Spin-Density-Matrix Analysis. Positivity Conditions and Eberhard-Good Theorem

PIERRE MINNAERT
Centre de Physique Théorique de l'Ecole Polytechnique, Paris, France
(Received 16 June 1966)

We derive the conditions that the positive-definiteness of the spin density matrix implies on its multipole parameters. We also derive the general conditions imposed by the Eberhard-Good theorem. General formulas are given for arbitrary spin j, and detailed results are worked out for $j=1$ and $j=\frac{3}{2}$. Some applications of these results to spin determination are considered.

INTRODUCTION

To study the production and decay of resonant states,[1] the most interesting quantity to consider is the spin density matrix of the resonant states. This matrix describes the spin distribution of the particle and carries all the information from the production process to the decay process. Thus, a determination of the multipole parameters of the density matrix,[2] through the decay angular distribution and through the polarization angular distributions of the decay products, allows one to assign a spin to the resonance[3] and to test the various models[4] proposed for the production process.

However, a complete determination of the density matrix is not always possible. For instance, in the case of a parity-conserving decay into 2 spinless particles ($K^* \to K\pi$, $\rho \to 2\pi$), no polarization measurement is possible. And in most cases, owing to large statistical errors in the determination of multipole parameters, one cannot decide uniquely the spin of the particle studied or draw definite conclusions about the dynamics of the production process.

One way to remove some of these ambiguities is to use the theoretical conditions imposed on multipole parameters by the general properties of the density matrix. The aim of this paper is to look for these conditions.[5] In the first section we review the genral properties of spin density matrices and multipole parameters. In Sec. II we derive the conditions imposed on the density matrix by its positive-definiteness and in Sec. III we discuss the conditions deduced from the Eberhard-Good theorem. In Secs. IV and V we show how one can obtain conditions on multipole parameters. Finally in Sec. VI we give some applications.

When dealing with multipole expansion of spin density matrices, we have found it very convenient to use the formalism of 3-j and 6-j symbols introduced by Wigner and Racah. We have used the formulas given in the textbooks of Edmonds[6] and Wigner.[7] In Appendix A we review some of the results relevant to this paper.

I. SPIN DENSITY MATRIX AND MULTIPOLE PARAMETERS

When the dynamical state of a quantum system is incompletely known, the state is not represented by a unique vector in the Hilbert space but by a statistical set of vectors. A convenient way to describe this set is to consider the density operator[8]

$$\rho=\sum_m |m\rangle p_m\langle m| \,. \tag{1.1}$$

When the vectors $|m\rangle$ are orthogonal and normalized to unity, p_m is the probability for the system to be in the state described by the vector $|m\rangle$ .Thus we have

$$0\leqslant p_m\leqslant 1 \quad \text{and} \quad \sum_m p_m=1\,. \tag{1.2}$$

The general properties of the density operator follow from its definition. It is a *self-conjugate*, *positive* operator with unit trace.

The dynamical state of a sample S of particles with fixed energy-momentum and fixed spin j is represented by a statistical mixture of pure polarization states. The number of pure states contributing to the description of the sample S depends on the way this sample is prepared; at most we can have $2j+1$ independent states. If we choose a frame of reference and a quantization axis, the $2j+1$ vectors $|jm\rangle$ (proper vectors of J^2 and J_z) form a basis in the spin space. The spin density operator ρ which describes the polarization state of the particles is represented in this basis by a trace-1, Hermitian, positive, $(2j+1)\times(2j+1)$ matrix.

[1] For a review of this subject see R. H. Dalitz, Ann. Rev. Nucl. Sci. **13**, 339 (1963); R. H. Dalitz, Lectures given at the International School of Physics on Strong Interactions, Varenne, July 1964 (to be published); J. D. Jackson, in *High Energy Physics 1965* (Gordon and Breach, Science Publishers, Inc., New York, 1966).

[2] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).

[3] For a review of spin determination see R. D. Tripp, Ann. Rev. Nucl. Sci. **15**, 325 (1965).

[4] J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965); J. D. Jackson, J. T. Donohue, K. Gottfried, R. Keyser, and B. E. Y. Svensson, Phys. Rev. **139**, B428 (1965).

[5] An analysis of spin density matrix has also been made by M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. **140**, B192 (1965); R. H. Dalitz, Nucl. Phys. **87**, 89 (1966).

[6] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[7] E. P. Wigner, *Group Theory and Its Application to the Quantum Theory of Atomic Spectra* (Academic Press Inc., New York, 1959).

[8] For a review of density matrix techniques see U. Fano, Rev. Mod. Phys. **29**, 74 (1957).

Let us denote by $\rho^m{}_n$ the matrix elements of ρ

$$\rho^m{}_n = \langle jm|\rho|jn\rangle. \quad (1.3)$$

If we rotate the frame of reference by a finite rotation $R(\alpha\beta\gamma)$, the basis vectors $|jm\rangle$ undergo a unitary transformation[9]:

$$|jm\rangle \to U(R)|jm\rangle = D^{(j)}[R(\alpha\beta\gamma)]^{m'}{}_m|jm'\rangle, \quad (1.4)$$

where $D^{(j)}(R)$ is the $(2j+1)$-dimensional representation of the rotation group. The corresponding unitary transformation of operators is

$$\rho \to \rho' = U(R)\rho U^{-1}(R)$$

or

$$\rho^m{}_n = D^{(j)}(R^{-1})^m{}_{m'} D^{(j)}(R)^{n'}{}_n \rho'^{m'}{}_{n'}. \quad (1.5)$$

By reducing the product of representations (Appendix A4) and making use of the orthogonality properties of 3-j symbols (A1), one obtains

$$\begin{pmatrix} n & L & j \\ j & M & m \end{pmatrix} \rho^m{}_n = D^{(L)}(R)^{M'}{}_M \begin{pmatrix} n' & L & j \\ j & M' & m' \end{pmatrix} \rho'^{m'}{}_{n'}, \quad (1.6)$$

where L runs on integer values $0 \leq L \leq 2j$. Formula (1.6) shows that for each value of L, the $2L+1$ quantities

$$t_M{}^{L*} \equiv (2j+1)^{1/2} \begin{pmatrix} n & L & j \\ j & M & m \end{pmatrix} \rho^m{}_n, \quad (1.7)$$

transform under rotation like the elements of an irreducible tensor:

$$t_M{}^{L*} = D^{(L)}(R)^{M'}{}_M t'_{M'}{}^{L*}. \quad (1.8)$$

The quantities $t_M{}^L$ are called multipole parameters. Owing to their transformation law (1.8) they are particularly suited to the study of spin density matrices.

By inversion[10] of the definition (1.7), one obtains the multipole expansion of the spin density matrix

$$\rho^m{}_n = (2j+1)^{-1} \sum_{L=0}^{2j} \sum_{M=-L}^{+L} (2L+1) t_M{}^{L*} (T_M{}^L)^m{}_n, \quad (1.9)$$

with

$$(T_M{}^L)^m{}_n = (-)^{2j}(2j+1)^{1/2} \begin{pmatrix} M & m & j \\ L & j & n \end{pmatrix}. \quad (1.10)$$

The quantities $(T_M{}^L)^m{}_n$ can be considered as the matrix elements in the basis $|jm\rangle$ of an irreducible tensorial operator $T_M{}^L$. This operator can be explicitly constructed[11] as a function of the angular momentum operators J_+, J_-, J_z. From the definition (1.10) and the properties of 3-j symbols one can derive the following properties of $T_M{}^L$:

$$T_M{}^L = (-)^M (T_{-M}{}^L)^\dagger, \quad (1.11)$$

$$\mathrm{Tr}(T_M{}^L) = (2j+1)\delta_{L0}\delta_{M0}, \quad (1.12a)$$

$$\mathrm{Tr}(T_M{}^L T_{M'}{}^{L'\dagger}) = \frac{2j+1}{2L+1}\delta_{LL'}\delta_{MM'}. \quad (1.12b)$$

The general properties of the density operator impose conditions on the multipole parameters. The Hermiticity of ρ implies

$$t_M{}^L = (-)^M t_{-M}{}^{L*}; \quad (1.13)$$

thus the parameters $t_0{}^L$ are real. From (1.12a) $\mathrm{Tr}\rho$ is found to be equal to the parameter $t_0{}^0$. Thus, the condition $\mathrm{Tr}\rho=1$ leads to

$$t_0{}^0 = \mathrm{Tr}\rho = 1. \quad (1.14)$$

From $\mathrm{Tr}\rho=1$ and from positivity one can easily derive the conditions

$$(2j+1)^{-1} \leqslant \mathrm{Tr}\rho^2 \leqslant 1. \quad (1.15)$$

To this end let us diagonalize the matrix ρ. Since $\mathrm{Tr}\rho$ is fixed to unity and the proper values are positive, $\mathrm{Tr}\rho^2$ has a maximum when one proper value is equal to 1, the others being null; then the matrix ρ describes a pure state of polarization and $\mathrm{Tr}\rho^2=1$. The minimum of $\mathrm{Tr}\rho^2$ is reached when all the proper values are equal to $(2j+1)^{-1}$; then the matrix describes a completely unpolarized state and $\mathrm{Tr}\rho^2=(2j+1)^{-1}$.

With formulas (1.9), (1.11), (1.12), (1.13), $\mathrm{Tr}\rho^2$ can be expressed as a function of the parameters $t_M{}^L$, and (1.15) becomes:

$$(2j+1)^{-1} \leqslant (2j+1)^{-1} \sum_{L,M} (2L+1)|t_M{}^L|^2 \leqslant 1. \quad (1.16)$$

For density matrices of spin-$\frac{1}{2}$ particles, conditions (1.15) or (1.16) are the only conditions induced by the positivity. But for $j>\frac{1}{2}$, further conditions are imposed on the density matrix and on the multipole parameters by the positivity property. The derivation of these conditions is performed in the next section.

II. POSITIVITY CONDITIONS FOR DENSITY MATRICES

In this section we first recall a result of matrix theory and then we show how the positivity property of the density matrix leads to conditions on the coefficients of the associated characteristic polynomial and on the traces of the powers of the matrix.

Let ρ be a Hermitian $n\times n$ matrix. Its characteristic polynomial is $\Delta(\lambda)=\det(\lambda 1-\rho)$

$$\begin{aligned}\Delta(\lambda) &= \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + (-)^{n-1} \\ &\quad \times c_{n-1}\lambda + (-)^n c_n, \quad (2.1) \\ &= (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n),\end{aligned}$$

[9] See A. R. Edmonds, Ref. 6. In our formulas the repeated magnetic-quantum-number indices m are to be summed over $(m=-j, -j+1, \cdots, +j)$. See Appendix A.

[10] This inversion is obtained by multiplying each member of Eq. (1.7) by a suitable 3-j symbol and summing over repeated indices.

[11] For an explicit construction of $T_M{}^L$ see C. Henry and E. de Rafael, Ann. Inst. Henri Poincaré 2, 87 (1965); E. de Rafael, thesis, Université de Paris (unpublished).

TABLE I. Relations between the coefficients c_l of the characteristic polynomial of a matrix ρ and the traces $s_k=\mathrm{Tr}\rho^k$ of the powers of the matrix.

$$2c_2=-s_2+(s_1)^2.$$
$$6c_3=2s_3-3s_2s_1+(s_1)^3.$$
$$24c_4=-6s_4+8s_3s_1+3(s_2)^2-6s_2(s_1)^2+(s_1)^4.$$
$$120c_5=24s_5-30s_4s_1+20s_3(s_1)^2-20s_3s_2+15(s_2)^2s_1-10s_2(s_1)^3+(s_1)^5.$$
$$720c_6=-120s_6+144s_5s_1+90s_4s_2-90s_4(s_1)^2+40(s_3)^2-120s_3s_2s_1+40s_3(s_1)^3-15(s_2)^3+45(s_2)^2(s_1)^2-15s_2(s_1)^4+(s_1)^6.$$

($\lambda_1, \lambda_2, \cdots\lambda_i, \cdots\lambda_n$ are the proper values of the matrix). Newton formulas express the coefficients c_l as symmetrical functions of the roots λ_i:

$$c_1=\sum_{i=1}^{n}\lambda_i=\mathrm{Tr}\rho\,,$$
$$c_2=\sum_{i_1 i_2}\lambda_{i_1}\lambda_{i_2}\,, \tag{2.2}$$
$$c_l=\sum_{i_1<i_2\cdots<i_l}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_l}\,,$$
$$c_n=\lambda_1\lambda_2\cdots\lambda_n=\det(\rho)\,.$$

If we denote by s_k the trace of the matrix ρ^k,

$$s_k\equiv\mathrm{Tr}\rho^k\,, \tag{2.3}$$

a simple result of matrix theory[12] expresses the coefficients c_l as functions of the traces s_k $(k\leq l)$:

$$c_l=(-)^{l+1}l^{-1}(s_l-c_1s_{l-1}+\cdots+(-)^{l-1}c_{l-1}s_1)\,. \tag{2.4}$$

Table I gives the expressions of the coefficients c_1, c_2, $\cdots$ c_6, obtained by repeated application of the formula (2.4).

We come now to the positivity property of ρ. By definition, a Hermitian matrix is said to be positive if its proper values are non-negative.

From the above Newton formulas one sees directly that a necessary and sufficient condition for the matrix ρ to be positive is that the coefficients c_l of its characteristic polynomial should be non-negative:

$$\rho \text{ positive} \rightleftarrows c_l\geqslant 0\,. \tag{2.5}$$

By using result (2.4) (or Table I), the positivity conditions (2.5) can be given as conditions on the traces s_k. Table II gives these conditions[13] for trace-one matrices of dimension $n\leq 6$. An n-dimensional, Hermitian, positive matrix satisfies the inequalities of the $n-1$ first rows in Table II.[14] The first condition, $-s_2+1\geq 0$, is the well-known condition (1.15) $\mathrm{Tr}\rho^2\leq 1$. We see that for particles of spin $j>\frac{1}{2}$ (i.e., for $n=2j+1>2$), the positivity property implies further conditions. As we shall see in Sec. V these supplementary conditions are more restrictive than the first condition.

[12] See, for instance, F. R. Gantmakher, *The Theory of Matrices* (Chelsea Publishing Company, New York, 1960), Vol. I, p. 87.

[13] These conditions have also been obtained by D. N. Williams (unpublished).

[14] An n-dimensional, Hermitian matrix satisfies identically the relations obtained by replacing the symbol $\geq$ by the sign $=$ in the following rows.

III. EBERHARD-GOOD THEOREM

In Sec. I, we saw that the dynamical state of a sample S of particles of fixed energy-momentum and spin j is generally represented by a statistical mixture of pure states of polarization. The number N of independent pure states contributing to the state of the system depends on the way the sample is prepared. If this number N is less than $2j+1$, the density matrix which describes the polarization state of the particles has to satisfy some conditions. Indeed, the N pure states of polarization span an N-dimensional subspace of the $(2j+1)$-dimensional spin space. Consequently the density matrix constructed with these N states can be transformed by a unitary transformation into a matrix whose only non-null elements are in the N first rows and columns. Otherwise stated, the rank of this matrix is at most N; i.e., the matrix has at most N non-null proper values. This result is an extension of a theorem by Eberhard and Good.[15]

With the notation introduced in the preceding section we can derive completely the conditions imposed by the Eberhard-Good theorem on density matrices. Consider the characteristic polynomial $\Delta(\lambda)$ of an n-dimensional density matrix:

$$\Delta(\lambda)=\lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+(-)^n c_n\,.$$

A necessary and sufficient condition for the matrix ρ to be of rank N (i.e. for its characteristic polynomial to have $n-N$ roots equal to 0) is that the $n-N$ last coefficients should be null:

$$\rho \text{ rank } N \rightleftarrows 0=c_n=c_{n-1}=\cdots=c_{N+1}\,. \tag{3.1}$$

By using result (2.4) (or Table I), these conditions can be expressed as conditions on the traces s_k.

In the first section we have seen that for a $(2j+1)$-dimensional matrix the positivity property and the

TABLE II. Positivity conditions in terms of the traces $s_k=\mathrm{Tr}\rho^k$ of the powers of the matrix ρ $(s_1=1)$. An n-dimensional matrix must satisfy the conditions of the $n-1$ first rows of this table.

j	n	Positivity conditions
$\frac{1}{2}$	2	$-s_2+1\geqslant 0.$
1	3	$2s_3-3s_2+1\geqslant 0.$
$\frac{3}{2}$	4	$-6s_4+8s_3+3(s_2)^2-6s_2+1\geqslant 0.$
2	5	$24s_5-30s_4+20s_3-20s_3s_2+15(s_2)^2-10s_2+1\geqslant 0.$
$\frac{5}{2}$	6	$-120s_6+144s_5+90s_4s_2-90s_4+40(s_3)^2-120s_3s_2+40s_3-15(s_2)^3+45(s_2)^2-15s_2+1\geqslant 0.$

[15] P. Eberhard and M. L. Good, Phys. Rev. **120**, 1442 (1960); R. H. Capps, *ibid.* **122**, 929 (1961).

property $\mathrm{Tr}\rho=1$ lead to the condition $\mathrm{Tr}\rho^2 \geq (2j+1)^{-1}$. When the rank of the matrix is N $(N<2j+1)$ the minimum of $\mathrm{Tr}\rho^2$ is reached when the N non-null proper values are equal to N^{-1}, and this minimum is N^{-1}; thus

$$\rho \text{ rank } N \Rightarrow \mathrm{Tr}\rho^2 \geqslant N^{-1}. \tag{3.2}$$

This result was first derived by Peshkin.[16] It is to be noticed that in relation (3.2) we do not have the converse ($\Leftarrow$). Thus, condition (3.2) is less restrictive than the set of conditions (3.1), and does not express completely the information contained in the Eberhard-Good theorem.

IV. CONDITIONS ON MULTIPOLE PARAMETERS

In Secs. II and III we have seen that the positivity property and the Eberhard-Good theorem imposes conditions on the traces s_k of the powers of the density matrices. But what is experimentally measured is the set of multipole parameters $t_M{}^L$ of the density matrix. Thus it is interesting to give conditions on the multipole parameters directly. To this end we must express the traces s_k as functions of $t_M{}^L$. This can be easily done by application of the Racah calculus to the multipole expansion of the density matrix.

$$\rho=(2j+1)^{-1}\sum_{L,M}(2L+1)t_M{}^{L*}T_M{}^L, \tag{4.1}$$

with

$$(T_M{}^L)^m{}_n=(-)^{2j}(2j+1)^{1/2}\begin{pmatrix} M & m & j \\ L & j & n \end{pmatrix}. \tag{4.2}$$

In Appendix B we derive compact expressions for the traces s_k directly by calculating the trace of a product of k $T_M{}^L$'s. However, the formulas are very intricate and to obtain explicit expressions one has to expand a new formula for each power k. We prefer to use a step-by-step method which has the advantage that explicit expressions for s_k can be obtained by repeated application of a single formula. Furthermore, the latter method is particularly suited when one does not want a full algebraic expression for s_k as a function of $t_M{}^L$, but only a way to compute s_k when the numerical values of the parameters are known.

Let us consider the product of two matrices $T_M{}^L$:

$$(T_M{}^L T_{M'}{}^{L'})^m{}_n=(2j+1)\begin{pmatrix} M & m & j \\ L & j & m' \end{pmatrix}\begin{pmatrix} M' & m' & j \\ L' & j & n \end{pmatrix}.$$

By using (A13) we transform the product of 3-j symbols and obtain

$$(T_M{}^L T_{M'}{}^{L'})^m{}_n=(2j+1)^{1/2}\sum_{J,\mu}(-)^J(2J+1) \times\begin{Bmatrix} L & L' & J \\ j & j & j \end{Bmatrix}\begin{pmatrix} M & M' & J \\ L & L' & \mu \end{pmatrix}(-)^{2j}(T_\mu{}^J)^m{}_n, \tag{4.3}$$

[16] M. Peshkin, Phys. Rev. **123**, 637 (1961).

where $(T_\mu{}^J)^m{}_n$ is defined by (4.2).[17] Formula (4.3) allows one to transform any product of $T_M{}^L$ into a sum and to calculate the powers of the matrix ρ as functions of the multipole parameters. Indeed, consider ρ^2:

$$(\rho^2)^m{}_n=(2j+1)^{-2}\sum_{\substack{L,M\\L',M'}}(2L+1)(2L'+1)t_M{}^{L*}t_{M'}{}^{L'*} \times(T_M{}^L T_{M'}{}^{L'})^m{}_n. \tag{4.4}$$

By using (4.3) ρ^2 can be put in the form of a multipole expansion identical to (4.1)

$$(\rho^2)^m{}_n=(2j+1)^{-1}\sum_{L_2,M_2}(2L_2+1)t(2)_{M_2}{}^{L_2*} \times(T_{M_2}{}^{L_2})^m{}_n. \tag{4.5}$$

The multipole parameters $t(2)_{M_2}{}^{L_2}$ of ρ^2 are related to to the $t_M{}^L$ by the formula

$$t(2)_{M_2}{}^{L_2}=(-)^{2j+L_2}(2j+1)^{-1/2}\sum_{\substack{L,M\\L',M'}}(2L+1)(2L'+1) \times\begin{Bmatrix} L & L' & L_2 \\ j & j & j \end{Bmatrix}\begin{pmatrix} M & M' & L_2 \\ L & L' & M_2 \end{pmatrix}t_M{}^L t_{M'}{}^{L'}. \tag{4.6}$$

Higher powers of the matrix ρ are obtained by iterating the preceding calculation. All matrices ρ^n can be put in the form of a multipole expansion where the multipole parameters $t(n)_{M_n}{}^{L_n}$ are functions of the multipole parameters of lower matrices. For instance, performing the product $\rho^p\rho^q$, one obtains the nth power of ρ $(n=p+q)$:

$$\rho^n=(2j+1)^{-1}\sum_{L_n,M_n}(2L_n+1)t(n)_{M_n}{}^{L_n*}T_{M_n}{}^{L_n}, \tag{4.7}$$

with

$$t(n)_{M_n}{}^{L_n}=(-)^{2j+M_n}(2j+1)^{-1/2}\sum_{\substack{L_p,M_p\\L_q,M_q}}(2L_p+1)(2L_q+1) \begin{Bmatrix} L_p & L_q & L_n \\ j & j & j \end{Bmatrix}\begin{pmatrix} L_p & L_q & L_n \\ M_p & M_q & -M_n \end{pmatrix} \times t(p)_{M_p}{}^{L_p}t(q)_{M_q}{}^{L_q}. \tag{4.8}$$

When a matrix is in the form of a multipole expansion, its trace is equal to $t_0{}^0$; thus

$$s_n=t(n)_0{}^0, \tag{4.9}$$

and formula (4.8) gives directly

$$s_n=(-)^{2j}(2j+1)^{-1/2}\sum_{L,M}(2L+1)^2 \times\begin{Bmatrix} L & L & 0 \\ j & j & j \end{Bmatrix}\begin{pmatrix} L & L & 0 \\ M & -M & 0 \end{pmatrix}t(p)_M{}^L t(q)_{-M}{}^L.$$

[17] Formula (4.3) is the extension to arbitrary spin of a well-known formula in the spin-$\frac{1}{2}$ case: $(\boldsymbol{\sigma}\cdot\mathbf{n}_1)(\boldsymbol{\sigma}\cdot\mathbf{n}_2)=(\mathbf{n}_1\cdot\mathbf{n}_2)+i\boldsymbol{\sigma}\cdot\mathbf{n}_1\times\mathbf{n}_2$ where $\boldsymbol{\sigma}$ are the Pauli matrices and $\mathbf{n}_1$ and $\mathbf{n}_2$ are arbitrary vectors. Formula (4.3) has been given by L. C. Biedenharn, Ann. Phys. (N. Y.) 4, 104 (1958).

TABLE III. Positivity condition for the density matrix of spin-1 particles.

$-s_2+1\geqslant 0.$ $\qquad$ $2s_3-3s_2+1\geqslant 0.$
$3s_3=t(2)_0{}^0t_0{}^0+3t(2)_0{}^1t_0{}^1-6\mathrm{Re}(t(2)_1{}^1t_{-1}{}^1)+5t(2)_0{}^2t_0{}^2 -10\mathrm{Re}(t(2)_1{}^2t_{-1}{}^2)+10\mathrm{Re}(t(2)_2{}^2t_{-2}{}^2).$
$3s_2=(t_0{}^0)^2+3(t_0{}^1)^2+6\|t_1{}^1\|^2+5(t_0{}^2)^2+10\|t_1{}^2\|^2+10\|t_2{}^2\|^2.$
$t(2)_0{}^0=s_2$
$t(2)_0{}^1=\frac{2}{3}t_0{}^0t_0{}^1+\frac{1}{3}(\sqrt{10})t_0{}^1t_0{}^2-\frac{1}{3}(\sqrt{30})\,\mathrm{Re}(t_1{}^1t_{-1}{}^2).$
$t(2)_1{}^1=\frac{2}{3}t_0{}^0t_1{}^1-\frac{1}{3}(\sqrt{15})t_{-1}{}^1t_2{}^2+\frac{1}{6}(\sqrt{30})t_0{}^1t_1{}^2-\frac{1}{6}(\sqrt{10})t_1{}^1t_0{}^2.$
$t(2)_0=\frac{1}{10}(\sqrt{10})(t_0{}^1)^2-\frac{1}{10}(\sqrt{10})\|t_1{}^1\|^2+\frac{2}{3}t_0{}^0t_0{}^2-\frac{1}{6}(\sqrt{10})(t_0{}^2)^2 +\frac{1}{3}(\sqrt{10})\|t_2{}^2\|^2-\frac{1}{6}(\sqrt{10})\|t_1{}^2\|^2.$
$t(2)_1{}^2=\frac{1}{10}(\sqrt{30})t_0{}^1t_1{}^1+\frac{2}{3}t_0{}^0t_1{}^2+\frac{1}{3}(\sqrt{15})t_2{}^2t_{-1}{}^2-\frac{1}{6}(\sqrt{10})t_1{}^2t_0{}^2.$
$t(2)_2{}^2=\frac{1}{10}(\sqrt{15})(t_1{}^1)^2+\frac{2}{3}t_0{}^0t_2{}^2+\frac{1}{3}(\sqrt{10})t_2{}^2t_0{}^2-\frac{1}{6}(\sqrt{15})(t_1{}^2)^2.$

Using explicit expressions (A9a, A15) for the 3-j and 6-j symbols, we obtain

$$s_n=(2j+1)^{-1}\sum_{L,M}(2L+1)t(p)_M{}^Lt(q)_{-M}{}^L(-)^M. \qquad (4.10)$$

Thus to calculate s_n as function of $t_M{}^L$ one has first to calculate $t(p)_{M_p}{}^{L_p}$ and $t(q)_{M_q}{}^{L_q}$ as functions of $t_M{}^L$, and this can be done by repeated application of the formula (4.8). Tables III and IV collect the results for density matrices of spin-1 and -$\frac{3}{2}$ particles.[18] For higher spin values, similar tables can be constructed, but the algebraic expressions obtained when expanding formula (4.8) are somewhat cumbersome and will not be given here.

The advantage of the step-by-step method set forth in this section over the global method shown in Appendix B is that in the former method we have only the rather simple formula (4.8) to expand, whereas in the latter we have an intricate formlua to expand for each trace s_k.

Conditions on $t_M{}^L$ for a Pure State

A density operator ρ describes a pure state if it is a projector $\rho=|m\rangle\langle m|$, i.e., if we have $\rho^2=\rho$. This condition can be easily expressed in terms of multipole parameters. To have $\rho^2=\rho$, it is necessary and sufficient that the parameters $t(2)_M{}^L$ of the multipole expansion of ρ^2 should be equal to the parameters $t_M{}^L$ of the multipole expansion of ρ. From formula (4.6), which gives $t(2)_M{}^L$ as functions of $t_M{}^L$, we conclude that the matrix ρ describes a pure state if its multipole parameters satisfy the set of conditions[19]

$$t_M{}^L=(-)^{2j+L}(2j+1)^{-1/2}\sum_{\substack{L',M'\\L'',M''}}(2L'+1)(2L''+1)\times\begin{Bmatrix}L' & L'' & L\\ j & j & j\end{Bmatrix}\begin{pmatrix}M' & M'' & L\\ L' & L'' & M\end{pmatrix}t_{M'}{}^{L'}t_{M''}{}^{L''}. \qquad (4.11)$$

It is known that a necessary and sufficient condition

TABLE IV. Positivity condition for the density matrix of spin-$\frac{3}{2}$ particles.

$-s_2+1\geqslant 0$ $\qquad$ $-6s_4+8s_3+3(s_2)^2-6s_2+1\geqslant 0$ $\qquad$ $2s_3-3s_2+1\geqslant 0$
$4s_4=(t(2)_0{}^0)^2+3(t(2)_0{}^1)^2+6\|t(2)_1{}^1\|^2+5(t(2)_0{}^2)^2+10\|t(2)_1{}^2\|^2+10\|t(2)_2{}^2\|^2+7(t(2)_0{}^3)^2+14\|t(2)_1{}^3\|^2+14\|t(2)_2{}^3\|^2+14\|t(2)_3{}^3\|^2$
$4_3s=t_0{}^0t(2)_0{}^0+3t_0{}^1t(2)_0{}^1-6\mathrm{Re}(t_1{}^1t(2)_{-1}{}^1)+5t_0{}^2t(2)_0{}^2-10\mathrm{Re}(t_1{}^2t(2)_{-1}{}^2)+10\mathrm{Re}(t_2{}^2t(2)_{-2}{}^2)+7t_0{}^3t(2)_0{}^3-14\mathrm{Re}(t_1{}^3t(2)_{-1}{}^3) +14Re(t_2{}^3t(2)_{-2}{}^3)-14\mathrm{Re}(t_3{}^3t(2)_{-3}{}^3).$
$4s_2=(t_0{}^0)^2+3(t_0{}^1)^2+6\|t_1{}^1\|^2+5(t_0{}^2)^2+10\|t_1{}^2\|^2+7(t_0{}^3)^2+14\|t_1{}^3\|^2+14\|t_2{}^3\|^2+14\|t_3{}^3\|^2.$
$t(2)_0{}^0=s_2$
$t(2)_0{}^1=\frac{1}{2}t_0{}^0t_0{}^1+\frac{2}{5}(\sqrt{5})t_0{}^1t_0{}^2-\frac{2}{5}(\sqrt{15})\mathrm{Re}(t_{-1}{}^1t_1{}^2)+\frac{1}{10}(\sqrt{105})t_0{}^2t_0{}^3-(2/15)(\sqrt{210})\mathrm{Re}(t_{-1}{}^2t_1{}^3)+\frac{1}{3}(\sqrt{21})\mathrm{Re}(\mathrm{t}_{-2}{}^2t_2{}^3).$
$t(2)_1{}^1=\frac{1}{2}t_0{}^0t_1{}^1-\frac{1}{5}(\sqrt{30})t_{-1}{}^1t_2{}^2+\frac{1}{5}(\sqrt{15})t_0{}^1t_1{}^2-\frac{1}{5}(\sqrt{5})t_1{}^1t_0{}^2+\frac{1}{2}(\sqrt{7})t_{-2}{}^2t_3{}^3 -\frac{1}{6}(\sqrt{42})t_{-1}{}^2t_2{}^3+\frac{1}{10}(\sqrt{70})t_0{}^2t_1{}^3-\frac{1}{10}(\sqrt{35})t_1{}^2t_0{}^3+1/30(\sqrt{105})t_2{}^2t_{-1}{}^3.$
$t(2)_0{}^2=\frac{1}{2}t_0{}^0t_0{}^2+(3/25)(\sqrt{5})(t_0{}^1)^2-(3/25)(\sqrt{5})\|t_1{}^1\|^2+(3/50)(\sqrt{105})t_0{}^1t_0{}^3-(3/25)(\sqrt{70})\mathrm{Re}(t_{-1}{}^1t_1{}^3)-(7/25)(\sqrt{5})(t_0{}^3)^2 -21/50(\sqrt{5})\|t_1{}^3\|^2+7/10(\sqrt{5})\|t_3{}^3\|^2.$
$t(2)_1{}^2=\frac{1}{2}t_0{}^0t_1{}^2+(3/25)(\sqrt{15})t_0{}^1t_1{}^1+(1/25)(\sqrt{210})t_0{}^1t_1{}^3-\frac{1}{10}(\sqrt{42})t_{-1}{}^1t_2{}^3-(3/50)(\sqrt{35})t_1{}^1t_0{}^3-(7/10)(\sqrt{5})t_3{}^3t_{-2}{}^3 +(7/10)(\sqrt{3})t_2{}^3t_{-1}{}^3-(7/50)(\sqrt{10})t_1{}^3t_0{}^3.$
$t(2)_2{}^2=\frac{1}{2}t_0{}^0t_2{}^2+(3/50)(\sqrt{30})(t_1{}^1)^2-(3/10)(\sqrt{7})t_{-1}{}^1t_3{}^3+\frac{1}{10}(\sqrt{21})t_0{}^1t_2{}^3-(1/50)(\sqrt{105})t_1{}^1t_1{}^3-(7/10(\sqrt{2})t_3{}^3t_{-1}{}^3 +(7/5)t_0{}^3t_2{}^3-(7/50)(\sqrt{30})(t_1{}^3)^2.$
$t(2)_0{}^3=(3/70)(\sqrt{105})t_0{}^1t_0{}^2+(3/35)(\sqrt{35})\mathrm{Re}(t_1{}^1t_{-1}{}^2)-\frac{2}{5}(\sqrt{5})t_0{}^2t_0{}^3+\frac{1}{5}(\sqrt{10})\mathrm{Re}(t_1{}^2t_{-1}{}^3)+2\mathrm{Re}(t_2{}^2t_{-2}{}^3).$
$t(2)_1{}^3=(1/70)(\sqrt{105})t_{-1}{}^1t_2{}^2+(1/35)(\sqrt{210})t_0{}^1t_1{}^2+(3/70)(\sqrt{70})t_1{}^1t_0{}^2+\frac{1}{2}(\sqrt{2})t_{-2}{}^2t_3{}^3+\frac{1}{2}(\sqrt{3})t_{-1}{}^2t_2{}^3-(3/10)(\sqrt{5})t_0{}^2t_1{}^3 -\frac{1}{10}(\sqrt{10})t_1{}^2t_0{}^3+\frac{1}{5}(\sqrt{30})t_2{}^2t_{-1}{}^3.$
$t(2)_2{}^3=(1/14)(\sqrt{21})t_0{}^1t_2{}^2+(1/14)(\sqrt{42})t_1{}^1t_1{}^2+\frac{1}{2}(\sqrt{5})t_{-1}{}^2t_3{}^3-\frac{1}{2}(\sqrt{3})t_1{}^2t_1{}^3+t_2{}^2t_0{}^3.$
$t(2)_3{}^3=(3/14)(\sqrt{7})t_1{}^1t_2{}^2+\frac{1}{2}(\sqrt{5})t_0{}^2t_3{}^3-\frac{1}{2}(\sqrt{5})t_1{}^2t_2{}^3+\frac{1}{2}(\sqrt{2})t_2{}^2t_1{}^3.$

[18] To construct these tables we have used the tables of 3-j and 6-j symbols established by M. Rotenberg, A. Bivins, N. Metropolis, and J. K. Wooten, Jr., *The 3-j and 6-j Symbols* (Crosby Lockwood & Son, Ltd., London, 1959).

[19] These relations have been obtained by L. C. Biedenharn (Ref. 17).

for a *positive* density matrix to describe a pure state is $\mathrm{Tr}\rho^2=\mathrm{Tr}\rho=1$. Thus, the set of conditions on $t_M{}^L$ expressing the fact that the density matrix is positive, together with the condition which expresses the restriction $\mathrm{Tr}\rho^2=1$, are equivalent to the set of conditions (4.11). But of course, the condition $\mathrm{Tr}\rho^2=1$ alone, without positivity conditions, is not sufficient to insure that we have a pure state.

V. THE POSITIVITY DOMAIN

Let us introduce some vocabulary. A given $n\times n$ Hermitian matrix is completely characterized by the set of its multipole parameters $t_M{}^L$. To each set of parameters, i.e., to each matrix, we associate a *representative point* of an abstract space, which we call the *space of parameters*. The Hermitian matrix will be positive if its multipole parameters satisfy the positivity conditions derived in the preceding section, i.e., if its representative point is in a given domain of the space of parameters. This domain will be called the *positivity domain*.

An interesting question to ask is: What is the respective contribution of each positivity condition to the determination of the positivity domain? To answer this question, we shall first study a simple example, the one of spin-1 matrix, and then consider the general case.

In a recent letter,[20] we have studied the positivity domain of the density matrix of spin-1 particles. We recall here the fundamental results. In fact, this simple example can be completely worked out, and it exhibits the features of the general case. For a spin-1 matrix the positivity conditions including all multipole parameters can be explicitly written with Table III. The results are not too complicated; however, to obtain more compact expressions one can take advantage of some restrictions imposed on $t_M{}^L$ by symmetry properties of the production process. We assume that the spin-1 particle is produced in a parity-conserving two-body reaction,[1] and we choose the axis of quantization along the normal to the production plane. Then, if the initial particles are unpolarized and if we average over the spin of the other produced particles, we have

$$t_M{}^L\equiv 0 \quad \text{for} \quad \text{odd } M. \tag{5.1}$$

Thus the density matrix is

$$\rho=\tfrac{1}{3}\begin{bmatrix} 1+(3/\sqrt{2})t_0{}^1+\tfrac{1}{2}(\sqrt{10})t_0{}^2 & 0 & (\sqrt{15})t_2{}^2 \\ 0 & 1-(\sqrt{10})t_0{}^2 & 0 \\ (\sqrt{15})t_2{}^{2*} & 0 & 1-(3/\sqrt{2})t_0{}^1+\tfrac{1}{2}(\sqrt{10})t_0{}^2 \end{bmatrix}. \tag{5.2}$$

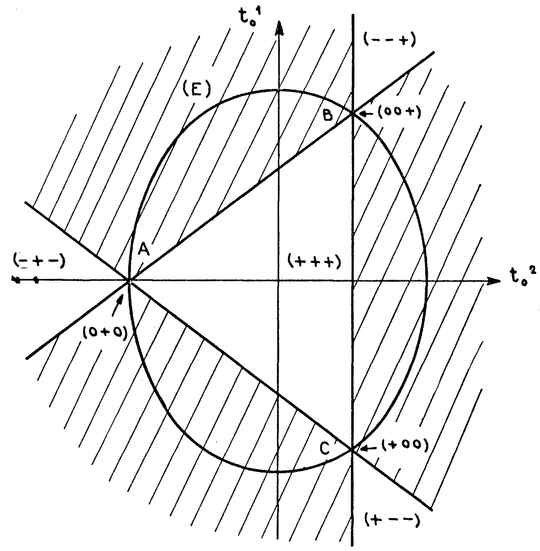

FIG. 1. Positivity domain for the density matrix of spin-1 particles described by the parameters $t_0{}^1$ and $t_0{}^2$. The straight lines correspond to $c_3=0$; the ellipse E corresponds to $c_2=0$. The points A, B, C are representatives of pure states.

From Table III, the first positivity condition ($c_2\geq 0$ or $\mathrm{Tr}\rho^2\leq 1$) can be written thus:

$$3(t_0{}^1)^2+5(t_0{}^2)^2+10|t_2{}^2|^2-2\leqslant 0. \tag{5.3}$$

The second positivity condition ($c_3\geq 0$ or $2\mathrm{Tr}\rho^3-3\mathrm{Tr}\rho^2+1\geq 0$) can also be obtained from Table III; but it is most easily derived by noting the $c_3=\det\rho$. From (5.2), we obtain directly the condition

$$[1-(\sqrt{10})t_0{}^2]\{[1+\tfrac{1}{2}(\sqrt{10})t_0{}^2]^2-\tfrac{9}{2}(t_0{}^1)^2-15|t_2{}^2|^2\}\geqslant 0. \tag{5.4}$$

Conditions (5.3) and (5.4) allow one to draw the positivity domain in the three-dimensional space of the parameters $t_0{}^1$, t_0, $^2|t_2{}^2|$. (See Fig. 1 of Ref. 20.) However, we can further simplify the expressions and the drawings without losing generality, by considering these conditions when $t_2{}^2=0$. Then, the space of parameters is reduced to a plane and the density matrix is diagonal. The second positivity condition ($c_3=\det\rho\geq 0$) becomes

$$\left(1+\frac{3}{\sqrt{2}}t_0{}^1+\tfrac{1}{2}(\sqrt{10})t_0{}^2\right)(1-(\sqrt{10})t_0{}^2)\times\left(1-\frac{3}{\sqrt{2}}t_0{}^1+\tfrac{1}{2}(\sqrt{10})t_0{}^2\right)\geqslant 0. \tag{5.5}$$

The equation $c_3=\det\rho=0$ is the equation of 3 straight lines which separate the plane ($t_0{}^1$, $t_0{}^2$) into several

[20] P. Minnaert, Phys. Rev. Letters **16**, 672 (1966). Results similar to those given in this letter had already been published by W. Lakin, Phys. Rev. **98**, 139 (1955), and displayed graphically by J. Raynal, Centre d'Etudes Nucléaires de Saclay, Report No. CEA 2287 (unpublished). We thank Dr. D. Zwanziger for pointing out these references. See also D. Zwanziger, Phys. Rev. **136**, B558 (1964).

domains, in each of which the set of proper values has a definite signature.[21] See Fig. 1. The shaded regions, in this figure, are those which are eliminated by (5.5). We see that (5.5) allows four regions in the plane (t_0^1,t_0^2) but only one region (the one connected with the origin) has the appropriate signature for the matrix to be positive. Indeed, the other three regions are eliminated by the first positivity condition,

$$3(t_0^1)^2+5(t_0^2)^2-2\leqslant 0\,,$$

which allows only the region inside the ellipse (E) circumscribed about the triangle ABC. Thus, *the bounds of the positivity domain are given by the equation* $c_3=0$; the first positivity condition is needed only to eliminate the region where $c_3>0$ but where the signature is not $(+++)$. That the points A, B, C should be on the ellipse (E) is obvious, for at these points two proper values of ρ are null; consequently not only the coefficient c_3, but also the coefficient c_2 of the characteristic polynomial is null. Since the equation of the ellipse (E) is $c_2=0$, the points A, B, C are on this ellipse.

From the example of the spin-1 matrices, one can see very well how the positivity conditions work in the general case. Let us consider the space of parameters associated with an $n\times n$ Hermitian matrix. The equation $c_n=\det\rho=0$ is the equation of a surface in this space. This surface divides the space into several domains, in each of which the set of proper values has a definite signature $[N_+,N_0,N_-]$ ($N_0\neq 0$ on the surface $c_n=0$). The positivity condition $c_n\geq 0$ eliminates the regions where N_- is odd and allows all the regions where N_- is even. However, the matrix will be positive only if $N_-=0$. Now it is obvious that the origin of the space of parameters ($t_M{}^L=0$, $L\neq 0$) corresponds to a positive matrix, and it is physically intuitive that the positivity domain is connected.[22] Consequently the density matrix will be positive if its representative point belongs to the domain connected to the origin limited by the surface $c_n=0$.

Thus as we have seen in the spin-1 case, *the bounds of the positivity domain are given by the equation* $c_n=\det\rho=0$. The other positivity conditions $c_{n-1}\geq 0$, $c_{n-2}\geq 0$, $\cdots$, $c_2=-\mathrm{Tr}\rho^2+1\geq 0$ eliminate all the regions where $c_n\geq 0$ but where $N_-\neq 0$.

Let us make a few remarks on the geometry of the surfaces $c_l=0$ in the space of parameters $t_M{}^L$. Generally the surface $c_n=\det\rho=0$ has several singularities. The type of each singularity can be characterized by the value of the number N_0 at this point. On the surface we have $N_0=1$; at a "double point singularity" we have $N_0=2$. Now, we have $N_0=2$ (2 proper values$=0$) if $c_{n-1}=0$. Thus the surface $c_{n-1}=0$ intersects the surface $c_n=0$ along the manifold of double point singularities. Similarly, the surface $c_{n-2}=0$ intersects the surface $c_n=0$ along the manifold of triple point singularities ($N_0=3$)$\cdots$ and the surface $c_2=0$ intersects the surface $c_n=0$ along the manifold on which $N_0=n-1$. Each of these manifolds corresponds to a matrix of definite rank. The last manifold ($N_0=n-1$) is the manifold of pure states. Its equations are explicitly given by (4.11).

It should be noticed that a given manifold ($N_0=\alpha$) may not intersect the subspace defined by some $t_M{}^L=0$. If this is the case, then a matrix defined by only a few parameters, the others being null, cannot have rank α. For instance, in the example of a spin-1 particle, the matrix defined by the single parameter t_0^1 cannot have rank 1, i.e., it cannot describe a pure state (see Fig. 1). We shall give further examples in Sec. VI.

VI. APPLICATIONS

A. Positivity Conditions and Eberhard-Good Theorem for the Density Matrix of Spin-$\frac{3}{2}$ Particles

Let us consider the spin-$\frac{3}{2}$ particles B^* [$N^*(1236)$, $Y^*(1385)$, $\Xi^*(1530)$] produced in reactions of the type

$$P+N\rightarrow B^*+P\,, \qquad (6.1)$$

where P stands for pseudoscalar mesons (K,π), and N for the nucleons. The common feature of these reactions is that the spin state of the B^* produced at a fixed angle is a statistical mixture of 2 pure polarization states, corresponding to the 2 states of the initial nucleon. Consequently the density matrix of the B^* is of rank 2, i.e., the coefficients c_4 and c_3 of its characteristic polynomial are null. The density matrix including all parameters $t_M{}^L$ (M even) is

$$\rho=\begin{bmatrix}\rho_{33} & 0 & \rho_{3-1} & 0\\ 0 & \rho_{11} & 0 & \rho_{1-3}\\ \rho_{3-1}{}^* & 0 & \rho_{-1-1} & 0\\ 0 & \rho_{1-3}{}^* & 0 & \rho_{-3-3}\end{bmatrix}, \qquad (6.2)$$

with

$$4\rho_{33}=1-(9/\sqrt{(15)})t_0^1+(\sqrt{5})t_0^2-(7/\sqrt{(35)})t_0^3\,,$$
$$4\rho_{33}=1-(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2-(21/\sqrt{(35)})t_0^3\,,$$
$$4\rho_{-1-1}=1+(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2-(21/\sqrt{(35)})t_0^3\,,$$
$$4\rho_{-3-3}=1+(9/\sqrt{(15)})t_0^1+(\sqrt{5})t_0^2+(7/\sqrt{(35)})t_0^3\,, \qquad (6.3a)$$

and

$$4\rho_{3-1}=(\sqrt{(10)})t_2^2-(\sqrt{(14)})t_2^3\,,$$
$$4\rho_{1-3}=(\sqrt{(10)})t_2^2+(\sqrt{(14)})t_2^3\,. \qquad (6.3b)$$

Writing explicitly the characteristic polynomial of ρ in terms of the matrix elements $\rho_{2m,2n}$ one obtains easily the coefficients c_4, c_3, and c_2.

---

[21] By the signature of the set of proper values we mean the set of signs of the proper values. We shall denote this signature either by the series of signs of the proper values, or by the triplet of numbers (N_+,N_0,N_-), where N_+ (N_0,N_-) is the number of positive (null, negative) proper values.

[22] The set of positive matrices is convex, i.e., if ρ_1 and ρ_2 are positive, the matrix $\alpha\rho_1+(1-\alpha)\rho_2$ $(0\leq\alpha\leq 1)$ is also positive. Consequently, the multipole parametrization being linear, the positivity domain is convex and *a fortiori* connected.

$$c_4=(\rho_{33}\rho_{-1-1}-|\rho_{3-1}|^2)(\rho_{11}\rho_{-3-3}-|\rho_{1-3}|^2), \qquad (6.4a)$$

$$c_3=(\rho_{33}+\rho_{-1-1})(\rho_{11}\rho_{-3-3}-|\rho_{1-3}|^2)+(\rho_{11}+\rho_{-3-3})(\rho_{33}\rho_{-1-1}-|\rho_{3-1}|^2), \qquad (6.4b)$$

$$c_2=(\rho_{33}\rho_{-1-1}-|\rho_{3-1}|^2)+(\rho_{11}\rho_{-3-3}-|\rho_{1-3}|^2)+(\rho_{33}+\rho_{-1-1})(\rho_{11}+\rho_{-3-3}). \qquad (6.4c)$$

The system of equations

$$c_4=0, \quad c_3=0, \qquad (6.5)$$

has 3 types of solutions.

(1) $\rho_{33}\rho_{-1-1}-|\rho_{3-1}|^2=0$ and $\rho_{33}+\rho_{-1-1}=0$, (6.6)

i.e., $\rho_{33}=-\rho_{-1-1}=0$ and $\rho_{3-1}=0$.

In terms of multipole parameters this solution is

$$\begin{aligned}1-(9/\sqrt{(15)})t_0^2+(\sqrt{5})t_0^2-(7/\sqrt{(35)})t_0^3&=0,\\ 1+(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2-(21/\sqrt{(35)})t_0^3&=0,\\ (\sqrt{(10)})t_2^2-(\sqrt{(14)})t_2^3&=0.\end{aligned} \qquad (6.7)$$

(2) $\rho_{11}\rho_{-3-3}-|\rho_{1-3}|^2=0$ and $\rho_{11}+\rho_{-3-3}=0$,

i.e., (6.8)

$\rho_{11}=-\rho_{-3-3}=0$ and $\rho_{1-3}=0$.

In terms of multipole parameters this solution is

$$\begin{aligned}1-(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2+(21/\sqrt{(35)})t_0^3&=0,\\ 1+(9/\sqrt{(15)})t_0^1+(\sqrt{5})t_0^2+(7/\sqrt{(35)})t_0^3&=0,\\ (\sqrt{(10)})t_2^2+(\sqrt{(14)})t_2^3&=0.\end{aligned} \qquad (6.9)$$

In both solutions (1) and (2) the relations between the parameters are such that the density matrix can be parametrized with one real parameter and one complex parameter. It is to be noticed that these two solutions are not basically different, since we go from one solution to the other by changing the signs of the odd-L parameters.

(3) $\rho_{33}\rho_{-1-1}-|\rho_{3-1}|^2=0$ and $\rho_{11}\rho_{3-3}-|\rho_{1-3}|^2=0$. (6.10)

In terms of multipole parameters this solution is

$$\begin{aligned}&[1-(9/\sqrt{(15)})t_0^1+(\sqrt{5})t_0^2-(7/\sqrt{(35)})t_0^3]\\ &\quad\times[1+(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2-(21/\sqrt{(35)})t_0^3]\\ &\quad-|\sqrt{(10)}t_2^2-\sqrt{(14)}t_2^3|^2=0, \qquad (6.11)\\ &[1-(3/\sqrt{(15)})t_0^1-(\sqrt{5})t_0^2+(21/\sqrt{(35)})t_0^3]\\ &\quad\times[1+(9/\sqrt{(15)})t_0^1+(\sqrt{5})t_0^2+(7/\sqrt{(35)})t_0^3]\\ &\quad-|\sqrt{(10)}t_2^2+\sqrt{(14)}t_2^3|^2=0.\end{aligned}$$

Moreover in all three solutions, the multipole parameters must satisfy the first positivity condition ($c_2\geq 0$ or $\mathrm{Tr}\rho^2\leq 1$)

$$3(t_0^1)^2+5(t_0^2)^2+10|t_2^2|^2+7(t_0^3)^2+14|t_2^3|^2-3\leqslant 0. \qquad (6.12)$$

Thus, the multipole parameters experimentally measured in reaction (6.1) must satisfy (6.12) and one of the three sets of relations (6.7), (6.9), and (6.11). Which one? This is fixed by the peculiar structure that must be exhibited by the density matrix of the B^* produced at a fixed angle in reaction (6.1). Indeed, Bohr's theorem[23] [invariance under the operation $R\equiv$(space reflection)$\times$(180° rotation around the normal to the production plane)] requires the 4×2 transition matrix in spin space T to have the form

$$T=\begin{bmatrix}0 & a\\ b & 0\\ 0 & c\\ d & 0\end{bmatrix}.$$

Thus, the density matrix of the B^* produced from an unpolarized target is

$$\rho=\tfrac{1}{2}TT^\dagger=\tfrac{1}{2}\begin{bmatrix}aa^* & 0 & ac^* & 0\\ 0 & bb^* & 0 & bd^*\\ ca^* & 0 & cc^* & 0\\ 0 & db^* & 0 & dd^*\end{bmatrix};$$

i.e., ρ is the sum of two rank-one submatrices.[5] Consequently, only the third solution (6.11), which expresses this peculiar structure of the matrix ρ, is physical. The other solutions are unphysical; however, if *in addition* to the relations (6.11), the parameters satisfy one of the sets (6.7) and (6.9), then one of the submatrices is null, and the rank of ρ is 1, i.e., the matrix ρ describes a pure state of polarization.

We emphasize that in the study of a production reaction like (6.1) we can only apply the Eberhard-Good theorem to the matrices describing particles B^* produced with fixed kinematics. This fact reduces the usefulness of the conditions in the case of poor statistics.

The results derived in this section can also be applied to the study of the formation experiments of the type

$$P+B\rightarrow B^*\rightarrow P+B.$$

The density matrix of the B^* formed must also have rank 2.

B. Positivity Conditions and the Method of Lee and Yang

Positivity conditions, in the form which we have given, can be applied only when one knows the density matrix completely. However, in 1958, Lee and Yang[24] derived several inequalities, very useful for the determination of spin of the baryons, whose application does not require a complete knowledge of the density matrix. The derivation of these inequalities implicitly makes use of the positivity property of the density matrix. As a matter of fact, Lee and Yang discuss the decay of a sample S of baryons, but they do not con-

[23] A. Bohr, Nucl. Phys. **10**, 486 (1959).

[24] T. D. Lee and C. N. Yang, Phys. Rev. **109**, 1755 (1958). This work was extended by L. Durand, III, L. F. Landovitz, and J. Leitner, Phys. Rev. **112**, 273 (1958) to the study of the polarization of the product particle.

sider the azimuthal distribution of the decay products. So the spin state of the sample S can be considered as an *incoherent* statistical mixture of states $|jm\rangle$, i.e., the density matrix of the sample is not completely determined; one knows only the diagonal elements.

The inequalities follow from the requirement $I_m \geq 0$, where I_m is the statistical weight of the state $|jm\rangle$ contributing to the incoherent mixture. In the language of density matrix, it is equivalent to requiring the diagonal matrix elements to be positive. It is very easy to show explicitly the equivalence of the Lee-Yang inequalities and the positivity conditions in the case $j=\frac{3}{2}$. The Lee-Yang inequalities are (see their relations 8 and 18)

$$I_m \geqslant 0, \quad (m=\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2}), \tag{6.13}$$

with

$$\begin{aligned} 4I_{3/2}&=1-(9/\alpha)\langle P_1\rangle-5\langle P_2\rangle+(7/3\alpha)\langle P_3\rangle,\\ 4I_{1/2}&=1-(3/\alpha)\langle P_1\rangle+5\langle P_2\rangle-(7/\alpha)\langle P_3\rangle,\\ 4I_{-1/2}&=1+(3/\alpha)\langle P_1\rangle+5\langle P_2\rangle+(7/\alpha)\langle P_3\rangle,\\ 4I_{-3/2}&=1+(9/\alpha)\langle P_1\rangle-5\langle P_2\rangle-(7/3\alpha)\langle P_3\rangle, \end{aligned} \tag{6.14}$$

where $\langle P_L\rangle$ is the average over the angular distribution of the Legendre polynomials P_L, and α is the asymmetry parameter of the decaying baryon.

The positivity conditions are

$$\rho_{2m,2m} \geqslant 0; \quad (m=\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2}) \tag{6.15}$$

with $\rho_{2m,2m}$ given by (6.3a).

To show the equivalence of the two sets of conditions (6.13) and (6.15), it suffices to use the relations

$$\begin{aligned} \langle P_1\rangle&=\alpha(1/\sqrt{(15)})t_0{}^1,\\ \langle P_2\rangle&=-(1/\sqrt{5})t_0{}^2,\\ \langle P_3\rangle&=-3\alpha(1/\sqrt{(35)})t_0{}^3, \end{aligned}$$

obtained from

$$P_L=[4\pi/(2L+1)]^{1/2}Y_0{}^L,$$

and from the Byers-Fenster[2] relations

$$\begin{aligned} \langle Y_0{}^L\rangle&=n_{L0}t_0{}^L, \quad L \text{ even},\\ \langle Y_0{}^L\rangle&=\alpha n_{L0}t_0{}^L, \quad L \text{ odd}, \end{aligned}$$

with

$$n_{L0}=(-)^{j-1/2}[(2j+1)/4\pi]^{1/2}(2L+1)^{1/2}\begin{pmatrix} j & j & L\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}.$$

In Ref. 2 of their paper, Lee and Yang noted that *if the azimuthal distribution is also studied, more inequalities can be derived.* These inequalities are precisely the positivity conditions derived in the present paper for complete density matrices.

Finally, let us make a remark on the use of the Lee and Yang inequalities. Generally one considers the test functions T_{jm} whose average over the angular distribution must be less than 1. The great advantage of these test functions is that they can be used without any knowledge of the asymmetry parameter α. However, when this asymmetry parameter is known (it is the case for Ξ hyperons) it is more interesting to consider directly the positivity condition $I_{jm}\geq 0$ which give more information than the conditions on the test functions $\langle T_{jm}\rangle \leq 1$.

For instance, an analysis of the decay of 182 Ξ^- has given[25]

$$\begin{aligned} \alpha_\Xi&=0.44\pm0.11,\\ \langle T_{\frac{3}{2}\frac{3}{2}}\rangle&=1.05\pm0.33. \end{aligned}$$

This result does not rule out $j=\frac{3}{2}$ with a significant probability (12%). Now, it we suppose that the major contribution to $\langle T_{jm}\rangle$ arises from $\langle P_1\rangle$ and if we use the less favorable value of α_Ξ ($\alpha_\Xi=0.44+(2.5\times0.11)=0.715$) we obtain

$$I_{\frac{3}{2}\frac{3}{2}}=-0.47\pm0.33.$$

(We suppose that the statistical error is not modified.) This value violates the condition $I_{\frac{3}{2}\frac{3}{2}}\geq 0$ by 1.40 standard deviations. Thus, with this rough calculation we eliminate $j=\frac{3}{2}$ with 84% probability, improving the above result considerably.

C. An Application of Eberhard-Good Theorem

The determination of the spin of the Ξ hyperons has been an unsolved problem for a long time. The more recent analyses of experimental data[26] favor considerably the spin-$\frac{1}{2}$ assignment, but they do not exclude absolutely the possibility that the spin is $\frac{3}{2}$. In this section we derive a result which can be of some help in this respect. We consider the density matrix of Ξ hyperons produced in the parity-conserving reaction

$$K^-+p\to K^++\Xi^-. \tag{6.16}$$

We suppose that we have chosen the axis of quantization such that $t_M{}^L=0$ for odd M [see Eq. (5.1)]. The experimental data are as follows: At each production angle the parameter $t_0{}^1$ is non-null (its magnitude, proportional to the vector polarization, depends on the K^- momentum and on the production angle) and all other parameters $t_M{}^L$ ($L>1$) are *compatible* with zero. We shall *suppose* that they are *rigorously null.*

One cannot draw any conclusion from these simple facts. Indeed, $t_M{}^L=0$ for $L>1$ is a necessary condition for the spin to be $\frac{1}{2}$, but we may have the case of a higher spin ($\frac{3}{2}$, $\frac{5}{2}$, $\cdots$) which by accident has less than the maximum possible complexity in its spin distribution. However, we have one more piece of information on the spin state of the Ξ's produced at a fixed angle in

[25] U. Nguyen-Khac, thesis, Ecole Polytechnique, Paris, 1964 (unpublished). We thank Dr. Louis Jauneau and Dr. Nguyen-Khac for a discussion of the experimental data.

[26] D. D. Carmony, G. M. Pjerrou, P. E. Schlein, W. E. Slater, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **12**, 482 (1964); J. Button-Shafer and D. W. Merrill, University of California Radiation Laboratory Report No. UCRL 11884, 1965 (unpublished).

the reaction (6.16). We know that it is a statistical mixture of 2 pure polarization states corresponding to the 2 initial proton states. Thus we know, from the Eberhard-Good theorem (Sec. III), that the density matrix of the Ξ's must have rank 2. Now the density matrix $\rho(j)$ (for spin j) constructed with the single parameter $t_0{}^1$ is diagonal, the diagonal matrix elements being [see (1.9), (1.10), and (A9b)]

$$\rho(j)^m{}_m=(2j+1)^{-1}\{1+(-)^{2j}3t_0{}^1m[j(j+1)]^{-1/2}\}. \quad (6.17)$$

From this form of the matrix elements, it is obvious that, whatever the value of $t_0{}^1$, we can have at most one null diagonal matrix element. Consequently the rank of the matrix $\rho(j)$ reduces at most from $2j+1$ to $2j$. Thus if the rank of $\rho(j)$ has to be 2, the dimension of the matrix is at most 2 or 3; i.e., the spin is $\frac{1}{2}$ or 1. Since in our case we have to decide between half-integer spins, we eliminate the spin 1 and we remain with the sole possibility that $j=\frac{1}{2}$. This result can be stated[27]:

If the polarization of the Y^* particles produced in the parity-conserving reaction

$$K^-+p\rightarrow Y^*+\pi(K)$$

is described by the parameter $t_0{}^1$ alone, then the spin of the Y^* is $\frac{1}{2}$.

Direct application of this result to Ξ hyperons is somewhat difficult, for the experimental parameters $t_M{}^L(L>1)$ are *not rigorously null.* However, if we suppose $j>\frac{1}{2}$, the requirement that the rank of the density matrix should be 2 imposes very strong conditions on the multipole parameters. These conditions were derived for $j=\frac{3}{2}$ in Eq. (6.11).

The present result is an illustration of the last remark of Sec. V, namely, when an n-dimensional matrix is defined by only a few parameters (the others being null) its rank cannot always take any arbitrary value. This can be understood in another way. An n-dimensional density matrix is defined by n^2-1 *real* parameters. Suppose that q of these parameters are null (i.e., the density matrix is defined by $p=(n^2-1)-q$ parameters) and that we ask that the matrix should have rank r (i.e., $n-r$ proper values are null); then we have $q+n-r$ relations between n^2-1 quantities. If $q+n-r>n^2-1$ i.e., if

$$n-r-p>0,$$

the system is impossible unless some of the $n-r$ relations are equivalent. Moreover, the parameters must of course satisfy the conditions prescribed by the positivity requirement.

Let us consider another illustration of these facts. We consider again the density matrix of Y^* produced in the reaction

$$K^-+p\rightarrow Y^*+\pi.$$

Let us suppose that for a given production angle the density matrix of the Y^* is completely defined by the real parameter $t_0{}^2$ and the complex parameter $t_2{}^2$ ($p=3$), i.e., the particle has only quadrupole polarization. The Eberhard-Good theorem provides $n-r$ relations ($r=2$) between the three real quantities $t_0{}^2$, $\mathrm{Re}t_2{}^2$, and $\mathrm{Im}t_2{}^2$. If we suppose $j=\frac{3}{2}$ ($n=4$), then $n-r-p=-1$ (<0), and the system is not completely determined. The two relations between the parameters are given by the set (6.11). It is easily seen that in this case both relations are identical and give

$$5(t_0{}^2)^2+10|t_2{}^2|^2-1=0.$$

This is compatible with the positivity condition

$$5(t_0{}^2)^2+10|t_2{}^2|^2-3\leqslant 0.$$

If we suppose $j=\frac{5}{2}$ ($n=6$), then $n-r-p=1$ (>0). The system is thus impossible unless some of the four Eberhard-Good relations are equivalent. Working out the calculation one shows that the former is effectively the case. The Eberhard-Good theorem provides only two relations between the parameters $t_0{}^2$ and $|t_2{}^2|$, but these *nonlinear* relations have *no real* solution. Consequently, the hypothesis $j=\frac{5}{2}$ has to be rejected, and by the same reasoning all higher spins are eliminated. This result can be stated:

If the particle Y^* produced in the parity-conserving two-body reaction

$$K^-+p\rightarrow Y^*+\pi$$

has only $L=2$ tensorial polarization, then the spin of Y^* is $\frac{3}{2}$ and we have the relation

$$5(t_0{}^2)^2+10|t_2{}^2|^2-1=0$$

between the parameters.

VII. CONCLUSIONS

The purpose of the present work was twofold. On the one hand we have shown that besides the well-known condition $\mathrm{Tr}\rho^2\leq 1$, the positive-definiteness imposes further conditions on the density matrix and on its multipole parameters. On the other hand, we have seen that in some physical situations, the Eberhard-Good theorem imposes strong relations between the elements of the density matrix.

Both sets of conditions may be of some help in the problem of determining the spin of the resonant states and in the study of production or formation processes.

For instance, we have shown how the Eberhard-Good theorem can be used to study reactions of the type $P+B\rightarrow B^*+P$ or $P+B\rightarrow B^*\rightarrow P+B$, and to determine the spin of Ξ hyperons. As for the positivity conditions, they can be very useful to remove the ambiguities which remain in the determination of spin by the method of Byers and Fenster. For instance, in his review article,[3] Tripp points out that, after an *extensive study*, the possibility $j=\frac{5}{2}$ for Y^* (1385) was

---

[27] A similar result has been given by M. Peshkin, Phys. Rev. **129**, 1864 (1963); however, his derivation is much more laborious than ours.

ruled out by the inequality $c_2 \geq 0$ ($\mathrm{Tr}\rho^2 \leq 1$). Had the complete set of positivity conditions ($c_6 \geq 0$, $c_5 \geq 0$, $c_4 \geq 0$, $c_3 \geq 0$, $c_2 \geq 0$) been used, the possibility $j=\frac{5}{2}$ would no doubt have been excluded earlier.

ACKNOWLEDGMENTS

We thank Dr. E. de Rafael for bringing the problem of positivity to our attention and for many helpful discussions. We are grateful to Professor Louis Michel for continual encouragement and for enlightening discussions on the Eberhard-Good theorem.

APPENDIX A

1. 3-*j* Symbols

The 3-j symbol is defined by the relation

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-)^{j_1-j_2-m}(2j+1)^{-1/2}(j_1m_1j_2m_2|j_1j_2j-m) \quad \text{(A1)}$$

where $(j_1m_1j_2m_2|j_1j_2j-m)$ is the Clebsch-Gordan coefficient of Condon and Shortley. The 3-j symbols are invariant under any even permutation of the columns. For any odd permutation we have

$$\begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} = (-)^{j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}. \quad \text{(A2)}$$

Another property of 3-j symbols is

$$\begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix} = (-)^{j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}. \quad \text{(A3)}$$

Wigner has introduced the very convenient concept of covariant and contravariant 3-j symbols. The metric tensor which allows one to raise and lower the magnetic-quantum-number indices is

$$C(j)^{mm'} = (-)^{j-m}\delta_{m,-m'}\,,$$
$$C(j)_{mm'} = (-)^{j+m}\delta_{m,-m'}\,, \quad \text{(A4)}$$

and we have

$$\begin{pmatrix} j_1 & j_2 & m \\ m_1 & m_2 & j \end{pmatrix} = C(j)^{mm'}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m' \end{pmatrix} = (-)^{j-m}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}. \quad \text{(A5)}$$

Throughout the paper we adopt the convention that the repeated magnetic quantum number indices are to be summed over ($m=-j, -j+1, \cdots, +j$). Summations over j when they occur are always indicated.

From (A5) one easily deduces the property

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m \\ j_1 & j_2 & j \end{pmatrix}. \quad \text{(A6)}$$

The orthogonality of Clebsch-Gordan coefficients leads to the relations

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} m_1 & m_2 & m' \\ j_1 & j_2 & j' \end{pmatrix} = (2j+1)^{-1}\delta_{jj'}\delta_{mm'}\,, \quad \text{(A7)}$$

$$\sum_j (2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} m'_1 & m'_2 & m \\ j_1 & j_2 & j \end{pmatrix} = \delta_{m_1m'_1}\delta_{m_2m'_2}. \quad \text{(A8)}$$

Finally, we give the values of some special 3-j symbols

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-)^{j-m}(2j+1)^{-1/2}, \quad \text{(A9a)}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} = (-)^{j-m}m[(2j+1)j(j+1)]^{-1/2}. \quad \text{(A9b)}$$

2. 6-*j* Symbols

Definition (Edmonds 6.2.3):

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} m_1 & \mu_2 & l_3 \\ j_1 & l_2 & \mu_3 \end{pmatrix} \times \begin{pmatrix} l_1 & m_2 & \mu_3 \\ \mu_1 & j_2 & l_3 \end{pmatrix}\begin{pmatrix} \mu_1 & l_2 & m_3 \\ l_1 & \mu_2 & j_3 \end{pmatrix}. \quad \text{(A10)}$$

The 6-j symbols are invariant under *any* permutation of the columns, and they are invariant under the following operation:

$$\begin{Bmatrix} j_1 & l_2 & l_3 \\ l_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}. \quad \text{(A11)}$$

Due to the orthogonality properties of 3-j symbols the following formulas are equivalent to the definition (A10)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \begin{pmatrix} m_1 & \mu_2 & l_3 \\ j_1 & l_2 & \mu_3 \end{pmatrix} \times \begin{pmatrix} m_2 & \mu_3 & l_1 \\ j_2 & l_3 & \mu_1 \end{pmatrix}\begin{pmatrix} m_3 & \mu_1 & l_2 \\ j_3 & l_1 & \mu_2 \end{pmatrix}. \quad \text{(A12)}$$

$$(-)^{2l_2}\sum_{j_3} (2j_3+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$$

$$\begin{pmatrix} j_3 & l_1 & \mu_2 \\ m_3 & \mu_1 & l_2 \end{pmatrix} = \begin{pmatrix} m_1 & \mu_2 & l_3 \\ j_1 & l_2 & \mu_3 \end{pmatrix}\begin{pmatrix} m_2 & \mu_3 & l_1 \\ j_2 & l_3 & \mu_1 \end{pmatrix},$$

or

$$\begin{pmatrix} m_1 & \mu_2 & l_3 \\ j_1 & l_2 & \mu_3 \end{pmatrix}\begin{pmatrix} m_2 & \mu_3 & l_1 \\ j_2 & l_3 & \mu_1 \end{pmatrix}$$

$$=\sum_{j_3} (2j_3+1)(-)^{l_1-l_2-j_3}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{pmatrix} m_1 & m_2 & j_3 \\ j_1 & j_2 & m_3 \end{pmatrix} \times\begin{pmatrix} m_3 & \mu_2 & l_1 \\ j_3 & l_2 & \mu_1 \end{pmatrix}, \quad \text{(A13)}$$

$$=\sum_{j_3} (2j_3+1)(-)^{l_1-l_2+m_3}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \times\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}\begin{pmatrix} m_3 & \mu_2 & l_1 \\ j_3 & l_2 & \mu_1 \end{pmatrix}. \quad \text{(A14)}$$

Finally, we give the value of a special 6-j symbol,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & j_3 & j_2 \end{Bmatrix}=(-)^{j_1+j_2+j_3}[(2j_2+1)(2j_3+1)]^{-1/2}. \quad \text{(A15)}$$

3. Properties of the Matrices $D^{(j)}(R)$

$$D^{(j)}(R^{-1})^m{}_{m'}=[D^{(j)}(R)^m{}_{m'}]^{-1}=[D^{(j)}(R)^m{}_{m'}]^\dagger =D^{(j)}(R)^{m'}{}_m{}^*, \quad \text{(A16)}$$

$$D^{(j)}[R(\alpha\beta\gamma)]^m{}_{m'}=e^{im\alpha}d^{(j)}(\beta)^m{}_{m'}e^{im'\gamma}. \quad \text{(A17)}$$

The matrix elements $d^{(j)}(\beta)^m{}_{m'}$ are real and have the properties

$$d^{(j)}(-\beta)^m{}_{m'}=d^{(j)}(\beta)^m{}_{m'}, \quad \text{(A18)}$$

$$d^{(j)}(\beta)^m{}_{m'}=(-)^{m-m'}d^{(j)}(\beta)^{m'}{}_m =(-)^{m-m'}d^{(j)}(\beta)^{-m}{}_{-m'}; \quad \text{(A19)}$$

thus

$$D^{(j)}(R)^m{}_{m'}{}^*=(-)^{m-m'}D^{(j)}(R)^{-m}{}_{-m'}. \quad \text{(A20)}$$

4. Reduction of a Product of Representations

Edmonds (4.3.2) gives the formula

$$D^{(j_1)}(R)^m{}_{m'}D^{(j_2)}(R)^n{}_{n'}=\sum_{L,M',M} (2L+1)\begin{pmatrix} j_1 & j_2 & L \\ m & n & M' \end{pmatrix} \times D^{(L)}(R)^{M'}{}_M{}^*\begin{pmatrix} j_1 & j_2 & L \\ m' & n' & M \end{pmatrix},$$

which can be transformed into

$$D^{(j_1)}(R)^m{}_{m'}D^{(j_2)}(R)^n{}_{n'} =\sum_{L,M',M} (2L+1)(-)^{2L}\begin{pmatrix} m & n & L \\ j_1 & j_2 & M' \end{pmatrix} \times D^{(L)}(R)^{M'}{}_M\begin{pmatrix} j_1 & j_2 & M \\ m' & n' & L \end{pmatrix}, \quad \text{(A21)}$$

or

$$\begin{pmatrix} m' & n' & L \\ j_1 & j_2 & M \end{pmatrix}D^{(j_1)}(R)^m{}_{m'}D^{(j_2)}(R)^n{}_{n'} =\begin{pmatrix} m & n & L \\ j_1 & j_2 & M' \end{pmatrix}D^{(L)}(R)^{M'}{}_M. \quad \text{(A22)}$$

APPENDIX B

In this Appendix[28] we derive compact expressions for the traces s_k directly by computing the trace of a product of k $T_M{}^L$'s. From the definition,

$$(T_M{}^L)^m{}_n=(-)^{2j}(2j+1)^{1/2}\begin{pmatrix} L & m & j \\ M & j & n \end{pmatrix} \quad \text{(B1)}$$

and formula (A14), we obtain the very useful relation

$$(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2})^m{}_n=(2j+1)^{1/2}\sum_J (2J+1)(-)^{2j+\mu} \times\begin{Bmatrix} L_1 & L_2 & J \\ j & j & j \end{Bmatrix}\begin{pmatrix} L_1 & L_2 & J \\ M_1 & M_2 & -\mu \end{pmatrix}(T_\mu{}^J)^m{}_n. \quad \text{(B2)}$$

By successive application of this formula one can transform any product of $T_M{}^L$ into a sum. Since we know the trace of $T_\mu{}^J$

$$\mathrm{Tr}(T_\mu{}^J)=(2j+1)\delta_{J0}\delta_{\mu0}, \quad \text{(B3)}$$

it is possible to calculate the trace of any product of $T_M{}^L$. For instance, by taking the trace of both sides of (B2) and using (A9a), (B3), and (A15), we obtain

$$\mathrm{Tr}(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2})=(-)^{M_1}\frac{2j+1}{2L_1+1}\delta_{L_1L_2}\delta_{M_1,-M_2}. \quad \text{(B4)}$$

Similarly, to compute the trace of a product of three $T_M{}^L$'s, we first transform

$$\mathrm{Tr}(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2}T_{M_3}{}^{L_3})=(2j+1)^{1/2}\sum_J (2J+1)(-)^{2j+\mu} \times\begin{Bmatrix} L_1 & L_2 & J \\ j & j & j \end{Bmatrix}\begin{pmatrix} L_1 & L_2 & J \\ M_1 & M_2 & -\mu \end{pmatrix}\mathrm{Tr}(T_\mu{}^JT_{M_3}{}^{L_3});$$

then we use (B4) and obtain

$$\mathrm{Tr}(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2}T_{M_3}{}^{L_3})=(-)^{2j}(2j+1)^{3/2} \times\begin{Bmatrix} L_1 & L_2 & L_3 \\ j & j & j \end{Bmatrix}\begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{pmatrix}. \quad \text{(B5)}$$

[28] Results similar to those of Sec. II and of this Appendix are given by M. C. Marinov, Moscow, 1965, unpublished report. We thank Dr. Marinov for sending us a copy of his work.

By the same procedure (first reducing with (B2), then using previous results) we obtain

$$\mathrm{Tr}(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2}T_{M_3}{}^{L_3}T_{M_4}{}^{L_4})=(2j+1)^2\sum_J(2J+1)(-)^J\begin{Bmatrix}L_1 & L_2 & J\\ j & j & j\end{Bmatrix}\begin{Bmatrix}L_3 & L_4 & J\\ j & j & j\end{Bmatrix}\begin{pmatrix}\mu & L_1 & L_2\\ J & M_1 & M_2\end{pmatrix}\begin{pmatrix}J & L_3 & L_4\\ \mu & M_3 & M_4\end{pmatrix}, \tag{B6}$$

$$\mathrm{Tr}(T_{M_1}{}^{L_1}T_{M_2}{}^{L_2}T_{M_3}{}^{L_3}T_{M_4}{}^{L_4}T_{M_5}{}^{L_5})=(2j+1)^{5/2}(-)^{2j}\sum_{JJ'}(-)^{J+J'}(2J+1)(2J'+1)\begin{Bmatrix}L_1 & L_2 & J\\ j & j & j\end{Bmatrix}\times\begin{Bmatrix}L_3 & L_4 & J\\ j & j & j\end{Bmatrix}\begin{Bmatrix}J & J' & L_5\\ j & j & j\end{Bmatrix}\begin{pmatrix}L_1 & L_2 & \mu\\ M_1 & M_2 & J\end{pmatrix}\begin{pmatrix}L_3 & L_4 & \mu'\\ M_3 & M_4 & J'\end{pmatrix}\begin{pmatrix}J & J' & L_5\\ \mu & \mu' & M_5\end{pmatrix}. \tag{B7}$$

With these formulas one easily obtains the traces s_k

$$s_2=(2j+1)^{-1}\sum_{L,M}(2L+1)\,|t_M{}^L|^2, \tag{B8}$$

$$s_3=(-)^{2j}(2j+1)^{-3/2}\sum_{\substack{L_1L_2L_3\\ M_1M_2M_3}}(2L_1+1)(2L_2+1)(2L_3+1)t_{M_1}{}^{L_1}t_{M_2}{}^{L_2}t_{M_3}{}^{L_3}\begin{Bmatrix}L_1 & L_2 & L_3\\ j & j & j\end{Bmatrix}\begin{pmatrix}L_1 & L_2 & L_3\\ M_1 & M_2 & M_3\end{pmatrix}, \tag{B9}$$

$$s_4=(2j+1)^{-2}\sum_{L_iM_i}\prod_{i=1}^{4}\{(2L_i+1)t_{M_i}{}^{L_i}\}\sum_J(-)^J(2J+1)\begin{Bmatrix}L_1 & L_2 & J\\ j & j & j\end{Bmatrix}\begin{Bmatrix}L_3 & L_4 & J\\ j & j & j\end{Bmatrix}\times\begin{pmatrix}L_1 & L_2 & \mu\\ M_1 & M_2 & J\end{pmatrix}\begin{pmatrix}L_3 & L_4 & J\\ M_3 & M_4 & \mu\end{pmatrix}, \tag{B10}$$

$$s_5=(-)^{2j}(2j+1)^{-5/2}\sum_{L_i,M_i}\prod_{i=1}^{5}\{(2L_i+1)t_{M_i}{}^{L_i}\}\sum_{JJ'}(-)^{J+J'}(2J+1)(2J'+1)\begin{Bmatrix}L_1 & L_2 & J\\ j & j & j\end{Bmatrix}\begin{Bmatrix}L_3 & L_4 & J'\\ j & j & j\end{Bmatrix}\begin{Bmatrix}J & J' & L_5\\ j & j & j\end{Bmatrix}\begin{pmatrix}L_1 & L_2 & \mu\\ M_1 & M_2 & J\end{pmatrix}\begin{pmatrix}L_3 & L_4 & \mu'\\ M_3 & M_4 & J'\end{pmatrix}\begin{pmatrix}J & J' & L_5\\ \mu & \mu' & M_5\end{pmatrix}. \tag{B11}$$

Owing to the rather impressive collection of indices to be summed over in these expressions, we cannot hope to expand each formula and obtain full algebraic expressions for the traces s_k as functions of $t_M{}^L$. For practical use we prefer the step-by-step method set forth in the text.