the single-pion production angular distributions. A more difficult test of the model would be a correct description of the nonresonating phase shifts—our hope and inclination is that careful treatment of kinematics and three-body unitarity would suffice for their description. Finally, interesting questions can be asked *in the context of this model*; for example, what are the effects of varying the S-wave $\pi\pi$ scattering length on a variety of pion nucleon phenomena?[24] Also one can examine the validity of Dalitz-type analyses in obtaining widths and positions of resonances—i.e., do the bumps in the production cross sections which are due to the presence of N^* and ρ in the final state resemble the input N^* and ρ? We are presently performing such calculations and attempting to answer such questions as those described above.

[24] For a possible significance of the sign of the $\pi\pi$ scattering length, see G. F. Chew, Phys. Rev. Letters **16**, 60 (1966).

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Spin Effects on Triangle Graphs*

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It is shown that the triangle amplitude can be written as the scalar graph multiplied by a factor which contains only the characteristics of the external particles. In the case where the spins of the external particles are summed, their angles averaged, and only one partial-wave set (that is, a set of relative orbital angular momenta among the final-state particles) retained, this multiplicative factor is just a product of appropriate 3-momenta. The case $K^+p \rightarrow K\pi\pi\pi p$ at 3 BeV/c is considered; and it is demonstrated that the correct inclusion of threshold factors does not diminish the effect calculated in our earlier work, where the shape of the "κ enhancement" was successfully described by a triangle graph.

I. INTRODUCTION

It is the primary purpose of this paper to study the influence of non-spin-zero particles on triangle amplitudes insofar as they impose specific threshold behavior upon the amplitudes. By thresholds we simply mean the various kinematical limits outlined by the available phase space. Our fundamental assumption is that the triangle graph is dominated by two major effects. First, there is a second-sheet Landau singularity,[1] denoted hereafter by LS, which in the cases of interest here approaches close to the physical region.[2,3] This effect is embodied in the scalar triangle graph. Second, the spins and parities of the particles involved in the graph impose upon the amplitude a minimal threshold behavior; that is, for a given graph, there is a set of lowest relative orbital angular momenta among the final-state particles. We will show that if one assumes a particular partial-wave amplitude (normally the one with lowest partial waves), then the triangle amplitude can be written as the product of the scalar graph and 3-momenta factors determined by the relative orbital angular momenta in the final state.

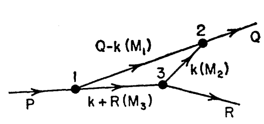

FIG. 1. The triangle graph. P, Q, and R are external 4-momenta. M_1, M_2, M_3 are internal masses. 1, 2, and 3 label the incident, sum, and difference vertices, respectively. k is an arbitrary 4-vector.

II. THE TRIANGLE GRAPH

In Fig. 1 is depicted the basic triangle mechanism. The vertices are labeled 1, 2, and 3 for incident, sum, and difference vertices, respectively. The masses of the internal particles (M_1, M_2, and M_3), as well as the external 4-momenta (P, Q, and R) are as designated in Fig. 1.

The scalar graph[1,4] is given by

$$J=\frac{1}{2}\int d^4k\{[(k+R)^2+M_3^2][(Q-k)^2+M_1^2]\times[k^2+M_2^2]\}^{-1}. \quad \text{(II.1)}$$

Or, introducing the Feynman parameters α_1, α_2, and

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[1] L. D. Landau, Nucl. Phys. **13**, 181 (1959).

[2] I. J. R. Aitchison, Phys. Rev. **133**, B1257 (1964).

[3] F. R. Halpern and H. L. Watson, Phys. Rev. **131**, 2674 (1963).

[4] C. Fronsdal and R. E. Norton, J. Math. Phys. **5**, 100 (1964); R. E. Cutkosky, *ibid.* **1**, 49 (1960); R. J. Eden, Proc. Roy. Soc. (London) **A210**, 338 (1952); R. J. Eden, Brandeis University Summer Institute in Theoretical Physics, 1960 (unpublished).

α_3, we obtain

$$J=\int_0^1 d\alpha\, \delta\Big(\sum_{i=1}^{3}\alpha_i-1\Big)\int d^4k\frac{1}{[k_L{}^2+\varphi]^3}\,, \tag{II.2}$$

where

$$d\alpha=d\alpha_1 d\alpha_2 d\alpha_3\,,$$

$$k_L=k+\alpha_3 R-\alpha_1 Q\,,$$

$$\varphi=\sum_{i=1}^{3}\alpha_i M_i{}^2-(\alpha_3 R-\alpha_1 Q)^2+\alpha_3 R^2+\alpha_1 Q^2\,.$$

Two important properties of the Landau singularity are[1,4]

$$k_L=0 \quad \text{at LS;} \tag{II.3}$$

$$\alpha_i\neq 0 \quad \text{for } i=1,\,2,\,3, \text{ at LS}. \tag{II.4}$$

Various authors[3,5,6] have shown how to extract the "singular behavior" from J. Using in part the work of Aitchison,[2] Halpern and Watson,[3] and Anisovich and Dakhno,[5] it is apparent that one can approximate the total "scalar" amplitude near LS by

$$I=c+\Delta\,, \tag{II.5}$$

where c is a background constant, and Δ is that part of the triangle amplitude which is singular near the physical region.

There are two points which should be emphasized: (1) We consistently consider the invariant functions at the triangle vertices to be unity. The justification for this is that at LS the intermediate particles are all on their mass shells. Thus, vertex functions are simply replaced by their mass-shell values. If an external variable is involved, we assume that the dependence of the invariant form factors on it can be approximated by a constant. We normalize arbitrarily, and this is incorporated later on. (2) We use the letter J to stand for a particular integral function, while the letter I stands for the scalar amplitude. They differ essentially by a constant. The point is that the true scalar amplitude, in general, has contributions from many processes. We assume, however, the dominance of the triangle amplitude, by which we mean that all other processes contribute by means of an additive constant. The "nonsingular" part of the triangle amplitude can be added to this constant. Thus we obtain Eq. (II.5).

No statement at this time can be made concerning either the normalization of I, or the relative strength of the c and Δ terms. At present these can only be determined from experiment.

Consider the quantity J^l given by

$$J^l=\int_0^1 d\alpha\, \delta\Big(\sum_{i=1}^{3}\alpha_i-1\Big)\int d^4k\frac{\alpha_l}{[k_L{}^2+\varphi]^3}\,. \tag{II.6}$$

Since $\alpha_l\neq 0$ at LS, then near the singularity, J^l (as well as J) will have the form of I in (II.5).

[5] V. V. Anisovich and L. G. Dakhno, Phys. Letters **10**, 221 (1964).

[6] Y. F. Chang and S. F. Tuan, Phys. Rev. **136**, B741 (1964).

Let us now include the spins and parities of the particles involved. The triangle amplitude can in general be approximated by a sum of terms of the form

$$A_T=B_{\alpha\beta\gamma\cdots}J_{\alpha\beta\gamma\cdots}\,, \tag{II.7}$$

where $B_{\alpha\beta\gamma\cdots}$ is a tensor formed from the 4-momenta of the external particles as well as tensors related to the spins of all the particles; while $J_{\alpha\beta\gamma\cdots}$ is a tensor formed as follows: In the integral for J [Eq. (II.2)], replace the 1 in the numerator of the integrand by a tensor made up of appropriate 4-momenta of the internal particles. [In Eq. (II.7) a summation over the indices α, β, γ, $\cdots$ is implied.]

Near the singularity LS, $J_{\alpha\beta\gamma\cdots}$ can be evaluated by making, in the numerator of the integrand, the replacement (see the Appendix):

$$k\rightarrow -\alpha_3 R+\alpha_1 Q\,. \tag{II.8}$$

Since the α's are not zero at the singularity, we deduce that near LS,

$$J_{\alpha\beta\gamma\cdots}\simeq C_{\alpha\beta\gamma\cdots}J\,, \tag{II.9}$$

where $C_{\alpha\beta\gamma\cdots}$ is a tensor formed from the 4-momenta of the external particles.

Hence the square of the triangle amplitude, "summed over spins" and "averaged over angles" is given by

$$A=T|I|^2\,, \tag{II.10}$$

where we have replaced J by I, the "true" amplitude; and T is essentially a function of the relative 3-momenta among the final-state particles. In particular, if one set of relative orbital angular momenta dominates the reaction, then T is simply a product of relative 3-momenta.

III. AN EXAMPLE

To exemplify the ideas expressed in the previous section, we apply the techniques to a particular case. In the reaction $K^+p\rightarrow K\pi\pi\pi p$ at 3 BeV/c, an enhancement is seen in the $K\pi$ mass spectrum at about 725 MeV.[7] It has been suggested[8] that the mechanism responsible for this effect is essentially the triangle diagram depicted in Fig. 2. However, in Ref. 8 full ac-

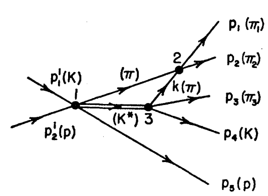

FIG. 2. Triangle diagram for $K^+p\rightarrow K\pi\pi\pi p$. The vertices are labeled 1, 2, and 3. The bracketed quantities identify the particles: π, K, K^*, and p stand for pion, K meson, $K^*(890)$ resonance, and proton, respectively. The subscripted p's are 4-momenta of the external particles as shown; k is an arbitrary 4-momentum.

[7] M. Ferro-Luzzi, R. George, Y. Goldschmidt-Clermont, V. P. Henri, B. Jongejans, D. W. G. Leith, G. R. Lynch, F. Muller, and J. M. Perreau, Phys. Letters **12**, 255 (1964).

[8] M. Month, Phys. Rev. **139**, B1093 (1965).

count of the K^* spin was not taken. Our endeavor here is to remedy this.

First we assume that the $\pi\pi$ interaction at the sum vertex (vertex 2 of Fig. 2) is s wave; and further that the $K\pi_3$ state has spin-parity 1^- (the 0^+ state is forbidden at the $K^* \to K\pi\pi$ vertex). To complete a description of the final state, one must specify the relative angular momenta between, on the one hand, the $(\pi_1\pi_2)$ and (π_3K) states and, on the other hand, the $(\pi_1\pi_2\pi_3K)$ and (p) states. In Ref. 8 these were both assumed to be s wave. Although the latter is both allowed and suitable, the former angular-momentum assignment cannot be 0^+. We will in fact prove that under the constraints that the state $(\pi_1\pi_2)$ be s wave and the state (π_3K) be p wave, it follows that the relative angular momentum between the states $(\pi_1\pi_2)$ and (π_3K) must be at least p wave. This is accomplished simply by computing the triangle graph.

The couplings are as follows: At vertex 1 there is a vector coupling X_μ; at vertex 2 there is a scalar coupling 1; and at vertex 3 there is a pseudovector coupling $\epsilon_{\mu\nu\alpha\beta}r_\nu p_\alpha k_\beta$. X_μ is a 4-vector characterizing the incident vertex. For s-wave production of the K^*, X_μ might for example be $p_{1\mu}{}^i$. The 4-vectors p and r are given by

$$\begin{aligned} p&=p_3+p_4\,,\\ r&=p_3-p_4\,. \end{aligned} \tag{III.1}$$

k is a 4-vector to be integrated over, while the 4-vectors p_1, p_2, p_3, p_4, p_5, $p_1{}^i$, $p_2{}^i$ are designated in Fig. 2. The invariant masses V, v, and s are defined by

$$\begin{aligned} V^2&=-(p_1{}^i+p_2{}^i-p_5)^2\,,\\ v^2&=-p^2\,,\\ s^2&=-(p_1+p_2)^2\,. \end{aligned} \tag{III.2}$$

The masses M_R, μ, μ_K, and m stand for the K^*, pion, K meson and proton, respectively. Also, we point out that hereafter we will refer to the final-state particles by π_1, π_2, π_3, K, p and 1, 2, 3, 4, 5 interchangeably.

Using the above couplings and Eqs. (II.1) and (II.2), the triangle graph can be written

$$G=\epsilon_{\mu\nu\alpha\beta}X_\mu r_\nu p_\alpha \int_0^1 d\alpha\,\delta\Big(\sum_{i=1}^{3}\alpha_i-1\Big) \times\int d^4k\frac{k_\beta}{[k_L{}^2+\varphi]^3}\,, \tag{III.3}$$

where

$$\begin{aligned} k_L&=k+\alpha_3p-\alpha_1q\,,\\ q&=p_1+p_2\,,\\ \varphi&=\alpha_3M_R{}^2+(\alpha_1+\alpha_2)\mu^2-(\alpha_3p-\alpha_1q)^2-\alpha_3v^2-\alpha_1s^2\,. \end{aligned}$$

Making the change of variable

$$k_L=k+\alpha_3p-\alpha_1q\,, \tag{III.4}$$

and using the fact that

$$\int d^4k_L\frac{k_L}{[k_L{}^2+\varphi]^3}=0\,, \tag{III.5}$$

we obtain

$$G=\epsilon_{\mu\nu\alpha\beta}X_\mu r_\nu p_\alpha q_\beta \int_0^1 d\alpha\,\delta\Big(\sum_{i=1}^{3}\alpha_i-1\Big) \times\int d^4k_L\frac{\alpha_1}{[k_L{}^2+\varphi]^3}\,. \tag{III.6}$$

Since $\alpha_1\neq 0$ at LS[1], then near the singularity

$$G\simeq\epsilon_{\mu\nu\alpha\beta}X_\mu r_\nu p_\alpha q_\beta I\,. \tag{III.7}$$

Hence we have that G is a product of the scalar triangle graph I and a threshold factor (this being the case, of course, because angular dependence and nucleon spin do not concern us here). Also, independent of the incident vertex, G is proportional to both q_{34} and $q_{(12)(34)}$. q_{34} is the relative momentum between particles 3 and 4 in the (34) center-of-mass system; and $q_{(12)(34)}$ is the relative momentum between the systems (12) and (34) in the center-of-mass of particles 1, 2, 3, and 4. This means that there is a unit of orbital angular momentum between particles 3 and 4, on the one hand, and between the systems (12) and (34), on the other hand. We have thus demonstrated the assertion we set out to prove.

IV. $K\pi$ MASS DISTRIBUTION IN $K^+p \to K\pi\pi\pi p$ AT 3 BeV/c

The triangle-amplitude contribution to the cross section is[8]

$$\frac{d\sigma}{dv}=\tau\simeq\int_{v+2\mu}^{W-m}dV\int_{2\mu}^{V-v}ds|I|^2\beta_1\beta_2p_s\,, \tag{IV.1}$$

where W is the incident energy in the center-of-mass system; I is the scalar amplitude given by Eqs. (II.8) and (II.9) of Ref. 8; p_s is a phase-space term given by Eq. (II.11) of Ref. 8; and the β's are threshold factors. β_i can be written

$$\beta_i=\frac{l_i^2}{l_i^2+\mu^2a_i^2}\,, \tag{IV.2}$$

where l_1 is the 3-momentum of the outgoing K meson in the (π_3K) center-of-mass system; l_2 is the 3-momentum of the two spectator pions π_1 and π_2, in the $(K\pi_1\pi_2\pi_3)$ center-of-mass system; and the a's are momentum cutoffs.

In our calculations, as in Ref. 8, we use an equivalent form instead of (IV.1) and (IV.2). We finally have

$$\tau=N\int_{v+2\mu}^{W-m}dV\int_{2\mu}^{V-v}ds|I|^2\gamma_1\gamma_2p_s\,, \tag{IV.3}$$

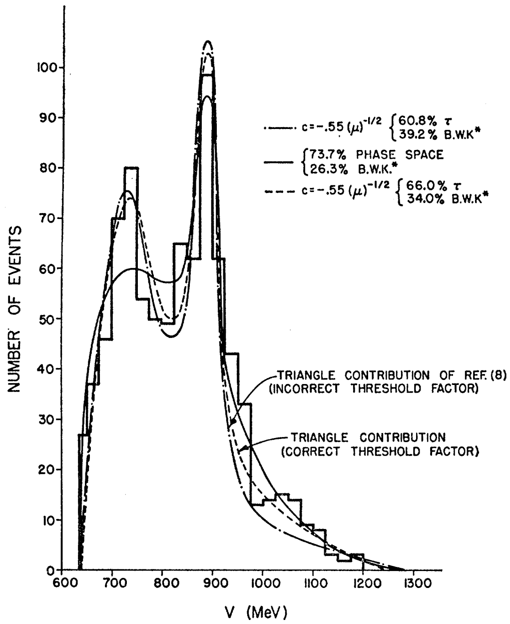

FIG. 3. Comparison with experiment of the $K\pi$ mass spectrum in $K^+p\to K\pi\pi\pi p$ at 3 BeV/c. The experimental data are those of Ferro-Luzzi *et al.* (Ref. 7). All the theoretical curves have a Breit-Wigner (B.W.) term for the $K^*(890)$ included. The solid curve is 73.7% phase space and 26.3% B.W. K^*, and contains no triangle contribution. The almost solid curve contains the triangle effect but does not have the correct threshold factor. The dashed curve [Eq. (IV.3)] corrects this by having p wave between the κ state (see Ref. 8) and the two spectator pions (π_1 and π_2 of Fig. 2). The interference background constant, c in Eq. (II.5), has the value $-0.55\,\mu^{-1/2}$, as in Ref. 8. v is the $K\pi$ mass.

where

$$\gamma_1=\frac{(v-\mu-\mu_K)}{(v-\mu-\mu_K)+\mu a_1^2/2}, \tag{IV.4}$$

$$\gamma_2=\frac{(V-s-v)}{(V-s-v)+\mu a_2^2/2}, \tag{IV.5}$$

and N is a normalization constant.

In Fig. 3, we compare our results with those of Ref. 8. We use the values $a_1=1.5$ and $a_2=2.5$. That is, the form factors, giving the momentum cutoffs, have a range of 1.5 and 2.5 pion masses, respectively. Our fit to the experimental data[7] is almost identical to that in Ref. 8, where a_1 had the value unity and γ_2 was not included. In other words, the introduction of the threshold factor γ_2 has no damping effect on the shape of the theoretical distribution.

APPENDIX

Consider the quantity

$$U_{\alpha\beta\gamma\cdots}=\int_0^1 d\alpha\,\delta(\sum_{i=1}^3\alpha_i-1)\int d^4k\frac{k_\alpha k_\beta k_\gamma\cdots}{[k_L^2+\varphi]^3}. \tag{A1}$$

(See Sec. II of the text for the definitions of these terms.) We wish to show that close to LS,

$$U_{\alpha\beta\gamma\cdots}\simeq V_{\alpha\beta\gamma\cdots}, \tag{A2}$$

where V is similar to U, the only difference being that the k vectors in the numerator of the integrand of U are replaced by $-\alpha_3R+\alpha_1Q$.

If we make the change of variables

$$k_L=k+\alpha_3R-\alpha_1Q, \tag{A3}$$

then

$$U_{\alpha\beta\gamma\cdots}=V_{\alpha\beta\gamma\cdots}+W_{\alpha\beta\gamma\cdots}, \tag{A4}$$

where W has at least one power of k_L in the numerator of the integrand. Thus we need only show that W is finite at LS, from which (A2) follows immediately.

We first dispose of terms having odd powers of k_L. These vanish because in these cases the integrand is odd over an obviously even interval in k_L. Secondly, each term with an even power of k_L in the integrand is proportional to

$$Y_n=\int_0^1 d\alpha\,\delta(\sum_{i=1}^3\alpha_i-1)\int d^4k_L\frac{k_L^{2n}}{[k_L^2+\varphi]^3}, \tag{A5}$$

where n is a nonzero positive integer denoting at most one-half the power of k_L in the integrand of (A1).

Recall now that at LS, $k_L=0$. It is thus obvious that the presence of k_L in the numerator of (A5) dampens the effect of the singularity. In fact, Landau[1] has shown that Y_n is finite at LS for $n\geq 1$.