Necessary Dependence of Currents on Fields They Generate*

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It is shown that in local (proper) Lorentz-invariant theories involving axial-vector, or tensor currents (conserved or not), the latter must vanish, if they commute at equal times with the fields they generate. The need for explicit field dependence of currents is demonstrated for gradient-coupled spinless and massive spin-one fields, as well as for electrodynamics with minimal or nonminimal coupling. The field-dependence requirement is distinct from that (already needed for free fields) of "spreading points" to make the current operators well-defined. The relation between the two, however, essentially fixes the form of this dependence. Applications are made to partially conserved currents, $\partial_\mu j^\mu = \alpha\varphi$; if j^0 commutes with φ, the latter vanishes.

I. INTRODUCTION

THE recent successes of current-algebra methods have revived interest in consistency problems[1,2] of current commutation relations. As was first emphasized in Ref. 1, vanishing of the equal-time commutator $[j^0, j^k]$ of a conserved current implies vanishing of the current operator itself. Since straightforward evaluation of $[j^0, j^k]$ in terms of the canonical relations for a spin-$\frac{1}{2}$ field, for example, yields zero, this difficulty is ascribed to the singular nature of the current as a product of two field operators at a common point. It is indeed removed by redefining the current to be the limit of a nonlocal two-point operator,

$$j^\mu(x) = \lim_{\epsilon\to 0} \tfrac{1}{2}[\bar{\psi}(x+\epsilon), \gamma^\mu\psi(x)]$$

("spreading the points"); the commutator then no longer vanishes. Actually, both the above "disease," and its cure, are not limited to electric currents, but are characteristic of any vector (or tensor) density, irrespective of whether it is conserved[3] or even coupled to any other field. This will be shown in Sec. II. The nonlocal redefinition of a current is adequate in the free-field case; when there is coupling, however, the situation must be re-examined. In particular, one must investigate whether the current may be regarded as kinematically independent of the fields with which it interacts, as is the case for the above nonlocal form. (An operator is independent of a particular canonical variable if, and only if, it commutes, at equal times, with the canonically conjugate variable.) Then, sources composed of spin-$\frac{1}{2}$ fields are usually assumed to be independent of any boson variables since there are no constraints on the Fermi fields, and these commute at equal times with the (kinematically independent) Bose variables. We denote by canonical variables those components (e.g., φ, φ^0 for spin 0; A^k, F^{0l} for spin 1) which are the independent dynamical variables of the Bose field in question, whether or not they obey canonical commutation relations.

One of our main results will be that nontrivial interaction requires that the current densities be made explicitly dependent on (fail to commute with) the fields they generate. Otherwise, e.g., if the simple free-field definition of the type

$$j^\mu(x) = \lim_{\epsilon\to 0} \tfrac{1}{2}[\psi(x+\epsilon), \gamma^\mu\psi(x)]$$

is kept, there is in fact no interaction. That such additional current-field dependence might be required was first noticed[1,2,4] in electrodynamics, from considerations of gauge invariance. Recently, Okubo[3] obtained a no-interaction result in gradient-coupled scalar-meson theory of the $ps(pv)$ type. Here, the current was spread, but no explicit meson field dependence introduced, there being, of course, no gauge-like argument for it in this case.

We shall show (Secs. III–V) that for spinless bosons with gradient coupling to a vector current, massive spin-one fields coupled to vector or tensor sources, and for electrodynamics with current and also magnetic tensor coupling, it will be necessary (quite independently of gauge arguments) to introduce field dependence in the sources. On the other hand, for scalar sources [as in direct $ps(ps)$ coupling], no such requirement arises, in general, unless the bosons are massless. It seems likely that similar results hold for higher spin fields as well.[5]

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[1] J. Schwinger, Phys. Rev. Letters 3, 296 (1959); Phys. Rev. 130, 406 (1963).

[2] K. Johnson, Nucl. Phys. 25, 431 (1961).

[3] This fact has also been noted recently by S. Okubo, International Conference for Theoretical Physics, Trieste, Report No. IC/66/10 (unpublished). An earlier demonstration was given by L. S. Brown (private communication, and to be published).

[4] D. G. Boulware, Phys. Rev. (to be published).

[5] One higher spin result is that the stress energy tensor $T^{\mu\nu}$ of any (proper) Lorentz-covariant field theory must depend on the metric $g^{\lambda\sigma}$ not only in the classical way, but also in the "explicit" fashion discussed here. The properties of $T^{\mu\nu}$ commutators will be presented elsewhere.

We assume that the theory is local (proper) Lorentz-invariant, has positive Hilbert-space metric, and possesses spectral representations at least for the renormalized fields. No assumptions are made concerning reflection properties,[6] internal symmetries, or any details of the currents' structures; nor is it necessary to use canonical commutation relations in establishing the general results. We shall also see explicitly how the introduction of field dependence will restore the interaction. In general, we will see that the form of this dependence is essentially fixed in terms of the original current-spreading prescription.

Analogous considerations apply to theories involving partially conserved currents. In Sec. VI, we show that the equation $\partial_\mu j^\mu = \alpha\varphi$ alone implies φ^0 dependence of j^0 if φ is not to vanish. In the conclusion, we comment on the significance and uniqueness of our prescriptions within the framework of Lagrangian field theory.

II. GENERAL CURRENT NONCOMMUTATION REQUIREMENTS

Since the necessary nonvanishing of current-charge-density commutators will be our starting point, we summarize the basis for it (quite independently of coupling), before proceeding to interacting fields. This will also introduce our notation and techniques. As was first pointed out by Schwinger,[1] a conserved vector current $j^\mu(x)$ which satisfies the equal-time vacuum expectation relation

$$\langle 0|[j^0(\mathbf{r}),j^k(0)]|0\rangle = 0 \tag{1}$$

must vanish. For a conserved current,

$$\nabla\cdot\mathbf{j}(\mathbf{r}) = -\partial_0 j^0(\mathbf{r}) = i[j^0(\mathbf{r}),H], \tag{2}$$

and hence by Eq. (2), and the fact that $H|0\rangle = 0$,

$$i\langle 0|[j^0(\mathbf{r}),\nabla\cdot\mathbf{j}(\mathbf{r}')]|0\rangle = \langle 0|j^0(\mathbf{r})Hj^0(\mathbf{r}')+j^0(\mathbf{r}')Hj^0(\mathbf{r})|0\rangle. \tag{3}$$

For the Hermitian operator j^0, Eq. (1) means that $j^0(x)|0\rangle = 0$ or, by covariance, that $j^\mu(x)|0\rangle = 0$. But the Federbush-Johnson theorem[7] then implies that $j^\mu(x)$ itself vanishes. While conservation seems to have played a strong role in the derivation, it is in fact irrelevant, as we now see.[3] Proper Lorentz invariance fixes the spectral form of the vacuum commutator of a vector operator to be[8]

$$\langle 0|[j^\mu(x),j^\nu(0)]|0\rangle = \int_0^\infty ds\{\rho_1(s)\theta^{\mu\nu} - \rho_0(s)\partial^\mu\partial^\nu s^{-1}\}\Delta(x,s). \tag{4}$$

Here $\theta^{\mu\nu} = \eta^{\mu\nu} - s^{-1}\partial^\mu\partial^\nu$, $\eta^{\mu\nu}$ is the Lorentz metric $(-1,1,1,1)$ and $\Delta(x,s)$ is the commutator function, whose Fourier transform is $\epsilon(p^0)\delta(p^2+s)$. It has the property that, for $x^0=0$

$$\Delta(x,s)=0, \quad \partial^0\Delta(x,s) = i\delta(\mathbf{r}). \tag{5}$$

The spectral functions ρ_1, and ρ_0 are necessarily non-negative, being, respectively, the spin-1 and spin-0 intermediate-state contribution of mass $\sqrt{s}$ from $j^\mu(x)|0\rangle$.[9] Note that the spin-1 part is identically conserved, the spin-0 contribution representing the non-conserved part. From Eq. (4), we learn that the only nonvanishing equal-time commutator is

$$\langle 0|[j^0(\mathbf{x}),j^k(0)]|0\rangle = -i\int_0^\infty ds\, s^{-1}[\rho_1(s)+\rho_0(s)]\partial^k\delta(\mathbf{r}). \tag{6}$$

In view of the non-negative value of ρ_1 and ρ_0, it follows that the commutator of Eq. (6) vanishes if and only if $\rho_1(s) = 0 = \rho_0(s)$. However, the Wightman product corresponding to (4) is determined by the same weight functions:

$$\langle 0|j^\mu(x)j^\nu(0)|0\rangle = \int_0^\infty ds\{\rho_1(s)\theta^{\mu\nu} - \rho_0(s)s^{-1}\partial^\mu\partial^\nu\}\Delta^{(+)}(x,s),$$

where $\Delta^{(+)}(x,s)$ is the positive frequency function whose Fourier transform is $\theta(p^0)\delta(p^2+s)$. Thus, whether or not j^μ is conserved (i.e., whether or not ρ_0 vanishes), if Eq. (1) holds, $\langle 0|j^\mu(x)j^\nu(0)|0\rangle = 0$ which, of course, implies that $j^\mu(x)=0$. Similar results hold for higher spin currents as well. An illustration which will prove relevant is furnished by an antisymmetric tensor density $S^{\mu\nu}$, such as a Pauli moment term. The general form for the commutator here is

$$\begin{aligned}\langle 0|[S^{\mu\nu}(x),S^{\lambda\sigma}(0)]|0\rangle &= \int_0^\infty ds\{\sigma_1(s)[\eta^{\mu\sigma}\partial^\nu\partial^\lambda - \eta^{\nu\sigma}\partial^\mu\partial^\lambda + \eta^{\nu\lambda}\partial^\mu\partial^\sigma - \eta^{\mu\lambda}\partial^\nu\partial^\sigma] \\ &\quad + \sigma_2(s)[-\epsilon^{\mu\nu\tau\rho}\epsilon^{\lambda\sigma\kappa}{}_\rho\partial_\tau\partial_\kappa]\}\Delta(x,s) \\ &\equiv \int_0^\infty ds\{\sigma_1(s)\theta_1{}^{\mu\nu\lambda\sigma} + \sigma_2(s)\theta_2{}^{\mu\nu\lambda\sigma}\}\Delta(x,s).\end{aligned} \tag{7a}$$

The weights σ_1 and σ_2 are non-negative, corresponding

[6] In particular, we need not distinguish between axial and vector currents or scalar and pseudoscalar fields.

[7] P. Federbush and K. Johnson, Phys. Rev. **120**, 1926 (1960).

[8] Here and throughout any internal-symmetry indices are suppressed because they are irrelevant to our considerations. All equations may be read as matrix equations in the internal-symmetry space (positive definiteness here being defined in the sense of a matrix). Equations of the type (4) are the commutator forms of the Lehmann-Källén representations. See H. Lehmann, Nuovo Cimento **11**, 342 (1954), and G. Källén, Helv. Phys. Acta **25**, 417 (1952).

[9] This may most readily be seen by considering the zero-spatial momentum term in the sum over states. The spin-1 states are coupled only to j^k, and yield δ^{kl}, which Lorentz transforms to $g^{\mu\nu}+(p^\mu p^\nu/S)$. The spin-0 states, on the other hand, couple only to j^0, which Lorentz transforms to $(p^\mu p^\nu/S)$.

to the intermediate vector and pseudovector states[10] linked to the vacuum by S^{0k} and S^{kl}. The only equal-time commutator in Eq. (7a) which does not vanish automatically is the one with an odd number of temporal indices, namely,

$$\langle 0|[S^{0k}(\mathbf{r}),S^{mn}(0)]|0\rangle = -i\int_0^\infty ds\{\sigma_1(s)+\sigma_2(s)\}(\delta^{kn}\partial^{ln}-\delta^{km}\partial^n)\delta(\mathbf{r}). \quad (7b)$$

Hence, it must be non-null to avoid the $\sigma_1=0=\sigma_2$ catastrophe which (for the same reasons as in the j^μ discussion), would lead to the vanishing of the $S^{\mu\nu}$ operator. It is easy to see that the spin-$\frac{1}{2}$ $S^{\mu\nu}=\bar{\psi}\sigma^{\mu\nu}\psi$ contradicts this requirement, if evaluated without making the nonlocal redefinition.

Since naive application of the equal-time canonical commutation relations $\{\psi^\dagger(\mathbf{r},x^0),\psi(\mathbf{r}',x^0)\}=\delta(\mathbf{r}-\mathbf{r}')$ to the commutator of spin-$\frac{1}{2}$ currents, $[\bar{\psi}\gamma^0\psi(\mathbf{r},x^0),\bar{\psi}\gamma^k\psi(\mathbf{r}',x^0)]$ implies the vanishing of the latter, one must first redefine j^μ to make it less singular before evaluating the commutator. The points are to be "split" in j^k according to the definition

$$j^k(x)=\lim_{\epsilon\to 0}\bar{\psi}(x+\boldsymbol{\epsilon})\gamma^k\psi(x). \quad (8)$$

In this definition, the limit is to be taken symmetrically (averaged over all directions of the spatial vector $\boldsymbol{\epsilon}$ at the instant in question) after commuting. The use of a spatial limit avoids introduction of time nonlocality in the theory. It is easily seen that the prescription (8) (whether or not j^0 is also defined nonlocally[11]) does lead to a commutator (6) with a nonzero (but divergent) coefficient of $\partial^k\delta(\mathbf{r})$ and hence to nonzero j^μ. Further, in electrodynamics, the general form (6) is consistent with the physical requirement that $[Q,j^k(\mathbf{r})]=0$, namely, that Q generate constant phase transformations. For this requirement is, of course, compatible with a general form $[j^0(\mathbf{r}),j^k(\mathbf{r}')]=-i\chi(\mathbf{r}')\partial^k\delta(\mathbf{r}-\mathbf{r}')$ for the density commutator. It is precisely the splitting (8), needed for kinematical reasons, which will permit (but does not of itself require) the introduction of explicit field dependence in the coupled case. In electrodynamics, gauge arguments were invoked[1] to introduce A^k dependence in j^k. For, if j^k is to remain gauge-invariant in the limiting process, the phase $\exp[-ie(\lambda(x+\boldsymbol{\epsilon})-\lambda(x))]$ arising in Eq. (8) under a local-gauge transformation may be cancelled by introducing an exponential factor into j^k:

$$j^k(x)\to\lim_{\epsilon\to 0}\bar{\psi}(x+\boldsymbol{\epsilon})\exp\left[ie\int_0^\epsilon d\mathbf{y}\cdot\mathbf{A}(x+\mathbf{y})\right]\gamma^k\psi(x). \quad (9a)$$

This dependence on $\mathbf{A}$ clearly implies noncommutation of $\mathbf{j}$ and the electric field (conjugate to $\mathbf{A}$) at a given time. As we shall see in Sec. V, the noncommutation is in fact forced directly by demanding a nonvanishing electrodynamic coupling. Gauge invariance alone (though not Lorentz invariance) could equally well have been restored by the term

$$\exp\left[ie\int_0^\epsilon d\mathbf{y}\cdot\mathbf{A}^L(x+\mathbf{y})\right], \quad (9b)$$

where $\mathbf{A}^L$ is the longitudinal (gauge) part of $\mathbf{A}$; however, the interaction would vanish in this case. Having noted the basis for a nonlocal definition of the type (8), we shall henceforth assume it has been performed and concentrate on the question of whether and how to introduce field dependence into such a limiting definition for the various cases to be discussed. We shall see that the gauge-like prescription is actually correct also for other than electromagnetic interactions.

III. THE SPIN-ZERO FIELD

We consider a spinless meson field $\varphi(x)$ coupled directly to a scalar source, $k(x)$, and with gradient coupling to a vector current $j^\mu(x)$. The field equations have the form[12]

$$\varphi^\mu(x)=\partial^\mu\varphi(x)-j^\mu(x), \quad \partial_\mu\varphi^\mu(x)-s_0\varphi(x)=k(x). \quad (10)$$

The required vacuum products may be expressed in terms of the weight functions of the following three quantities[13]:

$$\langle 0|\varphi(x)\varphi(0)|0\rangle=\int_0^\infty ds\,\rho(s)\Delta^{(+)}(x,s), \quad (11a)$$

$$\langle 0|j^\mu(x)\varphi(0)|0\rangle=-\int_0^\infty ds\,\rho_1(s)\partial^\mu\Delta^{(+)}(x,s), \quad (11b)$$

[10] We have assumed parity conservation, for simplicity. Otherwise, the final conditions are unchanged, though there are more terms in the equations. An alternative form of this equation is obtained by noting that

$$\theta_2{}^{\mu\nu\lambda\sigma}\Delta(x,s)=(\eta^{\mu\lambda}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\lambda})s\Delta(x,s)+\theta_1{}^{\mu\nu\lambda\sigma}\Delta(x,s).$$

However, the form used here directly shows the origin of the spectral functions in the two types of intermediate states. The additional terms which result from parity nonconservation may yield nonvanishing $\langle[S^{0k},S^{0l}]\rangle$ and $\langle[S^{kl},S^{mn}]\rangle$ commutators but do not affect the terms with which we are concerned. The structure, as in the case of the vector function, is most readily derived by considering the zero-spatial momentum term in the sum over states and Lorentz transforming. Here there are two states of opposite parity, the vector state coupled to S^{0k} and the axial-vector state to S^{kl} at zero momentum.

[11] In electrodynamics j^0 is essentially the generator of local-gauge transformations and is closely related to the time-derivative terms of the Lagrangian. A nonlocal j^0 (which is unnecessary to avoid the paradox) would necessitate a nonlocal time-derivative term, thus changing the commutation relations of the ψ field and negating the whole motivation of the point spreading. In general, however, there is no reason not to spread the points of all of a current's components and, in some cases, it may be necessary. If j^0 is nevertheless taken to be non-local, the resultant commutators and field dependence remain unchanged.

[12] As is well known, there are essentially two ways to introduce gradient coupling. The one chosen here corresponds to a "minimal" interaction $j^\mu\varphi_\mu$ in the Lagrangian. The other, as obtained from a $j^\mu\partial_\mu\varphi$ form, leads to equations in which only $\partial_\mu j^\mu$ appears; it is then effectively a "direct" coupling theory with $k=\partial_\mu j^\mu$.

[13] We have assumed here, and throughout, that the spectra unction for the renormalized operators exists.

$$\langle 0|j^\mu(x)j^\nu(0)|0\rangle$$

$$=\int_0^\infty ds[\sigma_1(s)\theta^{\mu\nu}-\sigma_0(s)s^{-1}\partial^\mu\partial^\nu]\Delta^{(+)}(x,s)\,, \quad (11c)$$

by means of Eqs. (10). The functions ρ_1, σ, and σ_0 are non-negative; (ρ is manifestly so, and σ_1, σ_0 are just the spin 1 and 0 weights of Sec. II.); ρ_1 is only known to be real. The latter property follows from the requirement that the various operators have local commutation relations. Since the equal-time commutator corresponding to the Wightman product (11b) is proportional to $\rho_1\partial^k\Delta^{(+)}-\rho_1^*\partial^k\Delta^{(-)}$, it will be local only if $\rho_1=\rho_1^*$. The additional products we need are, by Eqs. (10),

$$\langle 0|[\varphi^\mu(x),j^\nu(0)]|0\rangle$$

$$=-\int_0^\infty ds\{\sigma_1(s)\theta^{\mu\nu} -[\sigma_0(s)+s\rho_1(s)]s^{-1}\partial^\mu\partial^\nu\}\Delta(x,s)\,, \quad (12a)$$

$$\langle 0|[\varphi^\mu(x),k(0)]|0\rangle$$

$$=\int_0^\infty ds\{(s-s_0)[\rho(s)+\rho_1(s)] +s\rho_1(s)+\sigma_0(s)\}\partial^\mu\Delta(x,s)\,, \quad (12b)$$

$$\langle 0|[k(x),\varphi(0)]|0\rangle$$

$$=\int_0^\infty ds[(s-s_0)\rho(s)+s\rho_1(s)]\Delta(x,s)\,, \quad (12c)$$

$$\langle 0|[k(x),k(0)]|0\rangle$$

$$=\int_0^\infty ds[(s-s_0)^2\rho(s) +2s(s-s_0)\rho_1(s)+s\sigma_0(s)]\Delta(x,s)\,. \quad (12d)$$

No assumption has been made concerning canonical commutation relations for $[\varphi,\varphi^0]$; these would lead to a sum rule,

$$\int_0^\infty ds[\rho(s)+\rho_1(s)]=1\,,$$

which we will not need. Also, we have used no properties of j^μ and k other than their Lorentz-transformation character.

The field dependence of j^μ and k may be separately examined. For, Eq. (11b) determines $\langle[j^0(\mathbf{r}),\varphi(0)]\rangle$, while Eq. (12a) involves[14] $\langle[j^k(\mathbf{r}),\varphi^0(0)]\rangle$. Thus we have the implications

$$\langle 0|[j^0(\mathbf{r}),\varphi(0)]|0\rangle=0\Rightarrow\int_0^\infty ds\,\rho_1(s)=0\,, \quad (13a)$$

and

$$\langle 0|[j^k(\mathbf{r}),\varphi^0(0)]|0\rangle=0\Rightarrow$$

$$\int_0^\infty ds\{s^{-1}[\sigma_0(s)+\sigma_1(s)]+\rho_1(s)\}=0\,. \quad (13b)$$

Taken together, Eqs. (13) imply that if j^μ is independent of both φ and φ^0, then both σ_1 and σ_0 vanish.[15] But Eq. (11c) then also leads to the vanishing of j^μ itself (and also of ρ_1). If, further, there were no k current, the only remaining weight function ρ would have the free form $\rho\propto\delta(s-s_0)$. Turning next to $k(x)$ and Eqs. (12b) and (12c), we see that k and φ automatically have vanishing equal time commutators by Eq. (12c) and $\Delta(t=0)=0$. The commutation of k and φ^0, on the other hand, does yield a sum rule,

$$\langle 0|[k(\mathbf{r}),\varphi^0(0)]|0\rangle=0\Rightarrow$$

$$\int_0^\infty ds\{s[\rho(s)+2\rho_1(s)+s^{-1}\sigma_0(s)] -s_0[\rho(s)+\rho_1(s)]\}=0\,, \quad (14a)$$

but the latter is not strong enough to annihilate $k(x)$ on the basis of the spectral form (12d), basically because $(s-s_0)\rho$ is not positive-definite. However, for the special case of vanishing bare mass ($s_0=0$), k must formally depend on φ;

$$\langle 0|[k,\varphi^0]|0\rangle=0\Rightarrow$$

$$\int_0^\infty ds\; s(\rho(s)+2\rho_1(s)+s^{-1}\sigma_0(s))=0\,. \quad (14b)$$

On the other hand, Eq. (12d) implies that (for $s_0=0$), $s^2[\rho(s)+2\rho_1(s)+s^{-1}\sigma_0(s)]\geq 0$ and so also that $s[\rho(s)+2\rho_1(s)+s^{-1}\sigma_0(s)]\geq 0$. But Eq. (14a) then forces the latter form, and hence $\langle kk\rangle$ to vanish. {Note that the possibility that $s[\rho+2\rho_1+s^{-1}\sigma_0]\propto\delta(s)$ is excluded by (14b)}.

We now illustrate the way in which the j^μ current may be redefined to include field dependence, thereby allowing the relevant weight functions to be nonzero. Consider the charge-independent $ps(pv)$ theory with $j_\alpha{}^\mu=ig_0\bar\psi\gamma^5\gamma^\mu\tau_\alpha\psi$. We redefine $j_\alpha{}^\mu$ as the limit, at any instant,

$$j_\alpha{}^\mu(x)=\lim_{\epsilon\to 0} ig_0\bar\psi(x+\boldsymbol{\epsilon})\gamma^5\gamma^\mu\tau_\alpha \times\exp\left\{g_0\gamma^5\int_0^\epsilon dy_l\tau_\beta\varphi^\beta{}_{,l}(x+\mathbf{y})\right\}\psi(x)\,; \quad (15)$$

clearly φ still commutes with j^μ, but φ^0 no longer does.

[14] Equations (12) also govern $\langle[\varphi^k,j^0]\rangle$, but we emphasize the other form, as φ^0 (unlike φ^k) is a canonical variable.

[15] S. Okubo's derivation (Ref. 3) of a no-interaction theorem for the $ps(pv)$ theory is closely related to this result. For, by Eq. (10), j^0 is just $\varphi^0-\partial^0\varphi$. Hence if j^k commutes with φ^0 and $\partial^0\varphi$, it will commute with j^0 and hence vanish.

To compute the lack of commutation introduced by Eq. (15), we assume canonical commutation relations $[\varphi_\alpha{}^0(\mathbf{r},x^0),\varphi_\beta(\mathbf{r}',x^0)]=i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')$ and obtain

$$\langle 0|[\varphi_\alpha{}^0(\mathbf{r}),j_\beta{}^k(0)]|0\rangle =\lim_{\epsilon\to 0} 2\delta_{\alpha\beta}ig_0{}^2\,\mathrm{tr}\gamma^k S(-\boldsymbol{\varepsilon})\boldsymbol{\varepsilon}\cdot\nabla\delta(\mathbf{r}). \quad (16)$$

Here $S(-\boldsymbol{\varepsilon})$ is the interacting propagator for the ψ field:

$$S(x)=i\langle 0|T(\psi(x)\bar{\psi}(0))|0\rangle.$$

Note that $\langle[\varphi^0,j^0]\rangle$ still vanishes in the limit (since $\mathrm{tr}\gamma^0S(-\boldsymbol{\varepsilon})$ does) even though j^0 has also been made ostensibly φ-dependent. This is of course demanded for consistency with the equal-time vanishing of Eq. (12a). We may evaluate the trace if ψ obeys canonical anticommutation relations and obtain

$$\langle 0|[\varphi_\alpha{}^0(\mathbf{r}),j_\beta{}^k(0)]|0\rangle =i\delta_{\alpha\beta}(4/3\pi^2)g_0{}^2\epsilon^{-2}\partial^k\delta(\mathbf{r}) =i\delta_{\alpha\beta}\int_0^\infty ds\, s^{-1}[\sigma_1(s)+\sigma_0(s)]\partial^k\delta(\mathbf{r}). \quad (17)$$

a nonvanishing (if divergent) result for $\int_0^\infty ds\, s^{-1}[\sigma_1(s)+\sigma_0(s)]$. The current then no longer vanishes. The prescription (15), which was introduced in analogy to the electromagnetic definition (9a), is not of course determined by any gauge arguments, and its uniqueness must be investigated. If we compare the equal-time commutators of $\langle 0|[j^k,\varphi^0]|0\rangle$ and $\langle 0|[j^k,j^0]|0\rangle$, we see from Eqs. (11c) and (13b) that these are equal if we assume j^0 commutes with φ (i.e., is independent of the momentum φ^0); both are proportional to $\int_0^\infty ds\, s^{-1}[\sigma_1(s)+\sigma_0(s)]$. However, the leading term in the explicit evaluation of $\langle[j^k,j^0]\rangle$ comes from the part of Eq. (15) independent of the exponential. Hence the φ dependence of j^k must be so defined as to agree with the free result. If one assumes canonical commutation rules for $[\varphi,\varphi^0]$, this requirement forces precisely the form (15) for the exponential.[16] Of course, if j^0 is made dependent on φ^0, one would not expect the exponential form to hold; another (definite) field dependence would then follow. This argument is actually a general one for fixing the vacuum expectation value of the linear term of the field dependence. The exponential form, while necessary where a gauge argument applies, is not dictated by our considerations since we only treat the linear dependence on φ.

IV. MASSIVE VECTOR FIELD

The general massive spin-1 field equations, in the presence of both vector (j^μ) and antisymmetric tensor ($S^{\mu\nu}$) sources, are

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-S^{\mu\nu}, \quad (18a)$$

and

$$\partial_\nu F^{\mu\nu}+S_0A^\mu=j^\mu, \quad (18b)$$

where j^μ need not be conserved. Proceeding along the same lines as for spin 0, we take as independent commutators the following:

$$\langle 0|[A^\mu(x),A^\nu(0)]|0\rangle =\int_0^\infty ds[\tau_1(s)\theta^{\mu\nu}-\tau_0(s)s^{-1}\partial^\mu\partial^\nu]\Delta(x,s), \quad (19a)$$

$$\langle 0|[S^{\mu\nu}(x),A^\lambda(0)]|0\rangle =\int_0^\infty ds\,\rho_1(s)[\eta^{\mu\lambda}\partial^\nu-\eta^{\nu\lambda}\partial^\mu]\Delta(x,s), \quad (19b)$$

$$\langle 0|[S^{\mu\nu}(x),S^{\lambda\sigma}(0)]|0\rangle =\int_0^\infty ds\{\sigma_1(s)\theta_1{}^{\mu\nu\lambda\sigma}+\sigma_2(s)\theta_2{}^{\mu\nu\lambda\sigma}\}\Delta(x,s). \quad (19c)$$

Here the weight functions τ_1, and τ_0 are the non-negative intermediate spin-1 and -0 contributions; while σ_1 and σ_2 are the (likewise non-negative) intermediate vector and pseudovector weights of Eq. (7a); ρ_1 is real, but need not be positive. The remaining spectral forms we need follow from (19) and the field equations (18);

$$\langle 0|[S^{\mu\nu}(x),F^{\lambda\sigma}(0)]|0\rangle =-\int_0^\infty ds\{[\sigma_1(s)+\rho_1(s)]\theta_1{}^{\mu\nu\lambda\sigma} +\sigma_2(s)\theta_2{}^{\mu\nu\lambda\sigma}\}\Delta(x,s), \quad (20a)$$

$$\langle 0|[j^\mu(x),A^\lambda(0)]|0\rangle =-\int_0^\infty ds\{[(s-s_0)\tau_1(s)+s\rho_1(s)]\theta^{\mu\nu} +\tau_0(s)s_0s^{-1}\partial^\mu\partial^\lambda\}\Delta(x,s), \quad (20b)$$

$$\langle 0|[j^\mu(x),F^{\lambda\sigma}(0)]|0\rangle =\int_0^\infty ds[(s-s_0)(\tau_1(s)+\rho_1(s))+s(\rho_1(s)+\sigma_1(s))] \times[\eta^{\mu\sigma}\partial^\lambda-\eta^{\mu\lambda}\partial^\sigma]\Delta(x,s), \quad (20c)$$

$$\langle 0|[j^\mu(x),j^\lambda(0)]|0\rangle =\int_0^\infty ds\{[(s-s_0)^2\tau_1(s)+2s(s-s_0)\rho_1(s)+s^2\sigma_1(s)]\theta^{\mu\lambda} -\tau_0(s)s_0{}^2s^{-1}\partial^\mu\partial^\lambda\}\Delta(x,s). \quad (20d)$$

Turning first to the spin current, we note the consequences of its commutation with the dynamical variables A^k and F^{0k}

$$\langle 0|[S^{0k}(\mathbf{r}),A^l(0)]|0\rangle=0\Rightarrow\int_0^\infty ds\,\rho_1(s)=0, \quad (21a)$$

[16] One of us (S.D.) thanks R. Arnowitt for a discussion of this question.

$$\langle 0|[S^{kl}(\mathbf{r}),F^{0m}(0)]|0\rangle=0\Rightarrow$$

$$\int_0^\infty ds[\sigma_1(s)+\sigma_2(s)+\rho_1(s)]=0. \quad (21b)$$

If both these commutators (21) vanish, so must the positive weights σ_1 and σ_2, and hence as we have already seen in Sec. II, so must $S^{\mu\nu}$ itself. This result is independent of the properties of j^μ. For the current, the corresponding statements are

$$\langle 0|[j^0(\mathbf{r}),A^k(0)]|0\rangle=0\Rightarrow\int_0^\infty ds\{\tau_1(s)+\rho_1(s)$$
$$-s_0s^{-1}[\tau_0(s)+\tau_1(s)]\}=0, \quad (22a)$$

$$\langle 0|[j^k(\mathbf{r}),F^{0l}(0)]|0\rangle=0\Rightarrow$$
$$\int_0^\infty ds\{s[\tau_1(s)+2\rho_1(s)+\sigma_1(s)]$$
$$-s_0[\tau_1(s)+\rho_1(s)]\}=0. \quad (22b)$$

But we know that $(s-s_0)^2\tau_1+2s(s-s_0)\rho_1+s^2\sigma_1\geq 0$, and $\tau_0\geq 0$, since these quantities are the usual spin-1 and spin-0 weight functions in Eq. (20d) for $\langle j^\mu j^\nu\rangle$. Hence, the non-negative integral

$$I=\int_0^\infty ds\, s^{-1}\{(s-s_0)^2\tau_1(s)+2s(s-s_0)\rho_1(s)$$
$$+s^2\sigma_1(s)+s_0^2\tau_0(s)\}$$

will vanish if and only if j^μ itself does. But we may rewrite I as the sum of two terms,

$$I=I_1+I_2=\int_0^\infty ds\{s[\tau_1+2\rho_1+\sigma_1]-s_0[\tau_1+\rho_1]\}$$
$$+s_0\int_0^\infty ds\, s^{-1}[s_0(\tau_1+\tau_0)-s(\tau_1+\rho_1)],$$

the parts I_1 and I_2 vanish if j^μ is independent of F^{0k} and A^l, respectively; if both commutators of Eqs. (22) vanish, so does I, and therefore j^μ. Thus, both the vector and tensor sources of the massive vector field must separately depend on the latter if they are not to vanish. We consider the two commutators

$$\langle[j^k(\mathbf{r}),F^{0l}(0)]\rangle=i\int_0^\infty ds[(s-s_0)(\tau_1+\rho_1)$$
$$+s(\rho_1+\sigma_1)]\delta^{kl}\delta(\mathbf{r}),$$

$$\langle[j^k(\mathbf{r}),j^0(0)]\rangle=-i\int_0^\infty ds\, s^{-1}[(s-s_0)\tau_1+2s(s-s_0)\rho_1$$
$$+s^2\sigma_1+s_0^2\tau_0]\partial^k\delta(\mathbf{r}),$$

whose spectral integrals are *a priori* unequal (except if $s_0=0$, when we have the electromagnetic case $F^{0k}{}_{,k}=j^0$). However, if condition (22a) is imposed (which, as discussed, is reasonable), then it is easily verified that the integrals become identical, and we must therefore have the A dependence of j^k be such as to reproduce the $\langle[j^k,j^0]\rangle$ result. If we take the concrete model in which j^μ is the (spread-out) spin-$\frac{1}{2}$ current, and canonical commutation relations for $[A^k,F^{0l}]$, then we are forced to the prescription

$$j^\mu(x)=\lim_{\epsilon\to 0}g_0\bar{\psi}(x+\boldsymbol{\epsilon})\gamma^\mu\left[\exp\left(ig_0\int_0^\epsilon d\mathbf{y}\cdot\mathbf{A}(x+\mathbf{y})\right)\right]\psi(x),$$

where g_0 is the coupling constant. Naturally, if one dropped condition (22a) (assuming this is feasible), a different but fixed prescription for $j^\mu[A]$ would ensue.

The current-field relations (22) are also of interest in connection with the question of sum rules for the renormalized mass of vector mesons. If we take, for simplicity, $S^{\mu\nu}=0$ and a conserved[17] current, then ρ_1, τ_0, σ_1 and σ_2 vanish and each of Eqs. (22) can be viewed as an alternative sum rule for the physical mass

$$\int_0^\infty ds\, s^{-1}\tau_1(s)=s_0^{-1}\int_0^\infty ds\,\tau_1(s), \quad (23a)$$

$$\int_0^\infty ds\, s\tau_1(s)=s_0\int_0^\infty ds\,\tau_1(s). \quad (23b)$$

As we have seen, both rules cannot be correct, and indeed they can only be satisfied by $\tau_1(s)\propto\delta(s-s_0)$. Both could be incorrect if neither current commutator in Eq. (22) vanished. However, as we have seen, it is most natural to keep condition (22a) and hence (23a); the "reciprocal mass" sum rule is then favored.[18]

V. ELECTRODYNAMICS

Quantum electrodynamics differs from the massive-vector case in that it cannot be cast into a form which is simultaneously local and manifestly Lorentz-covariant. However, the spectral representation of the three-dimensional (radiation-gauge) form of electrodynamics can yield sufficient information for our purposes, because it possesses a positive Hilbert-space metric.

As was mentioned in Sec. II, gauge invariance already suggests that the spatial current will have to depend on the vector potential in order to maintain the invariance of the limiting definition of j^k. Another (related) argument for this dependence is the following. Since we must have $[j^0,j^k]\neq 0$, for the current to exist, it follows that $[\partial_kF^{0k},j^l]\neq 0$ or $[F^{0k\,L},j^l]\neq 0$, where $F^{0k\,L}$ is the longitudinal part of the electric field. Under a Lorentz transformation,[19] the longitudinal and transverse parts

[17] If the current is not conserved, spin-0 excitations are possible ($\partial_\mu A^\mu\neq 0$), and they contribute to the mass spectrum through a term $s_0\int_0^\infty ds\, s^{-1}\tau_0$ on the right of Eq. (23a).

[18] K. A. Johnson, Nucl. Phys. **31**, 464 (1962). The necessity of noncommutation of a conserved j^μ and F^{0k} was already noted in this reference.

[19] Note that three-dimensional rotation invariance is not sufficient for the argument, as longitudinal and transverse parts are irreducible, and hence do not mix, under spatial rotations alone.

of the field mix, so that one expects $[F^{0k\,T},j^l]$ not to vanish either. Thus the constraint equation $\partial_k F^{0k}=j^0$ (whose existence is of course connected with that of a gauge group) already implies that j^k depends on A^l.

To verify these expectations, we turn to the relevant spectral forms. In the presence of both a current j^μ and a magnetic-tensor source $S^{\mu\nu}$, the field equations are just Eqs. (18) of Sec. IV, with $s_0=0$. The lack of manifest covariance implicit in use of the radiation gauge ($\nabla\cdot\mathbf{A}=0$) is reflected in the appearance of the differential operator $\nabla^\lambda=(0,\nabla)$, and of the inverse Laplacian ∇^{-2}, in place of ∂^λ and the invariant mass s^{-1}, respectively, in the spectral representations. As in Sec. IV, we start with the three functions[20]

$$\langle 0|[A^\mu(x),A^\lambda(0)]|0\rangle =\int_0^\infty ds\,\rho(s)(\delta_\tau{}^\mu-\partial^\mu\nabla_\tau\nabla^{-2}) \times(\eta^{\tau\lambda}-\nabla^\tau\partial^\lambda\nabla^{-2})\Delta(x,s)\,, \quad (24a)$$

$$\langle 0|[A^\mu(x),S^{\lambda\sigma}(0)]|0\rangle =-\int_0^\infty ds\,\rho_1(s)(\eta^{\mu\nu}-\partial^\mu\nabla^\nu\nabla^{-2}) \times(\partial^\lambda\delta_\nu{}^\sigma-\partial^\sigma\delta_\nu{}^\lambda)\Delta(x,s)\,, \quad (24b)$$

$$\langle 0|[S^{\mu\nu}(x),S^{\lambda\sigma}(0)]|0\rangle =\int_0^\infty ds\{\sigma_1(s)\theta_1{}^{\mu\nu\lambda\sigma}+\sigma_2(s)\theta_2{}^{\mu\nu\lambda\sigma}\}\Delta(x,s)\,. \quad (24c)$$

The spectral functions ρ, σ_1, and σ_2 are non-negative, and ρ_1 is real. The remaining forms will then follow from these and the Maxwell equations, as in the massive case.

$$\langle 0|[S^{\mu\nu}(x),F^{\lambda\sigma}(0)]|0\rangle =-\int_0^\infty ds\{[\sigma_1(s)+\rho_1(s)]\theta_1{}^{\mu\nu\lambda\sigma} +\sigma_2(s)\theta_2{}^{\mu\nu\lambda\sigma}\}\Delta(x,s)\,, \quad (25a)$$

$$\langle 0|[j^\mu(x),A^\lambda(0)]|0\rangle =-\int_0^\infty ds\,s[\rho_1(s)+\rho(s)] \times(\eta^{\mu\lambda}-\nabla^\mu\partial^\lambda\nabla^{-2})\Delta(x,s)\,, \quad (25b)$$

$$\langle 0|[j^\mu(x),F^{\lambda\sigma}(0)]|0\rangle =\int_0^\infty ds\,s[\rho(s)+2\rho_1(s)+\sigma_1(s)] \times(\eta^{\mu\sigma}\partial^\lambda-\eta^{\mu\lambda}\partial^\sigma)\Delta(x,s)\,, \quad (25c)$$

$$\langle 0|[j^\mu(x),j^\lambda(0)]|0\rangle =\int_0^\infty ds\,s^2[\rho(s)+2\rho_1(s)+\sigma_1(s)]\theta^{\mu\lambda}\Delta(x,s)\,. \quad (25d)$$

[20] See for example Ref. 4, or K. A. Johnson *1964 Brandeis Lecture Notes* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965).

Turning first to the nonminimal source, we have the two relevant conditions from (24b) and (25a):

$$\langle 0|[S^{0k}(\mathbf{r}),A^l(0)]|0\rangle=0\Rightarrow\int_0^\infty ds\,\rho_1(s)=0\,, \quad (26a)$$

$$\langle 0|[S^{kl}(\mathbf{r}),F^{0m}(0)]|0\rangle=0\Rightarrow \int_0^\infty ds[\sigma_1(s)+\sigma_2(s)+\rho_1(s)]=0\,. \quad (26b)$$

Together, these would imply the vanishing of σ_1 and σ_2 and so of $S^{\mu\nu}$ itself. For the current, Eq. (25c) yields the implication

$$\langle 0|[j^l(\mathbf{r}),F^{0kT}(0)]|0\rangle=0\Rightarrow \int_0^\infty ds\,s[\rho(s)+2\rho_1(s)+\sigma_1(s)]=0\,, \quad (27)$$

which is precisely the requirement that $\langle 0|[j^0,j^k]|0\rangle$ vanish; j^k is thus necessarily $\mathbf{A}$-dependent. Note also that since Eqs. (25b) and (25c) assure us that $\langle[j^0,F^{0k}]\rangle =0=\langle[j^0,A^k]\rangle$ at equal times, no effective field dependence can be built into j^0 in agreement with the fact that j^0 need not be spread in the absence of interaction.[11] [Conditions (26) and (27) together also imply, of course, that only free photons survive. For then, $\sigma_1=0=\sigma_2=\rho_1$ and $\int_0^\infty ds\,s\rho(s)=0$. Hence $j^\mu=0$ and ρ must have the form $Z\delta(s)$].

When the charged fields are Bose fields, the constraint equations automatically produce field dependence of the type we have just found. As an example, we consider a charged spin-1 field interacting nonminimally with the electromagnetic field; both j^μ and $S^{\mu\nu}$ already depend on the electromagnetic field through the A^2 terms in the Lagrangian. Calling the charged field amplitudes $(\varphi^\mu,G^{\mu\nu})$, the Lagrangian is

$$\mathfrak{L}=\mathfrak{L}_m+\mathfrak{L}_B+ieA_\mu G^{\mu\nu}q\varphi_\nu+\tfrac{1}{2}i\lambda F_{\mu\nu}\varphi^\mu q\varphi^\nu\,, \quad (28)$$

where $\mathfrak{L}_m$ and $\mathfrak{L}_B$ are the free-photon and boson parts, q is the 2×2 charge matrix, and λ measures the anomalous magnetic-moment coupling. The associated current and magnetic sources are

$$j^\mu=ieG^{\mu\nu}q\varphi_\nu\,,\quad S^{\mu\nu}=i\lambda\varphi^\mu q\varphi^\nu\,. \quad (29)$$

Since the variable φ^0 is constrained to be

$$\varphi^0=(1/\mu^2)[i\lambda F^{0k}q\varphi_k-(\partial_k-ieqA_k)G^{0k}]\,, \quad (30)$$

both j^k and S^{0k} are field-dependent already at the "classical" level. Indeed, if we compute the commutators in (26a) and (27), we find, in terms of the canonical relations

$$[F^{0k}(\mathbf{r}),A^l(0)]=i\delta^{klT}(\mathbf{r})\equiv i[\delta^{kl}\delta(\mathbf{r})-\nabla^k\nabla^l 1/4\pi|\mathbf{r}|]\,,$$

that

$$[j^k(\mathbf{r}),F^{0l}(0)]=ie^2\{G^{0k}G^{0l}/\mu^2+\delta^{kl}\varphi^m\varphi_m-\varphi^k\varphi^l\}\delta(\mathbf{r}), \quad (31a)$$

$$[S^{0k}(\mathbf{r}),A^l(0)]=i(\lambda^2/\mu^2)\varphi^k(\mathbf{r})\varphi^m(\mathbf{r})\delta_m{}^l(\mathbf{r})^T, \quad (31b)$$

so that by taking vacuum expectation values, we have, in the notation of Sec. III,

$$\int_0^\infty ds\, s(\rho(s)+2\rho_1(s)+\sigma_1(s)] =\tfrac{1}{3}e^2\langle G^{0m}G_m{}^0/\mu^2+2\varphi^m\varphi_m\rangle, \quad (32a)$$

and

$$\int_0^\infty ds\,\rho_1(s)=-(\lambda^2/\mu^2)\tfrac{1}{3}\langle\varphi^m\varphi_m\rangle. \quad (32b)$$

We thus find that the classical dependence is of the form required by our theorem.

It may be thought that these considerations would indicate that the new field dependences we have discussed are only needed for spin-$\frac{1}{2}$ field currents, where there is no classical field dependence to start with. However, this is not always the case; additional field dependence may also be needed when classical field terms are present. One may also spread the points in j^μ and $S^{\mu\nu}$ to insert field dependence, and this may be necessary for consistency; that is, a nonclassical term may be required in any case, even when classical dependence is already present. If such dependence is inserted, it yields additional dependence on A in the same way as for the spin-$\frac{1}{2}$ currents. Since the constants are all divergent, it is impossible to draw any firm conclusions concerning the necessity of such dependence by considering the Lehmann forms, except to observe that if the points are spread in j^k, then nonclassical dependence must follow and conversely. A first-order calculation for the spin-0 field coupled to an external A indicates that it is possible to define the singular integrals so that such nonclassical dependence is not needed, but that, in general, current conservation and gauge invariance are most readily insured by inserting such dependence. In the case of electrodynamics, it is impossible to maintain any unnecessary A^k dependence in a Lorentz covariant limit, since the additional terms simply act to project out the gauge-invariant current conserving parts and, if the theory already has these properties, the additional dependence will disappear.[4]

VI. PARTIALLY CONSERVED CONSTANTS

As might be expected, the techniques of the previous sections have application to theories possessing partially conserved currents (e.g. PCAC), namely, to systems in which the relation

$$\partial_\mu j^\mu=\alpha\varphi \quad (33)$$

holds. There is one general condition, irrespective of the structure of j^μ or of φ. Starting from

$$\langle 0|\,j^\mu(x)j^\nu(0)\,|0\rangle =\int_0^\infty ds\{\rho_1(s)\theta^{\mu\nu}-\rho_0(s)s^{-1}\partial^\mu\partial^\nu\}\Delta^{(+)}(x,s), \quad (34)$$

it follows by successive differentiation that

$$\langle 0|\,j^\mu(x)\varphi(0)\,|0\rangle=\alpha^{-1}\int_0^\infty ds\,\rho_0(s)\partial^\mu\Delta^{(+)}(x,s), \quad (35)$$

and

$$\langle 0|\,\varphi(x)\varphi(0)\,|0\rangle=\alpha^{-2}\int_0^\infty ds\, s\rho_0(s)\Delta^{(+)}(x,s). \quad (36)$$

By Eq. (35), the nonvanishing equal-time commutation is

$$\langle 0|[j^0(\mathbf{r}),\varphi(0)]|0\rangle=i\alpha K\delta(\mathbf{r})=i\alpha^{-1}\int_0^\infty ds\,\rho_0(s)\delta(\mathbf{r}), \quad (37)$$

whose vanishing implies that the (non-negative) function ρ_0 vanishes. Thus, by Eq. (36), if j^0 commutes with φ at equal times, φ vanishes and the current is in fact totally conserved. This result is not surprising, since φ just represents the nonconserved part of j^μ and as was seen in Sec. II, the $[j^k,j^0]$ noncommutation requirement holds for the two parts of j^μ separately. If φ is a dynamical field, then by Eq. (36), the function $[\alpha^{-2}s\rho_0(s)]$ is by definition the spectral function ρ for $\langle 0|\,\varphi(x)\varphi(0)\,|0\rangle$, and we have a sum rule for the vacuum commutator K in terms of ρ: $K=\int_0^\infty ds\, s^{-1}\rho(s)$.

VII. CONCLUSIONS

We have seen in some detail the necessity of explicit field dependence in currents if their interaction is not to vanish. This meant, in particular, that fermion currents, ostensibly built up from spinor fields alone, must be redefined not only to be limits of spatially nonlocal forms but also they contain field-dependent factors. The only requirements involved were locality and that the currents transform as vectors or tensors under the proper Lorentz group. The nonlocal definition, needed to prevent j^μ from vanishing, then provides a natural and essentially unique way of inserting such field dependence, in agreement with gauge-invariance arguments where these apply.

The need for introducing explicit nonclassical terms is not restricted to fermion currents; for even if a boson current does carry explicit field dependence in a given case, this will not necessarily yield the required $\partial^k\delta(\mathbf{r})$ form. Similar considerations might be expected to apply to direct current-current couplings $\sim J_\lambda j^\lambda$ even in the absence of elementary boson fields. However, the present technique does not, at least superficially, yield any corresponding requirements for sources of Fermi fields. This problem is especially relevant to a pure quark framework, in which only fermions are supposed to appear as fundamental fields. What is the relation of the new definitions to the usually assumed completeness of Lagrangian field theory? Because of the singular nature of products of field operators at a point, a formal Lagrangian involving interactions is actually not well-defined except in terms of a limiting process similar to that performed on the currents. In the same sense, it must now be supplemented with field dependence as well (at least if it is to lead to nonvanishing interactions). Of course, both these prescriptions can be successfully neglected (in practice) in obtaining the Lagrange equations, just as they can be neglected in many applications of the latter. They become manifest only when detailed local current properties are probed, through the $\partial^k\delta(\mathbf{r})$ terms in vacuum-expectation values.[21] Possible physical consequences of these prescriptions should bear further investigation. In the one case which has been pursued in some detail,[20] electrodynamics, the exponential factor removes the quadratic photon self-energy divergence as well as a finite current nonconserving term in the photon-photon "box" diagram. It is not yet clear what role the corresponding factor plays for other theories.

Finally, one may ask whether the field dependence discussed here is not just a reflection of the necessity of counter terms in the Lagrangian. Except in the example of the scalar source coupled to a zero-mass and zero-spinfield, this is definitely not the case. First, we can apply the considerations to the renormalized fields and currents directly, yielding the same results. In order to see this more clearly, we consider the spin-0 case. If we attempt to say that j^k dependence on φ is through φ^k and that $j^k-\lambda\varphi^k$ has a vanishing commutator with φ^0, then we must use as the time component of the current $j^0-\lambda\varphi^0$, and we have renormalized φ^μ which now interacts with $\bar{j}^\mu=j^\mu-\lambda\varphi^\mu$. Now, we apply our considerations to $\bar{j}^\mu$. By construction, $\bar{j}^k$ is independent of φ. If, in addition, $\bar{j}^0$ is independent of φ^0, then $\bar{j}^\mu$ vanishes. Hence we can see that the dependence cannot be of such a trivial nature, and that in the process of removing φ dependence, we must introduce φ^0 dependence. This is avoidable only if j^μ is simply proportional to φ^μ, in which case there is no interaction associated with j^μ. We are forced to the conclusion that, for an interacting theory, the dependence must be noncovariant in the sense that different components display different dependences. The above argument breaks down in the case of the (one component) scalar source. On the other hand, there is no theorem either for this model, unless the bare mass is zero, when k must depend on φ. We could then consider $k+\lambda\varphi$ independent of φ, but then the equation becomes $(-\partial^2+\lambda)\varphi=k+\lambda\varphi$ instead of $-\partial^2\varphi=k$, and a mass term is introduced, nullifying the theorem.

Note added in proof. The considerations of this work deal exclusively with the vacuum expectation values of the commutators and shed no light on their operator properties. This question can only be dealt with by more detailed dynamical considerations beyond the scope of this paper; however, in the case of Fermion QED, there is apparently no operator dependence,[4] but in other theories there may be (and in general is) such operator dependence.

[21] For the stress-tensor commutators, the corresponding terms are of the type $\partial^k\partial^l\partial^m\delta(\mathbf{r})$.