Nuclear Scattering Lengths from the Spectroscopy of Muonic Molecules*

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The shift in energy levels of μ molecules produced by the nuclear interaction energy has been calculated. It is proportional to the probability of coincidence of the two nuclei and to their zero-energy scattering length. In $p\mu p$ it is 0.01 eV; in $p\mu d$ it is 0.005 eV. The effect may be detected by measuring the hyperfine splitting of $p\mu d$, of which it is a considerable fraction, and a null test for its presence is given. Its magnitude yields the difference of the nuclear scattering lengths in different spin states. As a by-product of our analysis of the $p\mu d$ hyperfine splitting, we discuss the possibility it presents, as compared to $p\mu p$, of studying the μ-p spin-spin interaction, which is a sensitive test of quantum electrodynamics.

I. INTRODUCTION

IN ordinary matter, nuclei are kept apart by their mutual Coulomb repulsion, so that the short-range nuclear forces acting between them have a negligible influence on molecular energy levels. However, in muonic molecules, which are made of two hydrogen isotopes and a muon, the mean nuclear separation is 200 times less, of order $(\alpha m_\mu)^{-1}$ instead of $(\alpha m_e)^{-1}$, and the probability of quantum-mechanical barrier penetration, whose dependence on the separation is roughly exponential, becomes significant. This gives rise to the familiar phenomenon of catalysis by muons of nuclear fusion reactions[1] that only take place in ordinary matter at thermonuclear temperatures. What does not seem to have been discussed before is that the nuclear forces also have a static effect: They produce shifts in μ-molecular energy levels.

It would be of interest to detect for the first time the effect of nuclear forces on molecular energy levels. We find, in fact, that the expected shift is a considerable fraction of the μ-molecular hyperfine splitting (and also changes the nuclear spin wave function by a few percent.) Its measurement has an immediate interpretation: It gives directly the difference between the zero-energy nuclear scattering lengths in different states of total nuclear spin. Determination of these scattering lengths by scattering experiments[2] is inherently less accurate and depends on extrapolating to zero energy the results of a phase-shift analysis at finite energies. To measure the μ-molecular hyperfine separation a resonance experiment is conceivable[3] in which change in population of the different hyperfine levels is detected by corresponding change in the spin-dependent fusion rate. Alternatively the muon decay asymmetry could perhaps be used in familiar fashion to detect changes in polarization induced by a laser beam, following a suggestion attributed to Novick.[4]

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[1] L. Alvarez *et al.*, Phys. Rev. **105**, 1127 (1957).
[2] R. A. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).
[3] Yale Study on High Intensity Proton Accelerators, Internal Report Y.12, by the Yale design study group, p. II–89 (unpublished).
[4] Ref. 3, p. II–91.

II. ENERGY SHIFT DUE TO NUCLEAR FORCES

We now calculate the molecular energy level shift produced by nuclear forces. The μ molecule is made up of two hydrogen isotopes and a muon. Let H_M be the molecular Hamiltonian which is the sum of the kinetic and Coulomb interaction energies of the constituent particles. Its eigenvalue E_M and corresponding normalized eigenvector $|M\rangle$ satisfy

$$H_M|M\rangle = E_M|M\rangle. \tag{1}$$

The total Hamiltonian is

$$H = H_M + V_N, \tag{2}$$

where V_N is the strong but short-range nuclear interaction Hamiltonian. We denote by P the projection operator onto the space orthogonal to $|M\rangle$,

$$P = I - |M\rangle\langle M|. \tag{3}$$

One easily verifies that the ket$[|M\rangle - (H_M + PV_N - E)^{-1} \times PV_N|M\rangle]$ is an eigenvector of H with eigenvalue E, provided E satisfies

$$E = E_M + \langle M|[V_N - V_N(H_M + PV_N - E)^{-1}PV_N]|M\rangle. \tag{4}$$

Let us now relate the energy-level shift $\Delta = E - E_M$ to the nuclear scattering length. Because V_N is strong it connects $|M\rangle$ to highly excited states orthogonal to $|M\rangle$ and the projection operator in P in Eq. (4) may be replaced by 1. The energy denominator is of order of nuclear energies (MeV) so that the contribution to the denominator of the kinetic and potential energies of the muon is completely negligible, and the denominator in Eq. (4) may be replaced by $(H_2 - E)^{-1}$, where H_2 is the total Hamiltonian of the two nuclei. We thus seek the expectation value in the molecular state of the operator $[V_N - V_N(H_2 - E)^{-1}V_N]$. It is independent of the muon coordinates and it vanishes unless the two nuclei are a nuclear distance apart. For this configuration, the molecular wave function, which is squared and integrated over muonic coordinates, may be replaced, apart from normalization, by the regular Coulombic

TABLE I. Energy levels and probabilities of total nuclear spin for the various muonic molecules.

	I	F	$\epsilon_F - E_0$	$P_{I>}$
$p\mu p$	0	$\frac{1}{2}$		
$p\mu d$[a]	$\frac{1}{2}$	0	$-\frac{1}{2}\Delta_{pd}-\frac{1}{2}\Delta_{\mu d}+\frac{1}{2}\Delta_{\mu p}$	
	$\frac{1}{2}, \frac{3}{2}$	$1'$	$\frac{1}{2}\lambda_{pd}$	$\frac{1}{2}+\frac{1}{2}\lambda_{pd}^{-1}[\Delta_{pd}-(7/9)\Delta_{\mu d}-\frac{1}{3}\Delta_{\mu p}]$
	$\frac{1}{2}, \frac{3}{2}$	1	$-\frac{1}{2}\lambda_{pd}$	$\frac{1}{2}-\frac{1}{2}\lambda_{pd}^{-1}[\Delta_{pd}-(7/9)\Delta_{\mu d}-\frac{1}{3}\Delta_{\mu p}]$
	$\frac{3}{2}$	2	$\frac{1}{2}\Delta_{pd}+\frac{1}{2}\Delta_{\mu d}+\frac{1}{2}\Delta_{\mu p}$	
$d\mu d$	0	$\frac{1}{2}$	$-\frac{1}{2}\Delta_{dd}+\frac{1}{3}\Delta_{\mu d}$	
	2	$\frac{3}{2}$	$\frac{1}{2}\Delta_{dd}-\frac{2}{3}\Delta_{\mu d}$	
	2	$\frac{5}{2}$	$\frac{1}{2}\Delta_{dd}+\Delta_{\mu d}$	
$p\mu t$[b]	0, 1	$\frac{1}{2}'$	$\frac{1}{2}\lambda_{pt}$	$\frac{1}{2}+\frac{1}{2}\lambda_{pt}^{-1}(\Delta_{pt}-\frac{1}{2}\Delta_{\mu p}-\frac{1}{2}\Delta_{\mu t})$
	0, 1	$\frac{1}{2}$	$-\frac{1}{2}\lambda_{pt}$	$\frac{1}{2}-\frac{1}{2}\lambda_{pt}^{-1}(\Delta_{pt}-\frac{1}{2}\Delta_{\mu p}-\frac{1}{2}\Delta_{\mu t})$
	1	$\frac{3}{2}$	$\frac{1}{2}\Delta_{pt}+\frac{1}{2}\Delta_{\mu p}+\frac{1}{2}\Delta_{\mu t}$	

[a] $\lambda_{pd}=[\Delta_{pd}^2+\Delta_{\mu p}^2+\Delta_{\mu d}^2-\frac{2}{3}\Delta_{\mu p}\Delta_{\mu d}-\frac{2}{3}\Delta_{pd}\Delta_{\mu p}-(14/9)\Delta_{pd}\Delta_{\mu d}]^{1/2}$.

[b] $\lambda_{pt}=[\Delta_{pt}^2+\Delta_{\mu p}^2+\Delta_{\mu t}^2-\Delta_{pt}\Delta_{\mu p}-\Delta_{pt}\Delta_{\mu t}-\Delta_{\mu p}\Delta_{\mu t}]^{1/2}$.

wave function $|F\rangle$. So we find

$$\Delta = E - E_M = G^2(0) \times \{\langle F|[V_N - V_N(H_2-E)^{-1}V_N]|F\rangle / C^2\}. \quad (5)$$

Here C^2 is the familiar Coulomb penetration factor which gives the probability, according to the Coulombic wave functions, for the two nuclei to coincide, and $G^2(0)$ is the same quantity, integrated over muonic coordinates, in the molecular state. The binding energy is about 2 keV (which is much closer to zero than one ever gets in the corresponding scattering experiments) so zero-energy scattering theory[5] is applicable. Apart from a factor $2m_r/4\pi$ (m_r=reduced mass) the matrix element is $-(k\cot\delta)^{-1}$, where δ is the additional phase shift induced in the Coulomb wave functions by the presence of the nuclear forces. At zero energy the nuclear scattering length a is given by[5] $C^2k\cot\delta = -1/a$ [apart from the familiar term involving $h(\eta)$ which is negligible at -2 keV], and so we obtain

$$\Delta_I = G_{n_1n_2}{}^2(0)(4\pi/2m_r)a_I. \quad (6)$$

We have introduced the subscript I as a reminder that the scattering length a, and consequently the level shift Δ, depends on the total spin I of the two nuclei, n_1 and n_2.

This is the desired result which is, in fact, intuitively obvious. It could have been obtained by introducing an effective nuclear interaction Hamiltonian, $V_{\text{eff}}=4\pi \times(2m_r)^{-1}a\delta(\mathbf{r})$, which gives the correct scattering amplitude in Born approximation. However, we have given a detailed derivation to make clear that the Coulomb interaction has been disentangled correctly. The quantity $G_{n_1n_2}{}^2(0)$ is important because it determines the rate of nuclear fusion reactions. It has been computed most recently for the ground state of $p\mu p$ and $p\mu d$ by Carter,[6] who finds

$$G_{pp}{}^2(0)=2.5\times10^{27}\ \text{cm}^{-3}, \quad G_{pd}{}^2(0)=1.0\times10^{27}\ \text{cm}^{-3} \quad (7)$$

(no estimate of error given). In the case of other μ molecules with higher reduced mass of the two nuclei, this quantity decreases further. For a p-p scattering length of -7.7 F, one finds in the $p\mu p$ molecule an energy level shift

$$\Delta_{pp} = -0.01\ \text{eV}. \quad (8)$$

For quartet and doublet p-d scattering lengths,[2] $a_{3/2}=12.5\pm0.1$ fm and $a_{1/2}=1.4\pm1$ fm, one finds in the $p\mu d$ molecule

$$\Delta_{3/2}=0.0049\pm0.0004\ \text{eV}. \quad (9a)$$

$$\Delta_{1/2}=0.0005\pm0.0004\ \text{eV}. \quad (9b)$$

These may be compared with a dissociation energy of the order of 200 eV.

III. MOLECULAR HYPERFINE SPLITTING

The most likely method of detecting such energy level shifts is to measure directly the molecular hyperfine splitting. This was first calculated in the thorough review article by Zel'dovich and Gershtein[7] and again by Carter,[6] neither of whom, however, included the effect of nuclear forces. We calculate the hyperfine splitting for spherically symmetric states, $K=0$ (K is the total orbital angular momentum of lepton and nuclei), of the various molecules.

In columns 2 and 3 of Table I, we list the total nuclear spin I and the total angular momentum F of the molecules. Except for $p\mu p$ and $t\mu t$ which have only $I=0$ because of the exclusion principle, they all have a

[5] J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. **22**, 77 (1950).

[6] B. P. Carter, Phys. Rev. **141**, 863 (1966) and Erratum (to be published). A more complete bibliography may be found here and in Ref. 7.

[7] Ya. B. Zel'dovich and S. S. Gershtein, Usp. Fiz. Nauk **71**, 581 (1960) [English transl.: Soviet Phys.—Usp. **3**, 593 (1961)].

hyperfine splitting and two possible values of I, which we distinguish by $I_>$ and $I_<$ for the greater and the lesser. In these cases the energy level shift given by Eq. (6) becomes an operator in spin space which we write as

$$H(n_1,n_2)=E_0(n_1,n_2)+\tfrac{1}{2}\Delta_{n_1n_2}(P_{I>}{}^{n_1n_2}-P_{I<}{}^{n_1n_2}), \quad (10)$$

where $E_0=\frac{1}{2}(\Delta_{I>}+\Delta_{I<})$, $\Delta=(\Delta_{I>}-\Delta_{I<})$, and n_1 and n_2 label the two nuclei, $n_i=p$, d, or t. The $P_I{}^{n_1n_2}$ are projection operators onto the states of total spin I of the two nuclei.

The remaining part of the hyperfine interaction is simply the familiar magnetic spin-spin interaction of the μ with each nucleus. (We neglect the magnetic spin-spin interaction of the two nuclei.) It has a tensor (D-wave) part which gives no contribution when evaluated in a spatially symmetric state ($K=0$), and an S-wave part

$$H(\mu,n)=G_{\mu n}{}^2(0)(4\pi\alpha g_n/3Mmi_n)\mathbf{i}_n\cdot\mathbf{s}, \quad (11)$$

where α is the fine-structure constant, g_n is the magnetic moment of the nucleus in nuclear magnetons ($g_p=2.79$, $g_d=0.857$, $g_t=2.98$), M is the nucleon mass, m is the muon mass, $\mathbf{i}_n$ is the spin operator of the nucleus, and $\mathbf{s}$ is the muon-spin operator. The overlap $G_{\mu n}{}^2(0)$ is the probability that the μ and the nucleus n coincide, integrated over the position of the other nucleus. Just as $G_{n_1n_2}{}^2(0)$ in Eq. (6) determines the rate of nuclear fusion, the quantity $G_{\mu n}{}^2(0)$ determines the rate of μ capture by the nucleus n. A comparison of Eq. (6) and Eq. (11), giving respectively the n_1-n_2 nuclear level shift and the μ-n spin-spin level shift, is instructive since they have a similar form. The weaker μ-n spin-spin forces are more than compensated for by the much larger μ-n overlap, $G_{\mu n}{}^2(0)$, which is enhanced by Coulomb attraction, whereas the n_1-n_2 overlap, $G_{n_1n_2}{}^2(0)$, is reduced by Coulomb repulsion.

We rewrite $H(\mu,n)$ in the form

$$H(\mu,n)=E_0(\mu,n)+\tfrac{1}{2}\Delta_{\mu n}(P_{i+1/2}{}^{\mu n}-P_{i-1/2}{}^{\mu n}), \quad (12)$$

where $i=$ spin of nucleus n, $E_0(\mu_1 n)=-\frac{1}{2}\Delta_{\mu n}/(2i+1)$, and $\Delta_{\mu n}=G_{\mu n}{}^2(0)(i+\frac{1}{2})(4\pi\alpha g_n/3Mmi)$. It is convenient to write $\Delta_{\mu n}$ as the product of the hyperfine splitting in the μn atom $\Delta\nu_{\mu n}$ and the relative overlap γ_n, which is the ratio of the overlap in the molecule $G_{\mu n}{}^2(0)$, to the overlap in the μn atom:

$$\Delta_{\mu n}=\gamma_\mu\Delta\nu_{\mu n}. \quad (13)$$

Values of the relative overlap are tabulated in Ref. 6 and are a bit larger than 0.5.

For the $n_1\mu n_2$ molecule, the total hyperfine Hamiltonian is the sum

$$\begin{aligned}H&=H(n_1,n_2)+H(\mu,n_1)+H(\mu,n_2),\\ H&=E_0+\tfrac{1}{2}\Delta_{n_1n_2}(P_{I>}{}^{n_1n_2}-P_{I<}{}^{n_1n_2})\\ &\quad+\tfrac{1}{2}\Delta_{\mu n_1}(P_{i_1+1/2}{}^{\mu n_1}-P_{i_1-1/2}{}^{\mu n_1})\\ &\quad+\tfrac{1}{2}\Delta_{\mu n_2}(P_{i_2+1/2}{}^{\mu n_2}-P_{i_2-1/2}{}^{\mu n_2}),\end{aligned} \quad (14)$$

where $E_0=E_0(n_1n_2)+E_0(\mu n_1)+E_0(\mu n_2)$. The matrix elements of this Hamiltonian are easily evaluated in a basis where a pair of P's are diagonal by simple recoupling of 3 angular momenta. The resulting eigenvalue problem is either one or two dimensional. The eigenvalues for all μ molecules are displayed in Table I. There the ground states of the various molecules are listed according to total nuclear spin I and total angular momentum F of each molecule. The entries for $d\mu t$ and $t\mu t$ are obtained from the corresponding entries for $p\mu d$ and $p\mu t$ by the substitution $p\rightarrow t$. The fourth column expresses the corresponding hyperfine energy ϵ_F produced by the nuclear and magnetic spin-spin interactions, over and above the energy levels produced by the Coulomb forces. When a state contains a superposition of two different total nuclear spins I, the last column gives the probability, $P_{I>}$, that the greater spin occurs, with $P_{I>}+P_{I<}=1$. These probabilities determine the fusion rate from the corresponding state, since the latter depends strongly on the total nuclear spin. The most convenient use of this table is, perhaps, to invert the formulas and express the two-body interaction energies, $\Delta_{\mu n}$ or $\Delta_{n_1n_2}$, in terms of the measurable hyperfine separations which are the differences between the hyperfine energy levels. Let us adopt for these differences the notation $\Delta\nu(F_1,F_2)=\epsilon_{F_1}-\epsilon_{F_2}$. Then for the $p\mu d$ system we find

$$\Delta_{\mu p}=\gamma_p\Delta\nu_{\mu p}=\Delta\nu(2,1')+\Delta\nu(0,1), \quad (15)$$

$$\begin{aligned}\Delta_{\mu d}=\gamma_d\Delta\nu_{\mu d}=\tfrac{1}{2}\Delta\nu(2,0)+\tfrac{3}{4}\sqrt{2}\{\Delta\nu^2(1,1')\\ -[\Delta\nu(2,1')+\Delta\nu(0,1)-\tfrac{1}{3}\Delta\nu(2,0)]^2\}^{1/2},\end{aligned} \quad (16)$$

$$\begin{aligned}\Delta_{pd}=\tfrac{1}{2}\Delta\nu(2,0)-\tfrac{3}{4}\sqrt{2}\{\Delta\nu^2(1,1')\\ -[\Delta\nu(2,1')+\Delta\nu(0,1)-\tfrac{1}{3}\Delta\nu(2,0)]^2\}^{1/2}.\end{aligned} \quad (17)$$

We have made use of the inequality $\Delta_{\mu d}>\Delta_{pd}$ [justified by the numerical estimates below, Eq. (21) and Eq. (23)] in determining the sign ambiguity of the square root. For the $d\mu d$ system we find

$$\Delta_{\mu d}=\gamma_d\Delta\nu_{\mu d}=\tfrac{3}{5}\Delta\nu(\tfrac{5}{2},\tfrac{3}{2}), \quad (18)$$

$$\Delta_{dd}=\Delta\nu(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{3}{5}\Delta\nu(\tfrac{5}{2},\tfrac{3}{2}). \quad (19)$$

For the $p\mu t$ system this inversion cannot be effected since the observable energy differences give only two linearly independent symmetric functions of the three unknowns $\Delta_{\mu p}$, $\Delta_{\mu t}$, Δ_{pt}. The result for the $d\mu t$ system is obtained from Eqs. (15) to (17) for $p\mu d$ by the substitution $p\rightarrow t$.

Let us now discuss the significance and use of Eqs. (15) to (17) and, *mutatis mutandis*, of Eqs. (18) and (19). We note first of all that Eq. (17) expresses the interaction energy due to nuclear forces, Δ_{pd}, entirely in terms of the measurable hyperfine separations. Consequently when the measurements are effected, a nonzero value for the right-hand side of Eq. (17) means that the effect of the nuclear forces has been detected, *independently of molecular wave-function calculations.*

This provides a null test for the effect of the nuclear forces. Of course knowledge of the overlap probability $G_{pd}^2(0)$ yields, via Eq. (6) the difference of scattering lengths:

$$\Delta_{pd}=\Delta_{3/2}{}^{pd}-\Delta_{1/2}{}^{pd}=G_{pd}{}^2(0)(2\pi)(m_p+m_d)\times m_p^{-1}m_d^{-1}(a_{3/2}{}^{pd}-a_{1/2}{}^{pd}). \quad (20)$$

As an estimate of the accuracy required in the measurement of the hyperfine separation in order to obtain a significant experimental value for Δ_{pd}, we give the calculated values of Δ_{pd}, $\Delta_{\mu p}$, and $\Delta_{\mu d}$. From Eqs. (9) we have

$$\Delta_{pd}=\Delta_{3/2}-\Delta_{1/2}=0.0044\pm0.0005 \text{ eV}, \quad (21)$$

The uncertainty due to molecular wave functions is not included here, or below, the estimated error being due entirely to the uncertainty in the measured value of the nuclear scattering length. From Eq. (13) and the values[6] $\gamma_p=0.507$, $\gamma_d=0.641$ we have

$$\Delta_{\mu p}=\gamma_p\Delta\nu_{\mu p}=0.0926 \text{ eV}, \quad (22)$$

$$\Delta_{\mu d}=\gamma_d\Delta\nu_{\mu d}=0.0315 \text{ eV}. \quad (23)$$

We thus have the ratios of nuclear- and magnetic-interaction energies: $\Delta_{pd}/\Delta_{\mu p}=5\%$, $\Delta_{pd}/\Delta_{\mu d}=14\%$. Because of the precision characteristic of spectroscopic determinations, if the hyperfine separations can be measured at all, they should yield significant values for Δ_{pd}. If these estimated values are inserted into the formulas of Table I, one obtains for the hyperfine energy levels in electron volts,[6]

$$\epsilon_0=0.0027,\ \epsilon_1=-0.0679,\ \epsilon_1'=0.0167,\ \epsilon_2=0.0386. \quad (24)$$

This includes the total change in energy, produced by nuclear- and magnetic-hyperfine interactions, with respect to the levels produced by purely Coulomb forces. The contribution by the nuclear forces to these four states are, respectively, in electron volts, 0.0005, 0.0040, 0.0014, and 0.0049. The states ϵ_0 and ϵ_2 are pure states of total nuclear spin, I, with $I=\frac{1}{2}$ and $I=\frac{3}{2}$, respectively. The states ϵ_1 and ϵ_1' are mixed states with probabilities of total nuclear spin

$$\begin{aligned} P_1(I=\tfrac{3}{2})&=80\%, \quad P_1(I=\tfrac{1}{2})=20\%,\\ P_1'(I=\tfrac{3}{2})&=20\%, \quad P_1'(I=\tfrac{1}{2})=80\%. \end{aligned} \quad (25)$$

The nuclear forces contribute 2% to these values.[8]

[8] The presence or absence of an electron bound to the $p\mu d$ (or $d\mu d$) system does not affect our hyperfine energies or spin wave functions significantly. This is because the spin-dependent part of the interaction energy of the electron with the rest of the molecule is of the order of usual atomic hyperfine energies which lie in the microwave region, whereas the energies found here lie in the near infrared.

IV. THE μ-p INTERACTION

The principal virtue of Eqs. (15), (16), and (17) is that they separately express the p-d and μ-p (and μ-d) interaction energies in terms of observables. These interaction energies may be related to fundamental quantities, provided that one knows, respectively, the p-d and μ-d overlap in the molecular wave function. The problem of calculating each of them is essentially independent.[9] Consequently it is fortunate that by means of Eqs. (15) to (17) the separate interaction energies are separately expressed in terms of observables.

Up to now we have been concerned with the p-d interaction. However, the μ-p hyperfine interaction is of very great interest because of the discrepancy of 45 ± 17 ppm in the hydrogen (e-p) hyperfine splitting.[10] It has been suggested[11] that the hyperfine separation of muonic hydrogen (μ-p) is 200 times more sensitive to the short-range phenomenon that is presumably the cause of the discrepancy, if it is genuine. However, as Professor Hughes has kindly explained to me, hyperfine measurements on μ molecules, however difficult, are more likely to be effected than on muonic hydrogen atoms (because in matter, the former are rapidly formed from the latter).

The first molecule that comes to mind is $p\mu p$. However, it is formed primarily in the lowest orthostate which is $K=1$. Therefore a calculation of the hyperfine splitting must also include the coupling to $\mathbf{K}$ and the tensor (D-wave) part of the dipole-dipole magnetic interaction. On the other hand, the $p\mu d$ molecule is formed in the $K=0$ state, and the molecular hyperfine separations are directly related by Eq. (15) to the atomic hyperfine splitting,

$$\gamma_p\Delta\nu_{\mu p}=\Delta\nu(2,1')+\Delta\nu(0,1). \quad (26)$$

Consequently the μ-p interaction is perhaps better observed in $p\mu d$ rather than in $p\mu p$, depending, of course, on experimental feasibility. In this case the relative μ-p overlap, γ_p, must be calculated to 0.3% if the comparison of theory and experiment is to reveal in the μ-p interaction the same phenomenon that may be the cause of the discrepancy in hydrogen.

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[9] In particular there exists a different variational calculation for the evaluation of the expectation value of each operator, in this case $\delta(\mathbf{r}_p-\mathbf{r}_d)$ and $\delta(\mathbf{r}_\mu-\mathbf{r}_p)$. I am grateful to Professor L. Spruch for pointing this out to me.

[10] W. E. Cleland, J. M. Bailey, M. Eckhause, V. W. Hughes, R. M. Mobley, R. Prepost, and J. E. Rothberg, Phys. Rev. Letters **13**, 202 (1964).

[11] Reference 3, p. II–90 and A. Verganelakis and D. Zwanziger, Nuovo Cimento **39**, 613 (1965).