current-carrying surface state) for which the number of enclosed fluxoids in the sample is zero ($b=0$). This surface state is magnetic irreversible (except over certain magnetic field intervals) whereas Abrikosov's mixed state is reversible. We believe that in general the surface state and the mixed state coexist simultaneously and that the surface state is of importance for thin specimen whereas the mixed state is of importance only for bulk specimen. In Sec. II we shall show from energy considerations that a surface-state solution is favored for $H_0<H_{c2}$ near H_{c2} for small samples. In Sec. III the exact giant-vortex solution is obtained for a cylinder of infinite length, radius $R=3\xi$, and $\kappa=1$. In Sec. IV we shall discuss the general behavior of the giant vortex with other values of R/ξ and κ. Section V is devoted to the conclusions.

II. COMPARISON OF THE MIXED STATE AND THE GIANT-VORTEX STATE

The energy of the mixed state is uniquely determined by the applied magnetic field. The magnetization per unit volume is finite below H_{c2} and is reversible as a function of applied field. It is determined by the condition of the lowest energy. In contrast to the mixed state, the giant vortex state is determined by the condition that the number of fluxoids enclosed in the specimen is conserved as long as the Gibbs free energy of the superconductor is smaller or equal to the Gibbs free energy of the normal state. This state is able to carry a total current which is a function of the previous magnetic history of the sample. When the specimen assumes the lowest energy for a given number of enclosed fluxoids, then the magnetization per unit volume $4\pi M$ is zero. It is known from experiments that this state can be achieved[7,11] for $H_0>H_{c2}$, and maybe also for small samples for other magnetic fields.

We compare the lowest energy of a cylinder in the mixed state (without a surface sheath) to that of the giant vortex state (without mixed state). We show that the lowest energy of the giant vortex state is under certain conditions lower than that of the mixed state for a finite range of magnetic fields smaller than H_{c2}.

The z axis is assumed parallel to the magnetic field and also parallel to the axis of the cylinder. We consider the Ginzburg-Landau equations[2] in the following normalization which is different from that used in Ref. 12. The order parameter is normalized with respect to its zero magnetic field value, has no z dependence as the cylinder is assumed to be of infinite length along the z direction. Its amplitude has no angular (θ) dependence which eliminates any modulation of the absolute value of the order parameter along $\boldsymbol{\theta}$ and therefore any structure similar to that of the mixed state, and it is assumed to be of the form

$$\Psi(r,\theta)=F(r)e^{ib\theta}, \tag{1}$$

where $r=\rho/R$, ρ is the distance from the symmetry axis of the cylinder and R is the radius of the cylinder. The constant b must be an integer in order that Ψ is single valued and it is equal to the number of the fluxoids enclosed in the cylinder. Further definitions are: The Ginzburg-Landau[2] κ value is: $\kappa=\lambda/\xi$, where λ is the low field penetration depth[2] and ξ the coherence length; $\chi=R/\xi$; $h=H/H_c$, where H is the internal magnetic field parallel to z direction and H_c is the thermodynamic critical field; and $h_0=H_0/H_c$. Further, the general form of the vector potential $\mathbf{A}\equiv(0;A_\theta;0)$ is assumed:

$$\frac{A_\theta}{H_c\lambda}=A_n=\frac{\chi}{2\kappa}\left[h_0r+\frac{\varphi(r)}{r}\right], \tag{2}$$

where the function $\varphi(r)$ is to be determined. From $\mathbf{H}=\operatorname{curl}\mathbf{A}$ it follows that

$$h=h_0+\frac{1}{2r}\frac{d\varphi}{dr}, \tag{3}$$

and therefore it follows from Maxwell's first equation

$$-\frac{4\pi}{c}\frac{\chi}{\kappa}j_n=\frac{1}{2}\frac{d}{dr}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)=\frac{dh}{dr}, \tag{4}$$

where $j_n=j(r)\lambda/H_c$. The magnetization per unit volume $4\pi M$ is defined by

$$\frac{4\pi M}{H_c}=2\int_0^1 r(h-h_0)dr=\varphi(1), \tag{5}$$

where $\varphi(0)=0$ was used which follows from the usual definition of the flux

$$\Phi=\iint\mathbf{H}\cdot d\mathbf{S}=\oint\mathbf{A}\cdot d\mathbf{l}. \tag{6}$$

With the above definitions the first and the second Ginzburg-Landau equations and the difference of the

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total Gibbs free energy between the normal and superconducting state is

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{dF(r)}{dr}\right\}=\left(\frac{\chi^2}{2\sqrt{2}\kappa}\right)^2\left[h_0r+\frac{1}{r}\left\{\varphi(r)-\frac{2\sqrt{2}\kappa}{\chi^2}b\right\}\right]^2F(r)$$
$$-\chi^2F(r)[1-F^2(r)], \quad (7)$$

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d\varphi(r)}{dr}\right\}=\left(\frac{\chi}{\kappa}\right)^2\left[h_0r+\frac{1}{r}\left\{\varphi(r)-\frac{2\sqrt{2}\kappa}{\chi^2}b\right\}\right]F^2(r), \quad (8)$$

$$\Delta g=\frac{G_{SH}-G_{NH}}{H_c^2V/2\pi}=\tfrac{1}{2}\int_0^1\{(h-h_0)^2-F^4\}rdr. \quad (9)$$

In Eq. (9), $F(r)$ and $h(r)$ are two independent variable functions of position and V is the volume. Assume that b and κ are held fixed, then Eq. (9) is a minimum with respect to the parameter h_0 (which is the normalized external field) when $(\partial\Delta G/\partial h_0)_{b,\kappa}=0$, where $\Delta G=G_{\mathrm{SH}}-G_{\mathrm{NH}}$. With this condition it follows from Eqs. (9) and (5), when the number of enclosed fluxoids in the specimen is held constant, that the lowest energy occurs when $4\pi M=0$. Hence the giant vortex state in its lowest energy, for a fixed number of fluxoids (fixed phase factor b of Ψ), has zero magnetization per unit volume though not necessarily zero total current [which follows by integrating Eq. (4)] unless $\chi\to\infty$. The total current is proportional to $h(0)-h_0$.

With the above equations and definitions we are able to compare the energies of the mixed state with the lowest energy of the giant vortex state. Though $\int_0^1 r(h-h_0)dr=0$ for the lowest energy of the giant vortex state for a fixed value of b and κ, the corresponding integral $\int_0^1 r(h-h_0)^2dr$ for the lowest energy (for the same values of b and κ) is in general not zero but small compared to $\int_0^1 rF^4dr$. Hence Eq. (9) may be written approximately

$$(\Delta G)_{\min}\approx-\frac{H_c^2V}{4\pi}\int_0^1 rF^4dr \quad (10)$$

$$\approx-\frac{H_c^2V}{4\pi}\int_0^1 F^4dr \quad \text{for } \chi\gg1, \quad (10a)$$

because $F(r)$ is only nonzero near the surface of the cylinder, namely at $r\approx1$. The thickness of the superconducting surface state Δ is defined by $(\chi\gg1)$

$$\frac{\Delta}{R}\approx\frac{1}{F^2(1)}\int_0^1 F^2(r)dr. \quad (11)$$

Further, we define an α value for $\chi\gg1$

$$\alpha=\int_0^1 F^4dr\bigg/\left(\int_0^1 F^2dr\right)^2 \quad (12)$$

which is of order unity. With these definitions one obtains

$$(\Delta G)_{\min}=-\frac{H_c^2V}{4\pi}\frac{\alpha}{\chi^2}\left\{\frac{\Delta}{\xi}F^2(1)\right\}^2. \quad (13)$$

Assume that we replace this giant vortex state completely by the mixed state and disregard any boundary effects, then it follows from Abrikosov's theory[1] that ΔG of the mixed state for $(H_{c2}-H_0)/H_{c2}\ll1$ is

$$(\Delta G)_A=-\frac{H_c^2V}{4\pi}\frac{\kappa^2}{(2\kappa^2-1)\beta}\left(1-\frac{H_0}{H_{c2}}\right)^2, \quad (14)$$

where Eq. (14) applies to applied magnetic fields $H_0<H_{c2}$ and[13] $\beta=1.16$.

We equate Eq. (14) to Eq. (13) and find the value of H_0 $(\equiv H_{0s})$ for which the energies are equal.

$$(\alpha\beta)^{1/2}\left\{\frac{\Delta}{\xi}F^2(1)\right\}=\frac{\kappa\chi}{(2\kappa^2-1)^{1/2}}\left(1-\frac{H_{0s}}{H_{c2}}\right), \quad (15)$$

where $(\Delta/\xi)F^2(1)$ is a function of H_0/H_{c2} and may be obtained indirectly from Ref. 12 or directly from Ref. 7. For a given κ value H_{0s} as a function of χ is obtained near H_{c2} with the following approximation of the left-hand side of Eq. (15)

$$\frac{\Delta}{\xi}F^2(1)\bigg|_{H_0/H_{c2}=1}+\left(1-\frac{H_0}{H_{c2}}\right)\left|\frac{d[(\Delta/\xi)F^2(1)]}{d(H_0/H_{c2})}\right|_{H_0/H_{c2}=1},$$

where we have assumed that $(\alpha\beta)^{1/2}=1$. With the numerical[7] values of the function $(\Delta/\xi)F^2(1)$, with $\chi=10$, and κ values between ∞ and 1.5 one obtains for $H_{0s}/H_{c2}\approx0.86$, and for $\kappa=1$ and $\chi=10$ one obtains $H_{0s}/H_{c2}\approx0.84$. For smaller κ values this ratio decreases. For $\chi\to\infty$, $H_{0s}/H_{c2}\to1$ for all appropriate κ values. For $H_0<H_{0s}$ the mixed state will probably be favored and for $H_0>H_{0s}$ the giant vortex state will probably be favored from the energetic point of view. As the mixed state is reversible and the giant vortex state magnetic irreversible, which means that the conservation of fluxoids may override the lowest energy state, it is likely that the giant vortex state is favored even for certain magnetic fields smaller than H_{0s}. Hence we may conclude that the giant vortex state will be more energetically favorable than the mixed state for certain magnetic field ranges below H_{c2} for cylinders which are "small," namely when the diameter is about 100ξ or smaller.

III. THE EXACT SOLUTION OF A GIANT VORTEX

We have obtained the exact solutions of the independent variable functions $F(r)$ and $\varphi(r)$ from Eqs. (7) and (8) and have also calculated the integral on the right-hand side of Eq. (9) [total energy per unit

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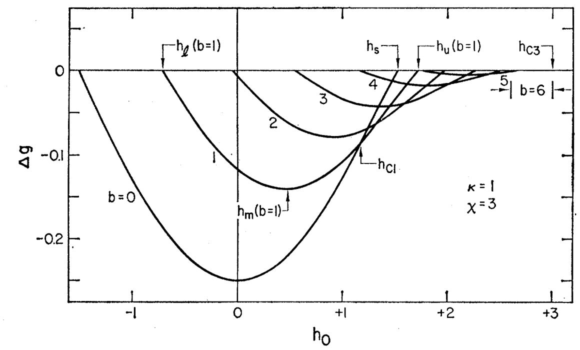

FIG. 1. The difference between the normalized magnetic Gibbs functions of the superconducting and normal state [Eq. (9)] is shown as a function of the normalized applied field $h_0(=H_0/H_c)$ for $\kappa(=\lambda/\xi)=1$, $\chi(=R/\xi)=3$, and for all possible numbers of enclosed fluxoids b. For $\Delta g(b=6)$ only the magnetic field interval is indicated as Δg is too small to be significant different from zero on the above scale. For details see text.

volume] as a function of h_0 for a given set of parameters κ, χ, and b. For a given set of parameters κ and χ we have varied the number of fluxoids from $b=0$ to $b=b_{\max}$ and obtained solutions for each value of the parameter b as a function of h_0.

Once $F(r)$ and $\varphi(r)$ for (κ,χ,b) are known the vector potential [Eq. (2)], the internal field [Eq. (3)], and the current density [Eq. (4)] are readily calculated. Also the magnetization per unit volume is readily available from $\varphi(1)$ [Eq. (5)], the total energy per unit volume from Eq. (9) and the value $h(0)$ which is proportional to the total persistent current flowing around the sample [see Eqs. (3) and (4)]. Other values such as $\int_0^1 rF^2dr$, $\int_0^1 rF^4dr$, etc., have also been calculated.

We have employed an analog computer for the above problem. Because of the two point boundary conditions at $r=0$ and $r=1$, an approximate finite difference technique was used for the computation of the nonlinear, coupled differential equations (7) and (8). The radial distance was divided into 10 equal segments and the average value of the functions was computed for each interval. For convenience on the analog computer the finite difference equations were computed in the form of time-dependent differential equations. This group of differential equations was programmed such that steady-state conditions could be obtained in a very short period of time on the computer. Continuous steady-state calculations were obtained for a linear variation of the parameter h_0 (normalized applied field). The rate of change of h_0 was adjusted to a value compatible with the steady-state calculations of the computer. The total energy, the magnetization per unit volume, $F(1)$, $\int_0^1 rF^2dr$ and others were plotted directly and simultaneously on x-y recorders as a function of h_0, while $F(r)$, $\varphi(r)$, and the current density were plotted at the intervals $r=0, r=0.1, r=0.2 \cdots r=1$ on multipen recorders as a function of time which was correlated to the sweep rate of h_0. Similarly $h(r)-h_0$ was plotted on a multipen recorder at $r=0.5$, $r=0.15$, $r=0.25$, $\cdots$ $r=0.95$.

The boundary conditions which were forced on the computer are

at $r=1$

$$dF/dr=0 \tag{16}$$

and

$$d\varphi/dr=0; \tag{17}$$

and at $r=0$

$$\varphi=0, \tag{18}$$

$$F=0 \quad \text{for} \quad b\neq 0, \tag{19}$$

and

$$dF/dr=0 \quad \text{for} \quad b=0. \tag{19a}$$

Equation (16) is the Ginzburg-Landau boundary condition[2] for a superconductor-vacuum interface; Eq. (17) follows from Eq. (3) [$h(1)=h_0$]; Eq. (18) follows from Eqs. (6) and (2), and Eqs. (19) and (19a) follow from the condition that Eq. (7) can be satisfied near $r=0$ with $F(r)\approx cr^{|b|}$, where b is the number of enclosed fluxoids in the cylinder and c is a constant, and also from the symmetry of the above problem. Because $h(r)$ must be an even function with respect to r, it follows from Eq. (3) that $\varphi(r)$ must be also even. As $\varphi(0)=0$, the term $\varphi(r)/r$ in Eqs. (2), (7), and (8) must approach 0 for $r\to 0$. For $b=0$ and r close to 0, $F(r)\approx c[1-(\chi/2)^2(1-c^2)r^2]$.

With the boundary conditions Eqs. (16) to (19a) all the other boundary conditions were automatically satisfied in the computation. These are for example: $j(0)=0$; $(d\varphi/dr)_{r=0}=0$; $F(1)\neq 0$; $F(0)\neq 0$ for $b=0$; $\{(1/r)(d\varphi/dr)\}_{r=0}\neq 0$; $j(1)\neq 0$; $\varphi(1)\neq 0$; the latter three boundary conditions are nonzero in general but can be zero for particular combinations of the parameters h_0, χ, κ, and b.

For our problem the Ginzburg-Landau Equations written in the form of Eqs. (7) and (8) have a high degree of symmetry. When one inverts the sign of the number of fluxoids $(b\to -b)$ then automatically $h_0\to -h_0$ and $\varphi(r)\to -\varphi(r)$. This means from Eqs. (8) and (4) that when the applied magnetic field is inverted in direction so is the current density and the phase of the order parameter Ψ. The r-dependent part of the order parameter $F(r)$ remains, however, unaffected by this sign inversion as can be seen readily from Eq. (7). From this it follows that $\Delta g(b)$—[Eq. (9)] has reflection symmetry when plotted as a function of h_0 with respect to the axis at $h_0=0$ when $b\to -b$ (which was verified by the computer solutions).

Though the computation was straightforward it was very complex. This required great accuracy of some of the potentiometer settings (one to two parts in 10^4) in order to obtain reliable results. Because of the complexity of this problem the over-all accuracy is probably not better than $\pm 2\%$. For an example we have chosen the parameters $\kappa=1$, $\chi=3$, and $b=0$ to

$b=b_{\max}=6$. Solutions of Eqs. (7) and (8) with other parameters (κ,X,b) will be discussed in Sec. IV.

Figure 1 shows the difference of the magnetic Gibbs functions between the superconducting and the normal state, normalized with respect to $H_c^2V/2\pi$ [Δg of Eq. (9)] as a function of the normalized applied magnetic field h_0. By definition the Meissner state is that state for which the number of enclosed fluxoids $b=0$. For this state Δg is symmetric with respect to h_0. For $b>0$ the energy curves are not symmetric with respect to their minima. Equation (9) has no solutions for $\Delta g>0$ with the above boundary conditions, and $\Delta g=0$ below a certain minimum value of $h_0=h_l(b)$ and above a certain maximum value of $h_0=h_u(b)$ as long as b is held constant. Within this interval of h_0, (h_u-h_l), the functions $F(r)$ and $\varphi(r)$ are finite but drop quickly to 0 for $h_0<h_l$ and for $h_0>h_u$. This will become more apparent from the figures and discussions below.

The intersection of the energy curves for $b=0$ and $b=1$ is defined as the lower critical field $h_{c1}(\equiv H_{c1}/H_c)$. This definition of h_{c1} is straightforward for $\kappa\gtrsim 1$ but it is somewhat more complex for smaller κ values and will be discussed in more detail in the next section. At this field (h_{c1}) it becomes energetically favorable for the superconductor to change the number of enclosed fluxoids from zero to one. It is often observed that "flux" is delayed[14,15] from entering the Meissner state beyond h_{c1}, which means that the conservation of the fluxoid (in this case $b=0$) is able to override the lowest energy condition. If the fluxoid $b=0$ is conserved beyond h_{c1}, the maximum applied magnetic field to which the state $b=0$ can be delayed from changing to $b>0$ is h_s $(\equiv H_s/H_c)$, at which magnetic field b must change because otherwise $\Delta g=0$ for $h_0>h_s$, and the sample would be driven into the normal state [$F(r)=\varphi(r)=0$ for $b=0$] which is not observed in bulk specimen when $\kappa>0.417$. When $X\rightarrow\infty$, the value of h_s approaches unity.[16] The size dependence of h_s will be discussed in the next section. When the number of fluxoids in the sample changes, then not only the absolute value of the order parameter Ψ and the vector potential $\mathbf{A}$ changes but also the phase of Ψ changes by $2\pi\Delta b$, where Δb is the number of fluxoids which enter the sample. Beyond a maximum value of $b=b_{\max}$, which in our case is 6, there exist no solutions of $F(r)$ and $\varphi(r)$ with the above boundary conditions and every point of the metal is driven into the normal state. The maximum magnetic field at which this happens for $b_{\max}$ is defined as h_{c3} which is related to the surface nucleation field.[5,17]

When the condition for the conservation of the number of fluxoids is stronger than the lowest energy condition, which is actually observed in a bulk specimen[7,8,11] above the upper bulk critical field H_{c2}, then for a small specimen as in Fig. 1 (which we assume was cooled in zero magnetic field through the transition temperature) a continuously increasing field h_0 will bring the sample along the $\Delta g(b=0)$ curve to h_s. Then a certain number of fluxoids will enter, one, two, three or four in our case (it is not clear how many at this point) and the process will be repeated over again on a different $\Delta g(b)$ curve when $h_u(b)$ is reached, and so on until h_{c3} is reached, at which field superconductivity in the sample is quenched everywhere. For continuously decreasing fields from h_{c3} to $h_0=0$ we can use similar arguments, namely that the number of fluxoids changes only at $h_l(b)$, and find that at $h_0=0$, two, one, or zero fluxoids will remain in the cylinder. If $b>0$ for $h_0=0$ we speak of "flux-locking". It is not known at present if the transition probability from the state $b=m$ to the state $b=n$ permits values of $|m-n|>1$ or if only values of unity are permitted for ideal type II-superconductors. Assuming that only $|m-n|=1$ is permitted, two fluxoids would be "locked-in" when h_0 is decreased from h_{c3} to zero. This corresponds to a large paramagnetic moment as will become obvious (below) from the magnetization per unit volume. When b is conserved for a change in the applied magnetic field, the magnetic properties of the sample are reversible. When the values of b are changed, for example by increasing h_0 from 0 to h_{c3} and then decreasing back to 0, the magnetic properties of our solution will be irreversible over the whole magnetic field range. This is to be compared to Abrikosov's solution of the ideal mixed state[1] which is magnetic reversible and which is, however, not always found in nature, in particular for small specimen and κ values of order unity. The above irreversible solutions have a certain similarity to the solutions of thin film (thickness $\ll\xi$) hollow cylinders,[18] but in our case we are discussing *solid* cylinders with $R>\xi$.

Figure 2 shows the dependence of the absolute value of $|\Psi|=F(r)$ on r for $\kappa=1$ and $X=3$ for the case $b=0$ and $b=3$. The curves labeled by M are computed at the value of the applied field $h_0=h_m(b)$ at which $\Delta g(b)$ is a minimum, the curves labeled by L are computed at the field $h_l(b)$, and those labeled by U are calculated at $h_u(b)$. For $b=0$ the values of $F(0)>F(1)$ for all values of h_0 except at zero field when $F(r)$ is unity. This behavior is expected for the Meissner state for any κ value except that for bulk samples the order parameter varies only near the surface and is constant over most of the sample, but this is only a matter of scaling. Figure 2(b) shows $F(r)$ when the number of enclosed fluxoids is three. For $h_0>h_u(b)$ and $h_0<h_l(b)$ the $F(r)$ functions become zero.

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[18] D. H. Douglass, Phys. Rev. **132**, 513 (1963).

At the minima of $\Delta g(b)$ the corresponding absolute values of Ψ are larger for all values of r than $|\Psi|$ at h_l or h_u. The shape of $|\Psi|$ at the magnetic field $h_u(b)$ is different from that at $h_l(b)$ due to the different direction of circulation of the internal currents with respect to the magnetic field. When the *total* current flows in a clockwise direction as to generate an average internal field which oppose the applied field such as at h_u, the order parameter is depressed at the surface, but reaches a maximum near the surface[19] as shown in Fig. 2(b). It should be noted that near $r=0$ the function $F(r)$ is proportional to $r^{|b|}$ as mentioned above. For $b=1$ there is no horizontal tangent at $r=0$, but for $b>1$ there is always a horizontal tangent at $r=0$, and $F(r)$ is depressed over a larger volume fraction near the center of the cylinder the larger the value of b is. This means that near $r=0$ superconductivity is effectively

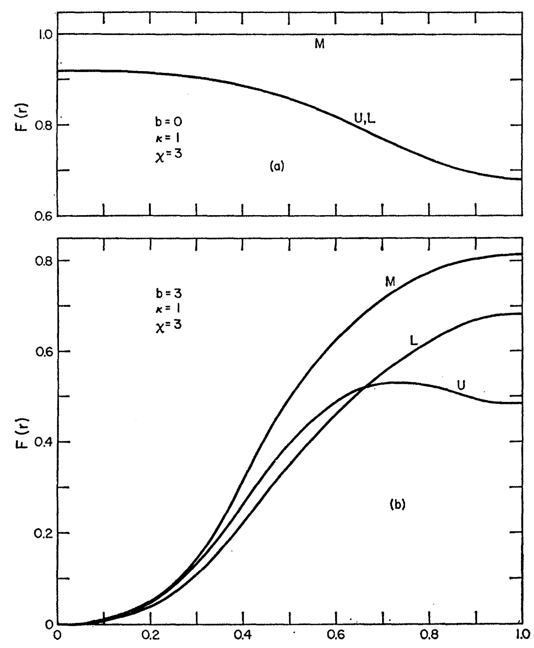

FIG. 2. The position dependence of the absolute value of the order parameter $F(r)$ at the applied magnetic fields $h_l(b)$ [$\equiv L$], $h_m(b)$ [$\equiv M$], and $h_u(b)$ [$\equiv U$] as shown in Fig. 1 for: (a) $b=0$, $\kappa=1$, and $\chi=3$; (b) $b=3$, $\kappa=1$, and $\chi=3$.

quenched over a larger fraction of the volume of the cylinder the larger b is, and for $b_{\max}$ superconductivity remains only very close to the surface.

The values of $F(1)$ for $\kappa=1$, $\chi=3$ are shown in Fig. 3 for all possible b values. They are asymmetric for $b>0$ as a function of h_0 with respect to their maximum value which does not exactly occur at that magnetic field h_0 at which $\Delta g(b)$ is a minimum. As the values of b are increased the $F(1)$ values decreases in general, but if h_0 is increased beyond h_u and b is held fixed, $F(1)$ collapses to zero.

[19] J. G. Park, Phys. Rev. Letters 15, 352 (1965).

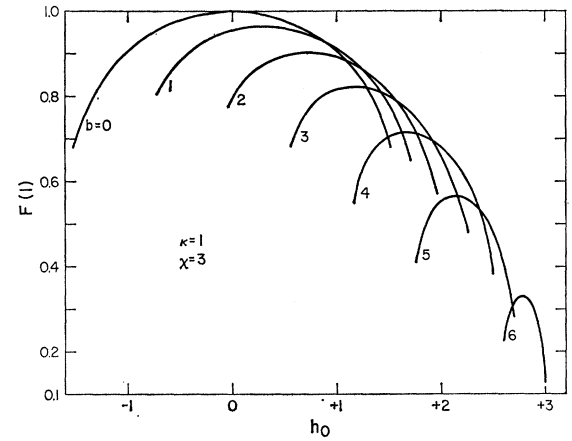

FIG. 3. The dependence of the absolute value of the order parameter at the surface of the cylinder ($r=1$) on the applied magnetic field h_0 for $\kappa=1$, $\chi=3$, and all possible values of the number of enclosed fluxoids b in the cylinder.

Figure 4 shows the radial dependence of part of the vector potential which is defined as the function $\varphi(r)$ in Eq. (2). Figure 4(a) shows $\varphi(r)$ for $b=0$, $\kappa=1$ and $\chi=3$ for magnetic fields $h_0=0$, $h_l(0)$ and $h_u(0)$ and Fig. 4(b) for $b=3$, $\kappa=1$, $\chi=3$ for the magnetic fields $h_l(3)$, $h_u(3)$ and for that value of h_0 for which $\Delta g(3)$ is a minimum [$h_m(3)$]. The spatial variation of $\varphi(r)$ near $r=0$ and $r=1$ is small as $\varphi(r)$ has horizontal tangents at these points. In particular, the spatial variation of the term $\varphi(r)/r$ of the vector potential in Eq. (2) does not vary very strongly near the surface. For $b=0$ the function $\varphi(r)$ at the magnetic fields $h_l(0)$ and $h_u(0)$ are the same except for the inversion of the sign, and $\varphi(r)=0$ at $h_0=0$. For $b>0$ the curves at $h_l(b)$ and $h_u(b)$ have no longer reflection symmetry with respect to the $\varphi(r)=0$ axis and the function $\varphi(r)$

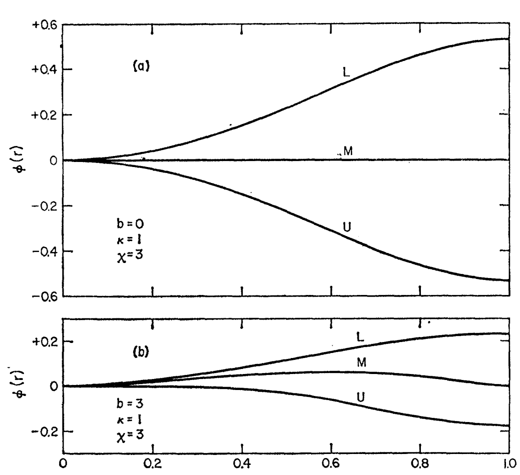

FIG. 4. The position dependence of the function $\varphi(r)$, which is part of the vector potential [see Eq. (2)], at the applied magnetic fields $h_l(b)$ [$\equiv L$], $h_m(b)$ [$\equiv M$], and $h_u(b)$ [$\equiv U$] as shown in Fig. 1 for: (a) $b=0$, $\kappa=1$, and $\chi=3$; (b) $b=3$, $\kappa=1$, and $\chi=3$.

at the magnetic field $h_m(b)$ is only zero at $r=0$ and $r=1$ but positive for all other values of r.

Figure 5 shows the values of $\varphi(1)$ for various values of b as a function of h_0. When $h_0=h_m(b)$ such that $\Delta g(b)$ is a minimum, then $\varphi(1)=0$ for this particular value of b as was pointed out above. This means that at $h_m(b)$ the magnetization per unit volume $4\pi M=0$ [see Eq. (5)]. Figure 5, therefore, represents direct measurable values of the magnetization per unit volume normalized with respect to H_c. The slope of the Meissner state $|4\pi dM/dH_0|$ is smaller than unity as the sample is small and the applied magnetic field penetrates appreciably into the specimen. If we follow the lowest energy envelope in Fig. 1 while h_0 is changed, then $\varphi(1)$ would follow the dot-dashed and solid lines in Fig. 5 which would be reversible. However, if the number of fluxoids is conserved, then $\varphi(1)$ would follow the dashed and solid lines and $\varphi(1)$ would be irreversible for increasing and decreasing magnetic fields. With the dashed lines as shown, the assumption was made that the number of fluxoids changes in steps of unity when $\Delta g(b)$ approaches zero. When h_0 is reversed from h_{c3} to zero two fluxoids remain "locked-in" in the cylinder. When h_0 is increased (we assume that the sample was cooled through T_c in zero field) and then reversed at h_x, for example, to $h_0=0$ the magnetization per unit volume would follow the $b=1$ curve and only one fluxoid would remain "locked-in" in zero field. When $b=b_{\max}$ the change of $4\pi M$ at h_{c3} appears to be of first order though $4\pi M$ is very small and becomes even smaller when $\chi>3$, so that one may speak of a second order transition at h_{c3} for all practical purposes when $\chi\gg1$. When, however, $\chi<3$ the assumption of a second order transition may be challenged. The reader should note the close similarity between the experimentally observed[7,11] hysteresis loops of a bulk type-II superconductor above the bulk critical field h_{c2} and Fig. 5. The curves in Fig. 5 are magnetic reversible as long as b is not changed. For a given value of h_0 the absolute value of the averaged diamagnetic envelope is always larger than the averaged paramagnetic envelope for the magnetic irreversible case.

Figure 6(a) shows the variation of the internal magnetic field h from the external magnetic field h_0, namely $h-h_0=(1/2r)d\varphi(r)/dr$ [Eq. (3)] as a function of distance for $b=0$, $\kappa=1$, and $\chi=3$. Near the center of the cylinder there is hardly any variation of h with distance and the internal field near $r=0$ is strongly reduced from the applied magnetic field, but it is not zero as the radius of the cylinder is comparable to the penetration depth λ ($R=3\lambda$). Figure 6(b) shows the same quantity for $b=3$. The internal field is an even function with respect to r. Note again the fairly constant field region near the center of the cylinder. This region becomes larger the larger the b values are. At $b=1$ the constant field region near $r=0$ is hardly noticeable. For $h_m(b)$ and $b>0$ the internal field near the surface is less than the applied field but larger in the center of

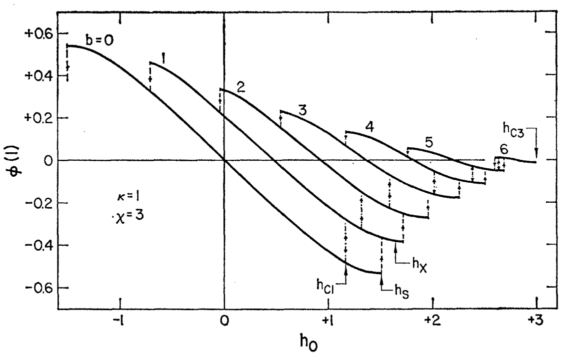

FIG. 5. The dependence of the function $\varphi(1)$, which is part of the vector potential at the surface of the cylinder [see Eq. (2)], on the applied magnetic field h_0 for $\kappa=1$, $\chi=3$, and all possible values of the number of enclosed fluxoids b in the cylinder. $\varphi(1)$ is related to the magnetization per unit volume [Eq. (5)]. For details see text.

the cylinder. At $h_u(b)$ the internal field is smaller than h_0 for all values of r and at $h_l(b)$ it is larger everywhere. Because for the same b value $h_u(b)$ is reached by increasing h_0, and $h_l(b)$ is reached by decrease h_0, the average internal field is generated by currents flowing around the axis of the cylinder which oppose the change of the external field h_0. The current density of these currents is given by Eq. (4), and the values of $\{(\kappa/\chi)^2(d/dr)\times[(1/r)(d\varphi/dr)]\}$ which are proportional to $j(r)$ are shown in Fig. 7 for various applied magnetic fields.

In Fig. 7(a) $(\kappa/\chi)^2 d/dr[(1/r)(d\varphi/dr)]$ is shown for $b=0$, $\kappa=1$, and $\chi=3$ and in Fig. 7(b) for $b=3$, $\kappa=1$, and

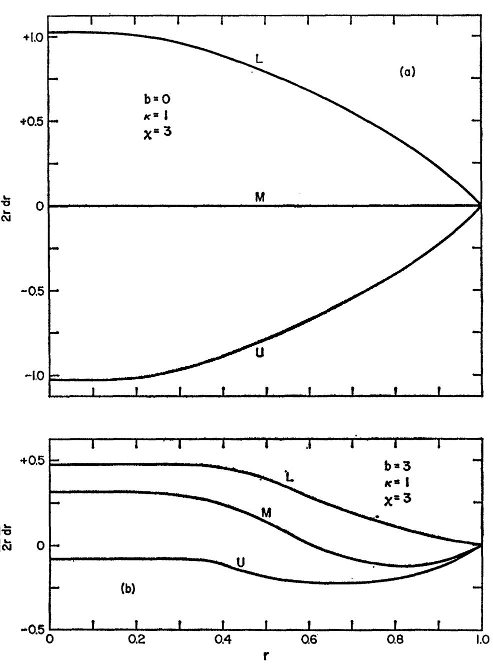

FIG. 6. The position dependence of the function $(1/2r)(d\varphi/dr)$, which is related to the internal field by Eq. (3), at the applied magnetic fields $h_l(b)$ [$\equiv L$], $h_m(b)$ [$\equiv M$], and $h_u(b)$ [$\equiv U$] as shown in Fig. 1 for: (a) $b=0$, $\kappa=1$, and $\chi=3$; (b) $b=3$, $\kappa=1$, and $\chi=3$.

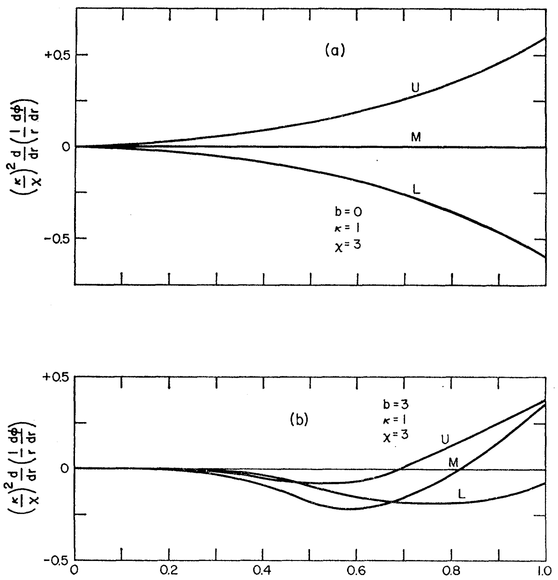

FIG. 7. The position dependence of the function $(\kappa/\chi)^2 d[(1/r) \times (d\varphi/dr)]/dr$, which is related to the current density by Eq. (4), at the applied magnetic fields $h_l(b)$ [$\equiv L$], $h_m(b)$ [$\equiv M$], and $h_u(b)$ [$\equiv U$] as shown in Fig. 1 for: (a) $b=0$, $\kappa=1$, $\chi=3$; (b) $b=3$, $\kappa=1$, and $\chi=3$.

$\chi=3$. For $b=0$ the values of $j(r)$ are the same but of opposite sign for the magnetic fields $h_u(0)$ and $h_l(0)$, and $j(r)=0$ for $h_0=0$. These currents are the usual Meissner currents. In Fig. 7(b) one sees that the current density may invert direction (for $b>0$) which, for example for the lowest energy state, leads to two opposing currents which make the magnetization per unit volume $4\pi M$ zero, and which is also true for a bulk specimen in its lowest energy state.[20] The current density $j(r)$ is an odd function with respect to r. There is no symmetry relation between the current densities at $h_u(b)$ and $h_l(b)$ with respect to the $j(r)=0$ axis except for $b=0$. For $b>3$ and $\chi=3$ the current density at the magnetic field $h_l(b)$ inverts also direction near the surface of the cylinder.

For completeness sake we also show $\int_0^1 rF^2(r)dr$ as a function of h_0 for various b values for $\kappa=1$ and $\chi=3$. This integral can be interpreted as being proportional to the total number of superconducting electrons n_s per unit length of the cylinder. The values of n_s decrease at $h_u(b)$ and $h_l(b)$ with respect to its value at $h_m(b)$ for a fixed value of b due to the extra kinetic energy of the persistent currents circulating around the axis of the cylinder at h_u and h_l.

IV. THE SIZE AND THE κ DEPENDENCE OF THE GIANT VORTEX

We have varied the parameter κ between 0.3 and 3 and χ between 2 and 12 and obtained similar solutions as in the above section. There are, however, some differences. When κ is decreased from unity and χ is kept constant the $F(1)$ values as a function of h_0 (see Fig. 3) become "stiffer" and the $4\pi M$ values become larger, whereas an increase in κ has the opposite effect. We noticed also that when χ is decreased for the same κ value that one can describe our results by "stiffer" $F(1)$ values and larger $4\pi M$ values compared to an increase in χ (for the same κ value) when $4\pi M$ becomes smaller and $F(1)$ becomes more "flexible".

Figure 9 shows Δg [Eq. (9)] for $\kappa=0.5$ and $\chi=3$ as a function of h_0 which is similar to Fig. 1. There is however, one important difference: the Δg curves for $1\leq b\leq 3$ are of higher energy than the Meissner state for all magnetic fields h_0 because they do not intersect the $\Delta g(b=0)$ curve. The $\Delta g(b)$ curves for the larger b values do, however, intersect the $\Delta g(0)$ curve and are similar to those shown in Fig. 1. This behavior is analogous to a bulk type-I superconductor with a sheath ($0.417<\kappa<0.707$). At h_s (or h_{c1}) the number of fluxoids in the sample changes by a number larger than unity, which means a relatively large jump in the $4\pi M$ curve at h_s or h_{c1} but not a jump to $4\pi M=0$ because superconductivity is quenched at h_{c3} which is larger than h_s. When the κ value is reduced to 0.3 (for example for $\chi=3$) then *all* the $\Delta g(b)$ curves for $b>0$ do not intersect themselves or the $\Delta g(0)$ curve and are of higher energy than the Meissner state. The highest $\Delta g(b)$ curve is again at $b=6$. At h_s the $4\pi M$ value drops sharply to zero when h_0 is increased beyond h_s and no similarity with the sheath is left for $h_0>h_s$. This behavior is similar for all χ values and appears to depend on κ only. When κ is increased to values larger than unity the $\Delta g(b)$ curves tend to space more evenly for all b values and no "bunching-up" of the $\Delta g(b)$ curves appear near h_s.

When $\chi\gtrsim 10$ an additional complication occurs. For intermediate b values no solutions appear which are consistent with the above described pattern. Over that range of b values the Abrikosov mixed state[1] or some other solution might be the appropriate description of the superconductor. Above and below this range of b values the giant vortex state might be the correct solution of which the Meissner state is one special case.

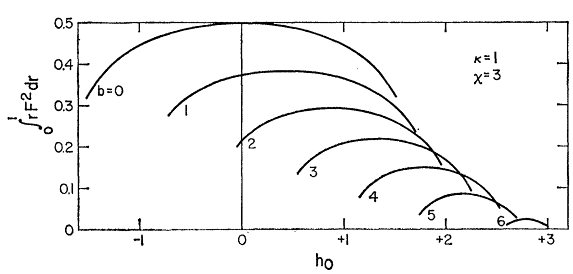

FIG. 8. Shown is the magnetic field dependence of $\int_0^1 rF^2 dr$ for $\kappa=1$, $\chi=3$, and all possible values of the number of enclosed fluxoids b. This integral is proportional to the number of superconducting electrons per unit length of the cylinder.

[20] H. J. Fink, Phys. Rev. Letters 14, 853 (1965).

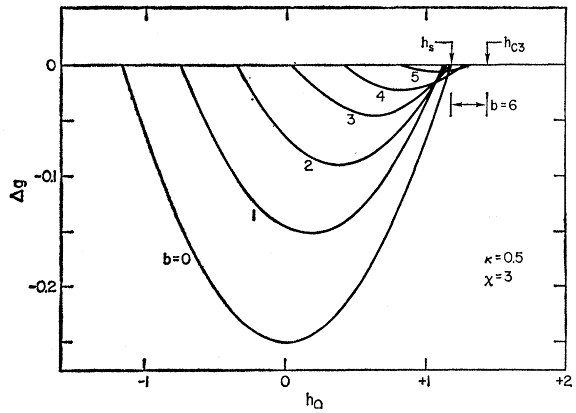

FIG. 9. This figure is similar to Fig. 1 except that $\kappa=0.5$ and $\chi=3$. $\Delta g(b=6)$ is too small on the above scale to be different from zero. Only the magnetic field interval is indicated. For details see text.

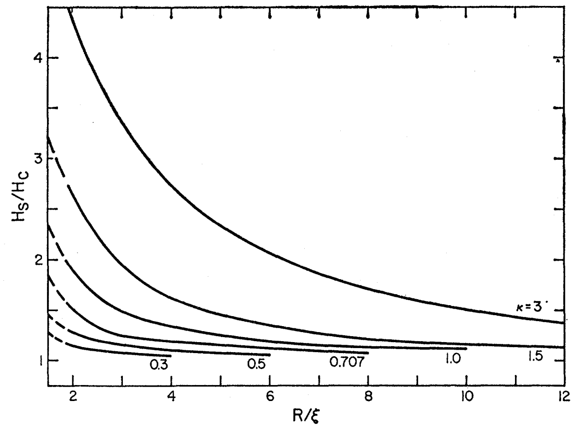

FIG. 10. The size dependence (R/ξ) of the maximum magnetic field $h_s(=H_s/H_c)$ up to which "flux" may be delayed from entering the Meissner state $(b=0)$ is shown for various values of κ. For $\chi\rightarrow\infty$ the value of $H_s\rightarrow H_c$ for all κ values. For the definition of h_s see Fig. 1.

Figure 10 shows the size dependence of the maximum magnetic field h_s to which "flux" may be delayed from entering the Meissner state as a function of R/ξ for various κ values, and Fig. 11 shows the magnetic field h_{c1} at which it is energetically favorable for "flux" to enter the Meissner state as a function of R/ξ for various κ values. For small values of R/ξ both fields h_{c1} and h_s increase. When $R/\xi\rightarrow\infty$ the h_{c1} values approach those given in Ref. 21 and all the $h_s(=H_s/H_c)$ values approach[16] unity. Cyclic variations of h_s and h_{c1} as a function of χ similar to those predicted[17] for h_{c3} were not observed within the present accuracy of the computation.

For the giant vortex state the bulk upper critical field $h_{c2}(=H_{c2}/H_c)$ has no meaning. For large values of χ and in large magnetic fields near h_{c3} the accuracy of our results is not very good due to the smallness of $F(r)$ and $\varphi(r)$. For more accurate values of h_{c3} as a function of χ one should resort to other calculations[17] which assume, however, even for χ values of order unity a second order phase transition at h_{c3}. From our computer calculations we were, however, able to obtain accurate values for $b_{\max}$ for small values of χ which are: for $\chi=1$, $b_{\max}=0$; for $\chi=2$, $b_{\max}=2$; for $\chi=3$, $b_{\max}=6$; for $\chi=4$, $b_{\max}=11$; for $\chi=5$, $b_{\max}\equiv 17$ to 18. These results agree with Ref. 17.

In the above model flux-locking at zero field is easily understood if one assumes that the conservation of the number of fluxoids is a real mechanism in superconductors. This mechanism is very similar to flux delay of the Meissner state though the measured $4\pi M$ values are quite different, namely paramagnetic for flux-locking and diamagnetic for flux delay. When in a *bulk* cylinder flux enters between H_{c1} and H_s, the flux will tend to concentrate near the center of the specimen due to symmetry considerations. Even if the flux is quantized in the form of flux tubes as in the mixed state, the sample as a whole will appear like a giant vortex and therefore show irreversible behavior, and this will lead to flux-locking provided the whole specimen (which is assumed to be ideal) can be described by a value of $\kappa>0.417$.

As mentioned above, the over-all accuracy of our calculations is probably not better than $\pm 2\%$. Furthermore, when we approached $\Delta g(b)\approx 0$, for values of h_0 near h_u and h_l, (see Fig. 1) we sometimes had a small undershoot or overshoot of $\Delta g(b)$ as a function of h_0. As this effect was usually small and strayed randomly, we believe that this was due to computational inaccuracies in the elements of the analog computer. For the present considerations we have used the extrapolated values of the variable functions when the small overshoot or undershoot at $\Delta g(b)=0$ was neglected. For $\kappa\lesssim 0.5$ the overshoot in $\Delta g(0)$ for $h_0>h_s$ (and

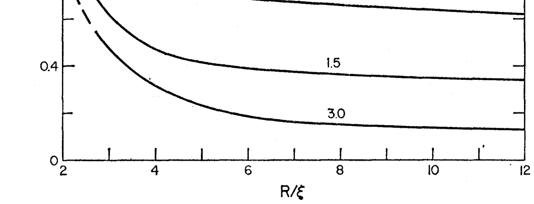

FIG. 11. The size dependence (R/ξ) of the normalized lower critical field $H_{c1}/\sqrt{2}\kappa H_c$ is shown for various values of κ. For $\kappa\lesssim 0.4$ the lower critical field $H_{c1}=H_s$ and for $0.4\lesssim\kappa\lesssim 0.7$ the values of H_{c1} are very close to, though smaller than, H_s. For $\kappa\lesssim 0.707$ and $\chi\rightarrow\infty$ the value of $H_{c1}\rightarrow H_c$.

[21] J. L. Harden and V. Arp, Cryogenics 3, 105 (1963).

$h_0 < -h_s$) is somewhat larger than the anticipated computational inaccuracies, but we do not know for certain if this is an inherent effect.

The Ginzburg-Landau equations are strictly valid only near the transition temperature T_c. The κ values of the above calculations apply strictly to those measured near T_c. However, the κ values do not vary strongly with temperature, and therefore, the above numerical work is approximately correct below T_c. Corrections for the temperature dependence of κ can be made.[22]

V. CONCLUSIONS

We have found solutions of the Ginzburg-Landau equations for a cylinder of infinite length in an axial magnetic field for various values of the Ginzburg-Landau parameter κ and the size parameter $X=R/\xi$. These solutions permit a total persistent current to flow around the axis of the cylinder and thereby affect the magnetic properties, which are in general irreversible and hysteretic. Over certain magnetic field ranges over which the number of fluxoids may be conserved the magnetic properties may be reversible. We have presented the exact solutions for $\kappa=1$, $X=3$ for all possible number of fluxoids b and discussed the behavior of similar solutions with other values of the parameters κ and X. We have obtained as a function of the applied magnetic field H_0, as a function of position, and as a function of the number of fluxoids b the absolute value of the order parameter $|\Psi(r)|$, (the phase of Ψ is related to b), the vector potential $\varphi(r)$ [see Eq. (2)], the internal magnetic field distribution and the current density. From these data one can obtain the total current. We have also calculated as a function of H_0 the total number of superconducting electrons per unit length of the cylinder and the magnetization per unit volume. The functions $|\Psi(r)|$ and $\varphi(r)$ (and its derivatives) depend on the magnitude and the direction of circulation of the total current with respect to the magnetic field which affects their shape in different ways at the magnetic fields $H_u(b)$ and $H_l(b)$ for $b>0$. The Meissner state is one particular solution of the above solutions, namely that for $b=0$.

When for constant b the total current is such as to make the difference between the total magnetic Gibbs functions of the superconducting and normal state zero, the functions $F(r)$ and $\varphi(r)$ (and all other functions derived from these) become zero in a step-function-like fashion, but we cannot say for certain whether these are step functions at $G_{SH}-G_{NH}=0$ because of computational inaccuracies. It appears, however, that no stable solutions of the above kind exist when $G_{SH}-G_{NH}>0$, which means that the superconducting pairs would break up when b is kept constant and $G_{SH}-G_{NH}$ is made larger than zero. This condition is similar to that of a superconducting ring with a maximum persistent current, but in our case we are dealing with a solid cylinder. We have also presented the lower critical field H_{c1} as a function of thickness R/ξ, and the field for maximum flux delay of the Meissner state H_s as a function of R/ξ. Both tend to increase rapidly for small values of R/ξ similar to H_{c3}. The bulk critical field H_{c2} does not occur in this solution.

When the applied magnetic field is reversed to zero after it was increased beyond the field where "flux" enters the specimen, flux is "locked-in" in the superconductor due to the conservation of the number of fluxoids which seems to override the lowest energy condition provided $\kappa \gtrsim 0.4$. The mechanism of flux-locking seems to be analogous to flux delay of the Meissner state except that one deals with different values of the number of fluxoids.

When R/ξ is increased beyond approximately 10 no solutions exist of the above type for a certain range of intermediate b values. Over the corresponding magnetic field interval another solution, for example the mixed state, may take over but we do not know for certain. The above solutions seem therefore favorable for thin specimen ($R/\xi \lesssim 10$) for all magnetic fields and might even be favorable for larger values of R/ξ, in particular over certain magnetic field ranges above H_{c1} and near H_{c2}.

The above magnetic irreversible (and hysteretic) solution is a counterpart to Abrikosov's mixed-state[1] magnetic reversible solution. Our irreversible solution does not depend on defects in the superconductor as it is the case in the Bean model.[23] The mixed state is dominant in very large specimen and the giant vortex state is dominant in thin specimen. In a finite specimen both states probably coexist. The giant vortex solution is related to the critical state solution of the surface sheath[6] of a bulk superconductor for $H_0 > H_{c2}$ when $\kappa > 0.417$.

ACKNOWLEDGMENTS

We would like to thank R. D. Kessinger for his assistance with the computation at the initial stage of this work and H. G. James for tabulating and plotting the numerical results. We acknowledge informative discussions with L. J. Barnes, R. R. Hake, and A. S. Joseph.

[22] K. Maki, Physics 1, 21 (1964); 1, 127 (1964); 1, 201 (1964); C. Caroli, M. Cyrot, and P. G. de Gennes, Solid State Commun. 4, 17 (1966).

[23] C. P. Bean, Rev. Mod. Phys. 36, 31 (1964).