Self-Consistent Three-Body Calculation of Pion-Nucleon Scattering

RONALD AARON*
Laboratory for Theoretical Studies, Goddard Space Flight Center, National Aeronautics and Space Administration, Greenbelt, Maryland

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Using an off-energy-shell static theory which treats three-particle unitarity exactly, but neglects left-hand cuts, we describe rather well the low-energy P-wave $N\pi$ system. We obtain a well-satisfied "bootstrap" requirement in the (3,3) channel and a possible explanation for the Roper resonance in the (1,1) channel.

I. INTRODUCTION

IN recent years most dynamical calculations in elementary-particle physics have involved the N/D method in the framework of S-matrix theory.[1] The usual approach in such calculations is to compute left-hand singularities as well as possible and, at best, to include inelasticity phenomenologically[2]; multiparticle unitarity cuts are neglected completely. We see no *a priori* justification for this procedure—for example, in the charged scalar static model Bronzon and Brown[3] conclude that "for strong coupling and particularly at higher energies, production has a decidedly greater effect on scattering than does crossing." Nevertheless, it is still true that one cannot obtain operational equations in a relativistic theory which includes spin, isospin etc., and *both* a left-hand cut and a three-particle unitarity cut, so a choice must be made as to which of the latter two effects to treat most carefully. In this paper, rather than the traditional one, we choose an alternative approach and examine the low-energy pion-nucleon system in a static off-energy-shell theory which treats three-particle unitarity exactly, but neglects crossed cuts. We obtain considerable success including a well-satisfied "bootstrap" requirement in the (3,3) channel and a possible explanation for the Roper resonance[4] in the (1,1) channel. Our calculation suggests that careful treatment of inelasticity may be important in low-energy baryon-meson systems, and also should encourage the use of relativistic off-energy-shell three-body equations to describe elementary-particle processes.

In Sec. II we describe our model, and in Secs. III and IV display the results of our calculations. In Sec. V we discuss these results and give conclusions.

II. THE MODEL

We study $N\pi$ scattering in an exactly soluble three-body model proposed by Amado[5] using an approach first suggested by Lovelace.[6] The nucleon is taken to be static and considered as a pion nucleon bound state, and therefore, the $N\pi$ system is a special case corresponding to bound-state scattering in the $N\pi\pi$ system. We use a second-quantized formalism in which the nucleons are static. Since the model lacks crossing symmetry, it is convenient to define two particles with the quantum numbers of a nucleon, N and N', and permit the reaction $N\to N'+\pi$, but not allow $N'\to N+\pi$. Our theory sums the infinite set of diagrams shown in Fig. 1. In the above diagrams the $\pi N'N$ and $\pi N'N^*$ vertices are described respectively by interaction terms $H_{1/2}$ and $H_{3/2}$ of the form[7]

$$H_x=\sqrt{3}g_x^{(0)}\int\frac{d^3k}{\sqrt{\omega_k}}f_x(k^2)\sum_{\substack{\mu,m,M\\ \lambda,\tau,T}}C(1,\tfrac{1}{2},x;\lambda,\tau,T)\times C(1,\tfrac{1}{2},x;\mu,m,M)\phi_T{}^{x}{}_M{}^{x}\Psi_\tau{}^{1/2}{}_M{}^{1/2}k_\mu a_\lambda(k^2)+\text{H.c.}, \qquad (1)$$

where $x=\frac{1}{2}$ or $\frac{3}{2}$. The operators $\phi^{1/2\,1/2}$, $\phi^{3/2\,3/2}$, Ψ, and $a(k^2)$ annihilate particles N, N^*, N', and π, respectively. $g_x^{(0)}$ is the unrenormalized coupling constant, and $f_x(k^2)$ is an arbitrary cutoff function. These quantities will be discussed in detail below.

The partial-wave amplitudes $T_{11}(\pi+N\to\pi+N)$ and $T_{21}(\pi+N^*\to\pi+N)$ satisfy coupled one-dimensional

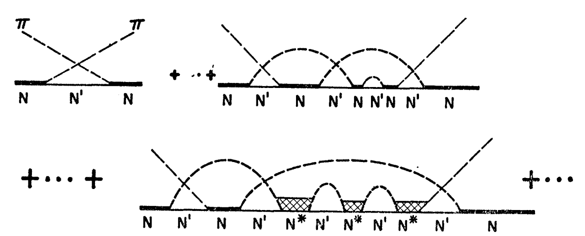

FIG. 1. Typical diagrams of our theory.

* National Academy of Sciences–National Research Council Postdoctoral Resident Research Associateship.
[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), pp. 48–50.
[2] G. Frye and R. Warnock, Phys. Rev. **130**, 478 (1963).
[3] J. B. Bronzon and R. W. Brown, Ann. Phys. (N.Y.) **39**, 335 (1966).
[4] L. Roper, Phys. Rev. Letters **12**, 340 (1964).
[5] R. D. Amado, Phys. Rev. **132**, 485 (1963). We use the term "exactly soluble" in the sense that all amplitudes can be obtained from the solution of one dimensional Fredholm integral equations.
[6] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[7] We use the notation of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), 3rd edition. For the case $X=\frac{1}{2}$, H_X is proportional to the usual pseudoscalar gradient-couple static interaction.

Fredholm integral equations which may be written

$$T_{ij}(k,k';E)=B_{ij}(k,k';E)+\frac{1}{(2\pi)^3}\sum_{m=1}^{2}\int_0^\infty dqB_{im}(k,q;E)\times G_m(q,E)T_{mj}(q,k';E)\,,\quad (2)$$

where E is the total incident energy. A diagrammatic representation of these equations is shown in Fig. 2. In the above equation, using units in which $\hbar$, c, and the pion mass are unity,

$$B_{ij}(k,k';E)=4\pi\Gamma_{ij}\frac{g_l f_l(k)}{\omega_k^{1/4}}\frac{kk'}{\omega_0-\omega_k-\omega_{k'}}\frac{g_r f_r(k')}{\omega_{k'}^{1/4}},\quad (3)$$

where $\omega_k=(k^2+1)^{1/2}$, ω_0 is the energy of the incident pion, $l=(2i-1)/2$, $r=(2j-1)/2$, and $g_{1/2}$ and $g_{3/2}$ are renormalized coupling constants (e.g., $g_{1/2}^2=Z_N g_{1/2}^{(0)2}$, where Z_N is the wave-function renormalization constant of the N particle).

$$\Gamma=\begin{pmatrix}1/9 & 8/9\\ 8/9 & 1/9\end{pmatrix},\ \begin{pmatrix}-2/9 & -2/9\times\sqrt{10}\\ -2/9\times\sqrt{10} & -2/9\end{pmatrix},\ \begin{pmatrix}4/9 & 5/9\\ 5/9 & 4/9\end{pmatrix},\quad (4)$$

in the (1,1), (1,3) or (3,1) and (3,3) channels, respectively. The Green's functions in Eq. (2) have the form

$$G_1(q,E)=(\omega_0-\omega_q)^{-1}\times\left[1-(\omega_0-\omega_q)\frac{g_{1/2}^2}{(2\pi)^3}\int\frac{d^3Q\, f_{1/2}^2(Q)Q^2}{\omega_Q^3(\omega_0-\omega_Q+i\epsilon)}\right]^{-1},\quad (5)$$

$$G_2(q,E)=\left[1+\frac{g_{3/2}^2}{(2\pi)^3}\int\frac{d^3Q\, f_{3/2}^2(Q)Q^2}{\omega_Q(\omega_0-\omega_Q+i\epsilon)}\right]^{-1}.\quad (6)$$

In the expression for G_1 we have anticipated and thus explicitly displayed the nucleon pole. The form of G_2 given above is only valid in the separable-potential limit of the $N^*N'\pi$ interaction ($Z_{N^*}=0$) but in the following calculations we consider only this limiting case.[8,9]

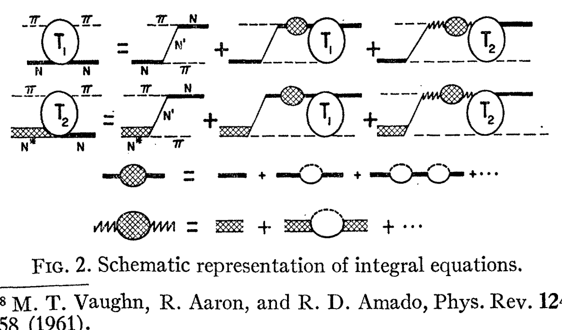

FIG. 2. Schematic representation of integral equations.

[8] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[9] The three body problem with separable potentials was first solved by A. N. Mitra, Nucl. Phys. **32**, 529 (1962).

We now perform two calculations, first omitting $\pi N'N^*$ coupling, and then including it.

III. CALCULATION I

We set $g_{3/2}^{(0)}\equiv 0$. Since, as we shall demonstrate in the next section, our results turn out to be essentially independent of the functional form of the cutoff function, we here arbitrarily choose $f_{1/2}(q^2)=\exp[(-q^2-1)/\beta_{1/2}^2]$, where q is the pion momentum. We first note that the amplitude for $\pi+N'\rightarrow\pi+N'$ can be solved trivially since it is N-θ scattering in a pseudoscalar Lee model.[10] There are three parameters in our theory—the cutoff parameter $\beta_{1/2}$, the renormalized coupling constant $g_{1/2}$ and the wave function renormalization constant of the N particle Z_N (recall that $g_{1/2}^2=Z_N g_{1/2}^{(0)2}$). We fix the parameters by demanding that our amplitude for $\pi N'$ scattering have a discrete state pole at the physical nucleon mass M with residue $g_{1/2}^2=1.48$ [11] subject to the condition $0\leq Z_N\leq 1$. Having determined the parameters in the two-body sector we now turn to the three-body sector of our theory.

We examine πN scattering in the (1,3) (3,1), and (3,3) channels which is the same as studying V-θ scattering in a pseudoscalar Lee model[10] or *equivalently* the Chew-Low static model[12] without crossing symmetry (i.e., $N'\nleftrightarrow N+\pi$) in the two-meson approximation. Three-body unitarity is satisfied exactly in our model. To obtain our πN amplitudes we solve the integral equations [Eq. (2)] numerically by methods discussed extensively elsewhere.[13] We find no (3,3) resonance for any value of $\beta_{1/2}$ such that $0\leq Z_N\leq 1$. The largest phase shifts are obtained for $Z_N=0$ and are shown together with phase shifts for $Z_N=0.144$ in Fig. 3 (curves No. 1 and No. 2). The corresponding (1,3) and (3,1) phase shifts (which are equal in our model) are shown in Fig. 4. It should be emphasized that one cannot obtain a resonance in the (3,3) channel for any allowed value of $\beta_{1/2}(0\leq Z_N\leq 1)$.

Further discussion is necessary before presenting the (1,1) results. We should note that in the (1,1) channel πN scattering is *not* equivalent to that in the Chew-Low model without crossing symmetry, for with the interaction Hamiltonian of Eq. (1) we are neglecting single-nucleon intermediate states (Fig. 5). Including this effect, the model can still be solved with a moderate amount of further computations involving integrals over off-energy-shell amplitudes already obtained, but

[10] T. D. Lee, Phys. Rev. **95**, 1329 (1954). The pseudoscalar Lee model is discussed by H. Chew, *ibid.* **132**, 2756 (1963).

[11] In more conventional units this choice of $g_{1/2}^2$ corresponds to $g_{\bar{N}N\pi}^2/4\pi=14.6$, where $g_{\bar{N}N\pi}^2$ is defined, for example, in P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).

[12] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1955). We should like to emphasize that the Chew-Low model gives nonlinear off-energy-shell integral equations to solve, while our model gives linear equations of the Lippmann-Schwinger type at the expense of exact crossing symmetry.

[13] J. H. Hetherington and L. H. Schick, Phys. Rev. **135**, B935 (1965). R. Aaron and R. D. Amado, *ibid.* **150**, 857 (1966).

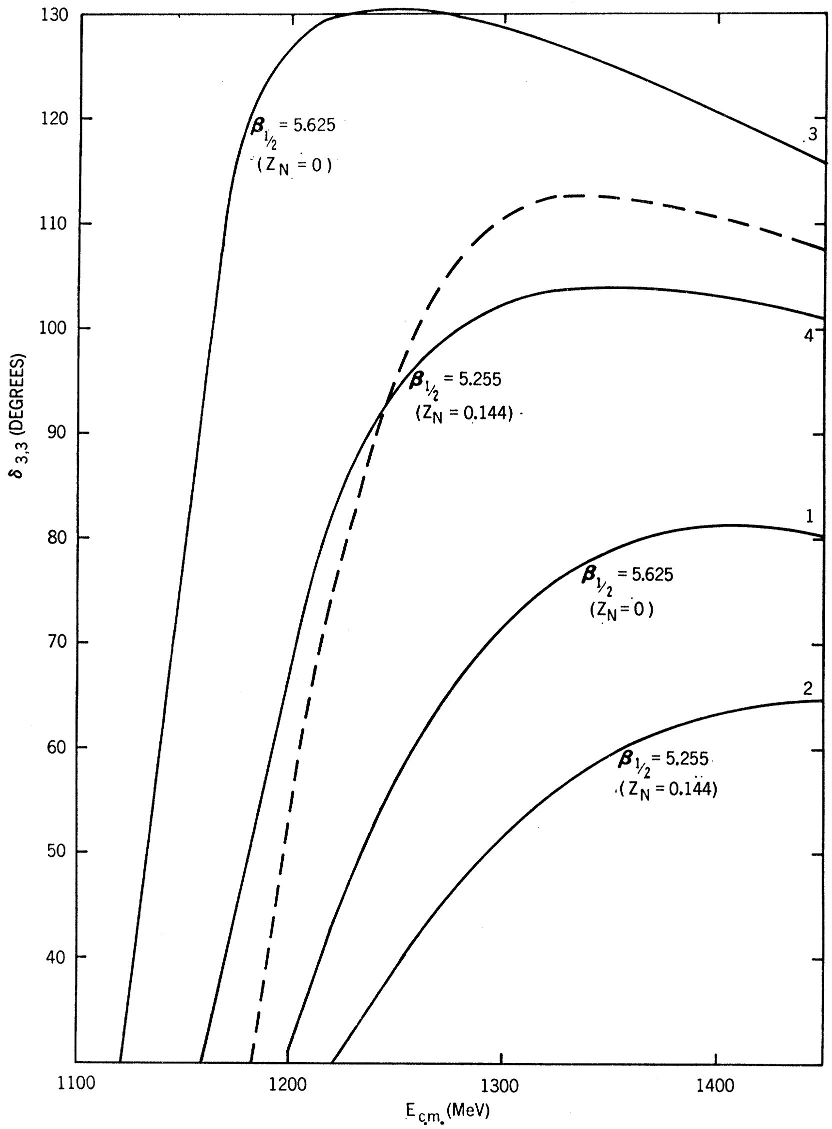

FIG. 3. We plot the real part of the (3,3) phase shift δ_{33} versus total center-of-mass energy $E_{\mathrm{c.m.}}$. Curves 1 and 2 represent behavior with no N^* included. Curves 3 and 4 have N^* included and $\beta_{3/2}=5.625$ and $g_{3/2}^2=0.1559$. The dashed curve is the input (3,3) phase shift corresponding to curve 4.

no further matrix inversions.[14] In view of their deficiencies, nevertheless, we present (1,1) πN phase shifts as calculated in our model, without the above modifications, in Fig. 6(a).

Some idea of the influence of single particle intermediate states is obtained by examining the behavior of the $\pi N'$ phase shift δ_{11}^0 [see Fig. 6(b)]. All the δ's of Fig. 6(b) pass through $-\pi/2$ in the vicinity of 1780 MeV as a consequence of Levinson's theorem which requires that for $Z_N=0$ the phase shift is $-\pi$ at infinite energy, while for $Z_N>0$ it is 0 at infinite energy.[8] It is interesting to note that δ_{11} is a more sensitive function of Z_N than δ_{11}'.

Inelasticity is very small in all channels, η_{33}, η_{13}, η_{31}, η_{11} being almost unity.

[14] This point was explained to us by R. D. Amado (private communication). For a similar problem in the Lee model see J. B. Bronzan, Phys. Rev. **139**, B751 (1965).

The significance of the above results for the various channels will be discussed in Sec. V.

IV. CALCULATION II

We now permit $N^*N'\pi$ coupling in our Hamiltonian, part of the rationale being that we are phenomenologically including more crossing symmetry in our model. For example, by introducing an elementary N^* field we include diagrams of the form shown in Fig. 7(a). The diagram of Fig. 7(b) is included in Chew-Low theory because of crossing symmetry, but was not present in our calculation I. It is clear, however, that the circled part of Fig. 7(b) is contained in some sense in the N^* bubble of Fig. 7(a). Note, however, that the diagram of Fig. 7(a) does not contribute a left-hand cut to our πN amplitude, so we are still neglecting the crossed πN cut.

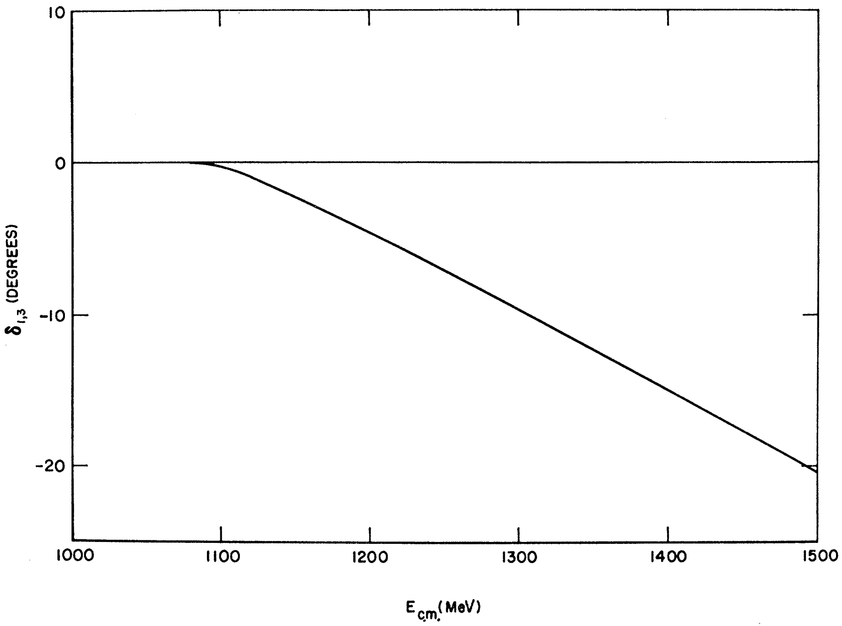

FIG. 4. Real part of the (1,3) shift δ_{13} versus total c.m. energy $E_{c.m.}$ for both $Z_N=0.$ and $Z_N=0.144$. The sensitivity to Z_N is so small it cannot be shown on this scale.

We first choose both $f_{1/2}$ and $f_{3/2}$ to be Gaussian. We treat the $N^*N'\pi$ interaction in the separable potential limit,[8] choosing the two independent parameters $g_{3/2}$ and $\beta_{3/2}$ so that $\pi N'$ scattering has a (3,3) resonance with approximately correct width and position. With $g_{1/2}^2=1.48$ and for several values of $\beta_{1/2}$ we once again solve the three-body equations for the πN scattering amplitude. This time we do obtain a resonance—in fact, for reasonable (3,3) widths and positions, the resonance reproduces itself rather well. An example is shown in Fig. 8. (In this figure and in the following ones, unless stated otherwise, $g_{1/2}^2=1.48$.) We have satisfied ourselves that the results are essentially independent of the functional form of the cutoff function and we demonstrate this fact in Fig. 9 where input and output (3,3) resonances are shown for a cutoff function $f(q^2)=[(\beta^2-1)/(q^2+\beta^2)]^2$. All remaining results are for Gaussian cutoffs. It is interesting to note that in order for there to be a resonance Z_N must be small. In our model the condition $Z_N=0$ is the integral condition that must be satisfied approximately

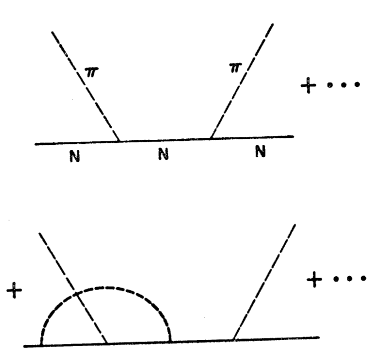

FIG. 5. Some diagrams neglected in our present model but included in the Chew-Low theory.

so that there can be a resonance in the Chew-Low model.[15] In Fig. 3 we plot the (3,3) phase shift for two values of Z_N (curves 3 and 4). Note the extreme sensitivity to this parameter. Roughly Z_N measures the fraction of time the nucleon is not a bound state of a pion and a nucleon.

We have varied all our parameters including $g_{1/2}^2$. Typical effects of such variations are shown in Fig. 10 for an input (3,3) resonance with position 1262 MeV and width 143 MeV. Figure 11 shows a case in which the input and the output are almost identical.

In Fig. 12 we plot the (3,1) phase shifts and in Fig. 13 the (1,1) phase shifts for the parameters used to obtain curves 3 and 4 of Fig. 3. In Fig. 14 the inelasticity factors η are plotted. The above results are all discussed in the following section.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

We first rather briefly discuss calculation I. Recall that the (3,3) phase shift in curves 1 and 2 of Fig. 3 would not pass through $\pi/2$ for any allowed values of the parameters. Our Born approximation used as input in an N/D calculation with elastic unitarity would of course give a resonance since there would be no restrictions on $\beta_{1/2}$. The lack thereof is a self-consistency requirement associated with inelasticity and

[15] See, for example, the discussion of G. Salzman, Phys. Rev. **99**, 973 (1955). In our model,

$$Z_N=1-\frac{g_{1/2}^2}{(2\pi)^3}\int d^3Q\frac{Q^2f_{1/2}^2(Q)}{\omega_Q^3}.$$

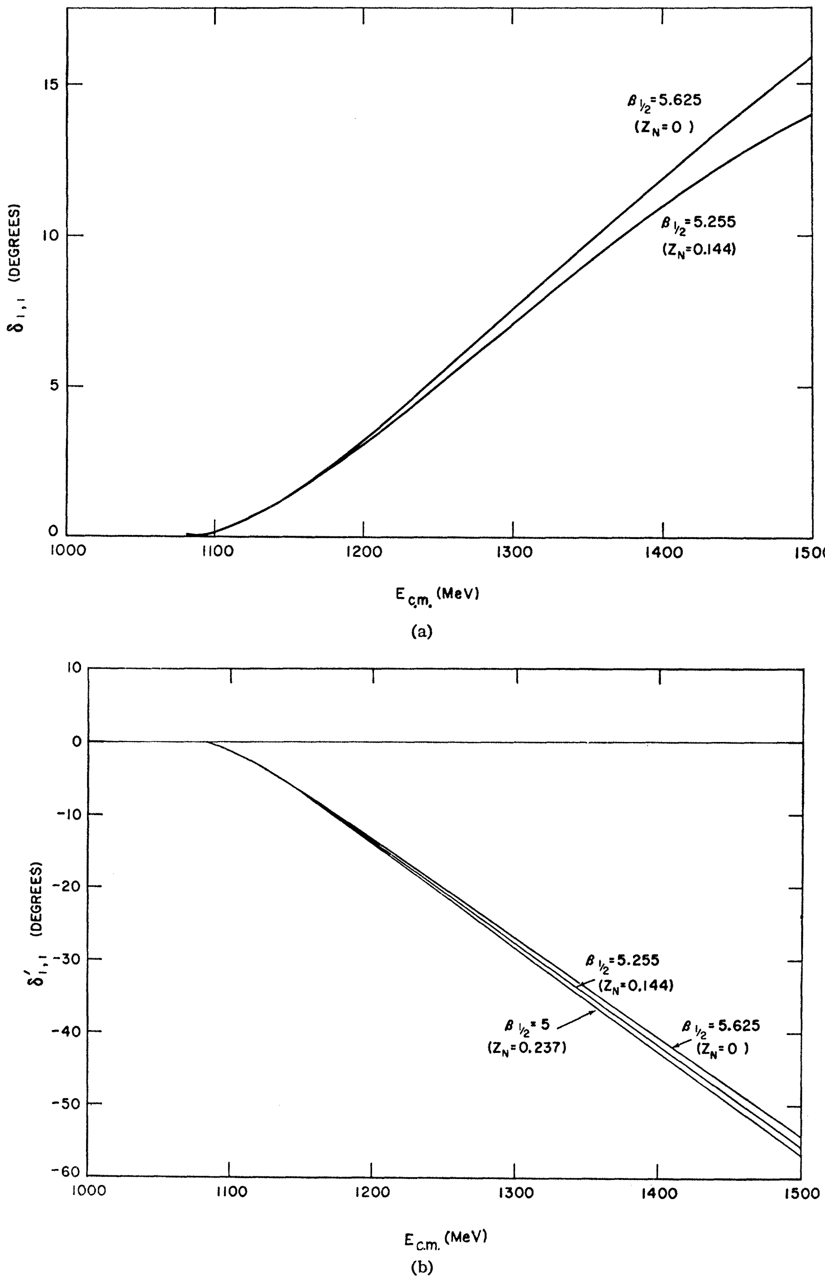

FIG. 6. (a) Real part of (1,1) phase shift versus total c.m. energy $E_{c.m.}$ for $Z_N=0$ and $Z_N=0.144$. (b) The $\pi N'$ phase shift δ_{11}' versus total c.m. energy $E_{c.m.}$ for various values of the cutoff parameter $\beta_{1/2}$. The scattering length is $\sim$0.56 F for all three choices.

contradicts no previous static calculations.[16] It is clear from this calculation that *even in a static theory which contains only pions and nucleons*, either the left-hand

[16] G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957). G. Schwartz, *ibid.* **137**, B212 (1965). Our conclusions differ from those of Lovelace (Ref. 2) although we are solving identical integral equations in the limit $Z_N=0$. The reason for this disagreement is that the approximation which Lovelace uses to obtain separable integral equations (although valid in certain energy regions) neglects consistency imposed by three body unitarity, setting the "sum of bubbles" in Fig. 2 equal to unity. For example, in order to obtain a resonance he must use values of the cutoff parameter for which his two-body (1,1) amplitude does not have a nucleon pole with correct position and residue. This difficulty is related to the problem of ghosts ($Z_N<0$) in our analysis.

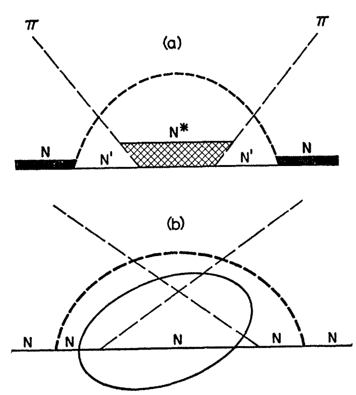

FIG. 7. Phenomenological inclusion of crossing symmetry.

cut or right-hand contributions from multiparticle states contribute significantly to the (3,3) resonance, a fact which, if realized, has certainly not received wide publicity.

The (1,3) and (3,1) phase shifts are small, uninteresting and consistent with previous static calculations. Our (1,1) phase shifts are small and have the wrong sign, but this error is to be expected since the sign of the scattering length is determined by the sign of the Born term, and by neglecting a direct nucleon pole in the three-body πN channel we have made our Born term eight times too small with the wrong sign.

We now discuss calculation II. Here we found that if a (3,3) resonance were included as input in the two body $\pi N'$ sector, it could be obtained self-consistently as output in the three body πN sector. (Figs. 9, 10, 11.) We may think of the above phenomenon as an unorthodox bootstrap calculation in which the input is on the right-hand cut. Note that including an N^* virtually on the right-hand cut is sufficient to give self-consistently a reasonable description of the (3,3) resonance. Calculation I showed that either the presence of the crossed πN cut or a more careful treatment of inelasticity was *necessary* to obtain a resonance. Calculation II suggests that *it is the proper treatment of inelasticity that is required for obtaining the (3,3) resonance, the*

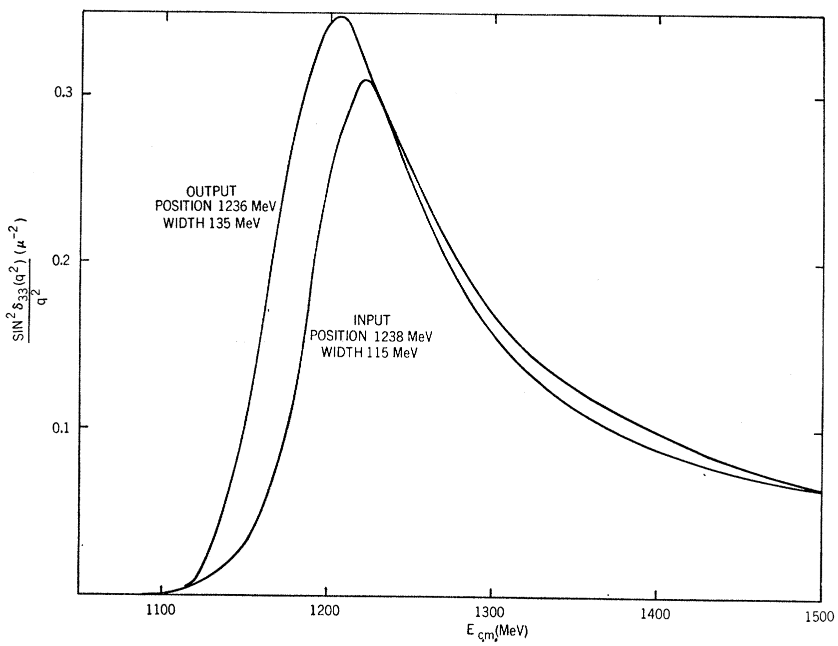

FIG. 8. Input and output (3,3) amplitudes in units of inverse pion masses (μ^{-2}) plotted against the total c.m. energy $E_{\text{c.m.}}$ for the case $\beta_{1/2}=5.255$ and $\beta_{3/2}=5.625$, $g_{3/2}=0.1559$.

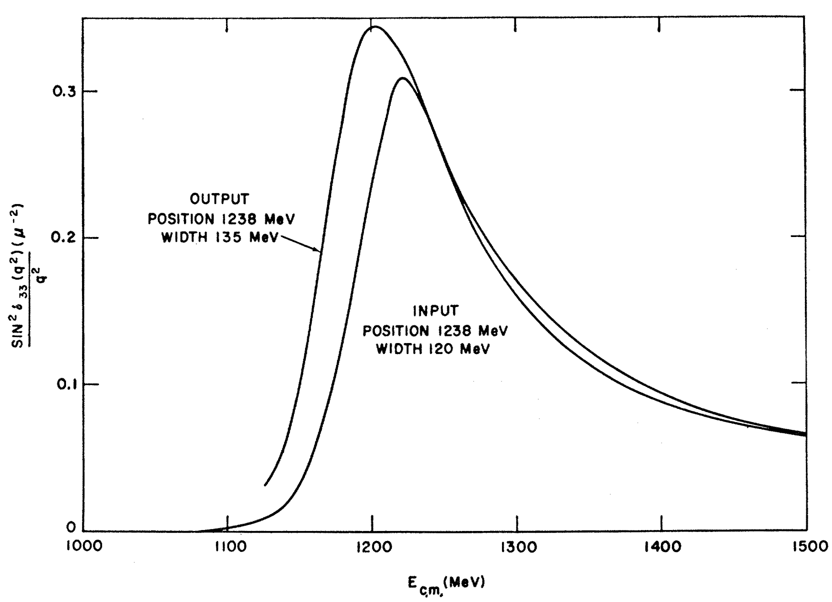

FIG. 9. For $f(q^2)\alpha(q^2+\beta^2)^{-2}$ we plot input and output (3,3) amplitudes versus total c.m. energy $E_{\text{c.m.}}$ for the case $\beta_{1/2}=9.238$ ($Z_N=0.118$) and $\beta_{3/2}=9.747$, $g_{3/2}^2=0.1457$.

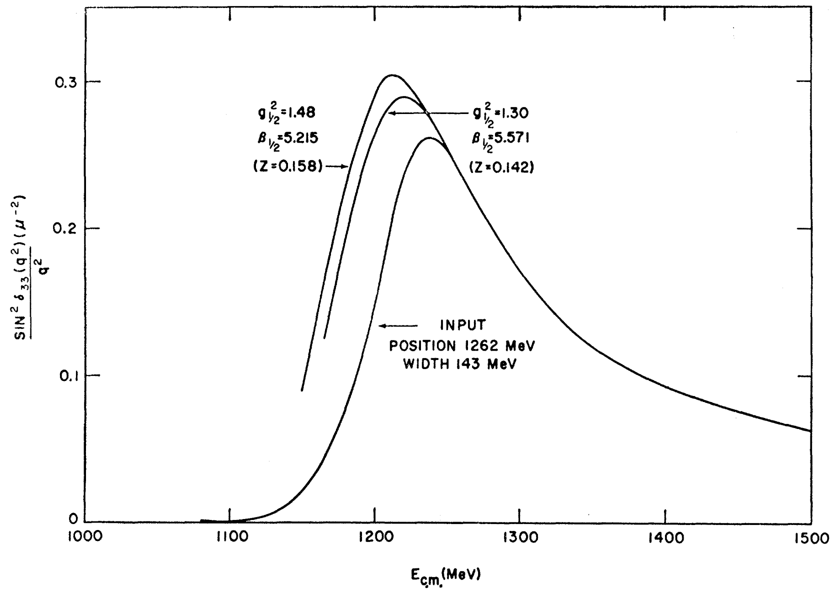

FIG. 10. Output (3,3) amplitudes corresponding to an input position of 1262 MeV and width 143 MeV ($\beta_{3/2}=5.612$, $g_{3/2}^2=0.1506$).

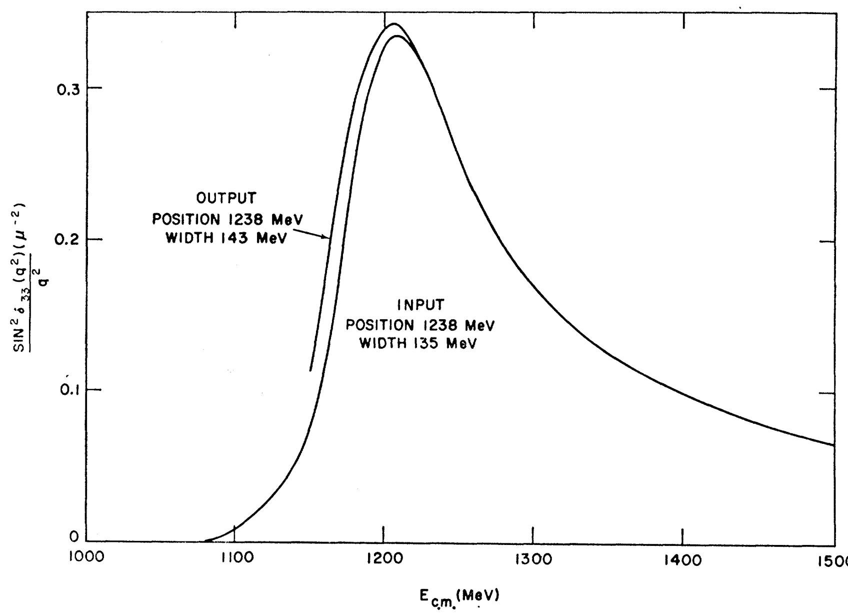

FIG. 11. Input and output (3,3) amplitudes plotted against total c.m. energy $E_{\text{c.m.}}$ for the case $\beta_{1/2}=5.313$ $(Z_N=0.118)$, $\beta_{3/2}=4.472$, $g_{3/2}^2=0.2845$.

effect of the crossed πN cut and other unknown factors being described by the single parameter Z_N.

The behavior of the (1,1) πN channel with the inclusion of πN^* intermediate states is quite dramatic (the phase shift once again has the wrong sign at low energies because we have not included a direct nucleon pole)—for example, with $Z_N=0.144$ the (1,1) phase shift rises rapidly through $\pi/2$ at 1310 MeV (Fig. 13). We speculate that the Roper resonance[4] is a manifestation of this behavior, although in contradiction to phase-shift analyses[17] η_{11} is almost unity. We would hope that the inclusion of recoil, single nucleon intermediate states *and* an $I=0$, $J=0\pi\pi$ interaction (all three effects can be included exactly in this model, the latter as a separable potential) would give the resonance at correct energy with correct inelasticity.[18] On the other hand

[17] L. Roper and R. Wright, Phys. Rev. **138**, B921 (1965) P. Auvil, C. Lovelace, A. Donnachie, and A. Lea, Phys. Letters, **12**, 76 (1964).

[18] The above speculation, assuming that one mechanism gives the resonance while another is largely responsible for the inelasticity is not without precedent. For example, Cook and Lee [L. F. Cook and B. W. Lee, Phys. Rev. **127**, 297 (1962)] obtained the 600- and 900-MeV πN resonances assuming a mechanism of virtual ρ production by single π exchange. This mechanism, however, gave poor inelastic cross sections. V. Teplitz [thesis, University of Maryland, 1962 (unpublished)] showed that by including coupling to πN^* states in addition to the above mentioned ρ mechanism, one obtained much better inelastic cross sections while maintaining the resonances without much change in width or position.

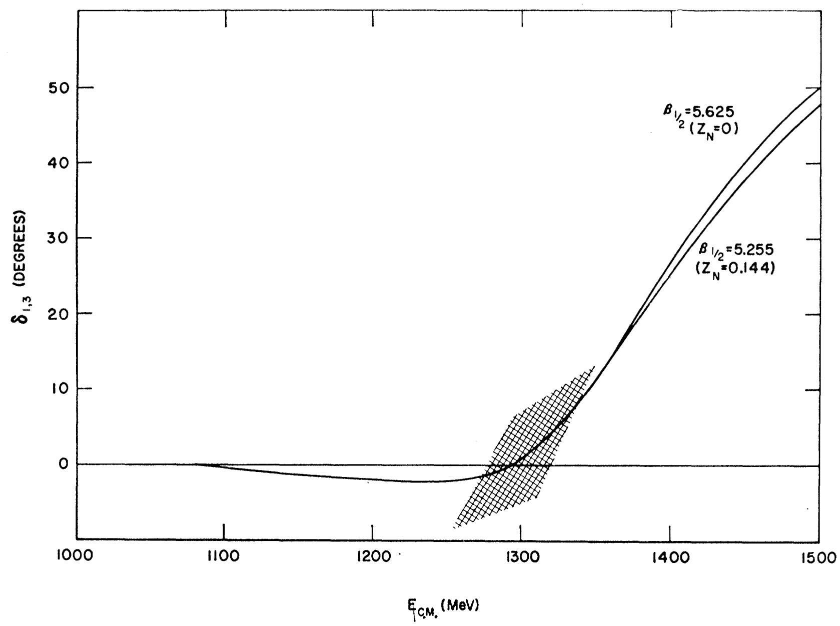

FIG. 12. Real part of the (1,3) phase shift δ_{13} versus total c.m. energy $E_{\text{c.m.}}$ for the cases $\beta_{1/2}=5.625$ ($Z_N=0.$) and $\beta_{1/2}=5.255$ ($Z_N=0.144$) with $\beta_{3/2}=5.625$, $g_{3/2}^2=0.1559$. There are computational difficulties in the shaded region.

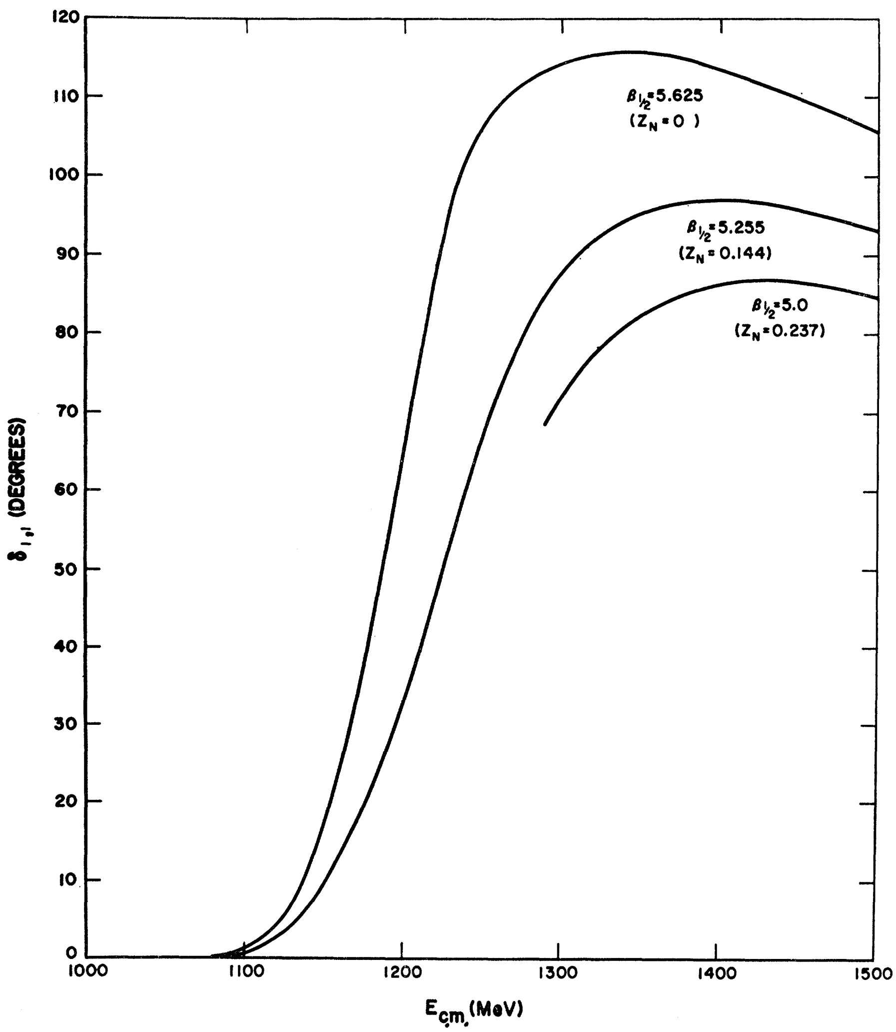

FIG. 13. Real part of the (1,1) phase shift δ_{11} versus total c.m. energy $E_{\text{c.m.}}$ for $\beta_{1/2}=5.625$ ($Z_N=0.$), $\beta_{1/2}=5.255$ ($Z_N=0.144$) and $\beta_{1/2}=5.0$ ($Z_N=0.237$) with $\beta_{3/2}=5.625$, $g_{3/2}^2=0.1559$.

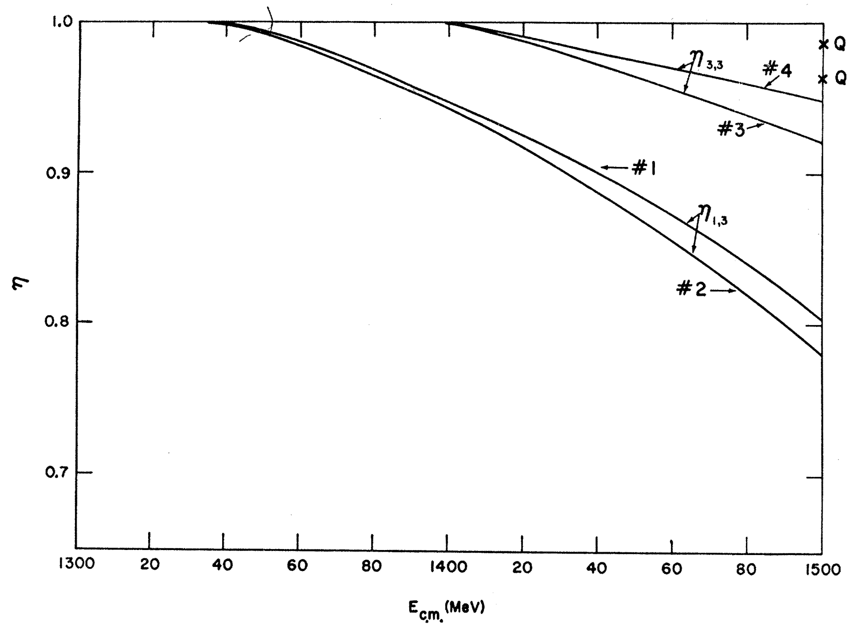

FIG. 14. We plot inelasticity factors η versus total c.m. energy $E_{c.m.}$ with $\beta_{1/2}=5.625$, $g_{3/2}^2=0.1559$. Curves 1 and 3 correspond to $\beta_{1/2}=5.255$ ($Z_N=0.144$) and curves 2 and 4 correspond to $\beta_{1/2}=5.625$ ($Z_N=0.$). The point Q represents η_{11} ($Z_N=0.144$) and Q' represents η_{11} ($Z_N=0.$).

Fig. 6(b) shows roughly that the effect of single-nucleon intermediate states is opposite to that of πN^* states and the competing effects could conceivably cancel. We are presently performing a calculation including these processes. In any case, the evidence seems to indicate that three-particle intermediate states consisting of a pion and a correlated (3,3) pion-nucleon pair contribute significantly to the (1,1) phase shift at low energies.

Finally, the (1,3) and (3,1) phase shifts (Fig. 12) are small at low energies, but in contradiction to phase-shift analyses change sign at higher energies. We do not attempt to defend this behavior, but hope that inclusion of further effects such as those described in the previous paragraph would improve these phase shifts.

In conclusion, using an off-energy-shell static theory which treats three-particle unitarity exactly, but neglects left-hand cuts, we have described rather well the low-energy P-wave $N\pi$ system. At the expense of going off the energy shell and neglecting the left-hand cuts, or amplitudes become solutions of linear Fredholm integral equations (the Chew-Low equation is nonlinear). Although the theory is not crossing-symmetric, some crossing can be included as inelasticity in a reasonable manner. An important asset of this model is that it can be extended to include other important effects and still be solved exactly. For example, at the expense of one further coupled integral equation we can introduce an $I=0$, $J=0\pi\pi$ interaction in the form of a separable potential and examine its effect on the low-energy $N\pi$ system. An $I=1$, $J=I\pi\pi$ interaction (ρ meson) could be included similarly.[19]

Relativistic off-energy-shell three-body equations have been proposed recently by several authors.[20] In the πN problem, for practical purposes, these equations can only be solved by using separable approximations such as we have used,[21] although the required calculations are somewhat more complicated than ours. In the static limit with separable approximations these equations are identical to those that we solve. We feel that using such relativistic equations and including $\pi\pi$ interactions [σ ("ABC"?), and ρ mesons] in addition to those interactions already considered in this calculation, one could successfully describe pion nucleon scattering below 1 BeV, obtaining most of the phase shifts and inelasticity factors for $J\leq\frac{5}{2}$ *and* production angular distributions. This possibly extravagant claim is based upon several factors. First, we note the success of the present calculation in obtaining a self-consistent description of the (3,3) resonance and providing a possible mechanism for the Roper resonance.[22] Second, we would include in a very tractable model exactly those mechanisms which Cook and Lee and Teplitz have shown in relatively crude calculations to be responsible for the 600- and 900-MeV resonances. Third, we would include exactly *with three-body unitarity* the isobar mechanism which Yodh and Olsson[23] have shown to give rather well

[19] A successful calculation using an approximate version of a relativistic off-energy-shell three-body theory with ρ-N intermediate states was reported by R. W. Finkel and L. Rosenberg, Bull. Am. Phys. Soc. **11**, 382 (1966).

[20] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965); R. Blankenbecler and R. Sugar, *ibid.* **142**, 1051 (1966).

[21] J. L. Basdevant and R. E. Kreps, Phys. Rev. **141**, 1398 (1966).

[22] Upon completion of this work a paper by D. Atkinson and M. B. Halpern, Phys. Rev. (to be published) was brought to our attention. It is interesting to note that these authors, using a completely different approach than we do, come to similar conclusions concerning the role of πN^* and σN intermediate states in the (1,1) channel.

[23] M. Olsson and G. B. Yodh, Phys. Rev. Letters **10**, 353 (1963).

the single-pion production angular distributions. A more difficult test of the model would be a correct description of the nonresonating phase shifts—our hope and inclination is that careful treatment of kinematics and three-body unitarity would suffice for their description. Finally, interesting questions can be asked *in the context of this model*; for example, what are the effects of varying the S-wave $\pi\pi$ scattering length on a variety of pion nucleon phenomena?[24] Also one can examine the validity of Dalitz-type analyses in obtaining widths and positions of resonances—i.e., do the bumps in the production cross sections which are due to the presence of N^* and ρ in the final state resemble the input N^* and ρ? We are presently performing such calculations and attempting to answer such questions as those described above.

[24] For a possible significance of the sign of the $\pi\pi$ scattering length, see G. F. Chew, Phys. Rev. Letters **16**, 60 (1966).

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Spin Effects on Triangle Graphs*

MELVIN MONTH
Department of Physics, University of Illinois, Urbana, Illinois
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It is shown that the triangle amplitude can be written as the scalar graph multiplied by a factor which contains only the characteristics of the external particles. In the case where the spins of the external particles are summed, their angles averaged, and only one partial-wave set (that is, a set of relative orbital angular momenta among the final-state particles) retained, this multiplicative factor is just a product of appropriate 3-momenta. The case $K^+p \to K\pi\pi\pi p$ at 3 BeV/c is considered; and it is demonstrated that the correct inclusion of threshold factors does not diminish the effect calculated in our earlier work, where the shape of the "κ enhancement" was successfully described by a triangle graph.

I. INTRODUCTION

IT is the primary purpose of this paper to study the influence of non-spin-zero particles on triangle amplitudes insofar as they impose specific threshold behavior upon the amplitudes. By thresholds we simply mean the various kinematical limits outlined by the available phase space. Our fundamental assumption is that the triangle graph is dominated by two major effects. First, there is a second-sheet Landau singularity,[1] denoted hereafter by LS, which in the cases of interest here approaches close to the physical region.[2,3] This effect is embodied in the scalar triangle graph. Second, the spins and parities of the particles involved in the graph impose upon the amplitude a minimal threshold behavior; that is, for a given graph, there is a set of lowest relative orbital angular momenta among the final-state particles. We will show that if one assumes a particular partial-wave amplitude (normally the one with lowest partial waves), then the triangle amplitude can be written as the product of the scalar graph and 3-momenta factors determined by the relative orbital angular momenta in the final state.

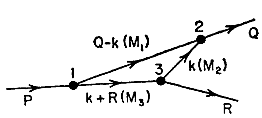

FIG. 1. The triangle graph. P, Q, and R are external 4-momenta. M_1, M_2, M_3 are internal masses. 1, 2, and 3 label the incident, sum, and difference vertices, respectively. k is an arbitrary 4-vector.

II. THE TRIANGLE GRAPH

In Fig. 1 is depicted the basic triangle mechanism. The vertices are labeled 1, 2, and 3 for incident, sum, and difference vertices, respectively. The masses of the internal particles (M_1, M_2, and M_3), as well as the external 4-momenta (P, Q, and R) are as designated in Fig. 1.

The scalar graph[1,4] is given by

$$J=\frac{1}{2}\int d^4k\{[(k+R)^2+M_3{}^2][(Q-k)^2+M_1{}^2]\times[k^2+M_2{}^2]\}^{-1}. \quad \text{(II.1)}$$

Or, introducing the Feynman parameters α_1, α_2, and

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