Low-Energy Theorem for the Weak Axial-Vector Vertex*

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A low-energy theorem is derived for the weak axial-vector vertex. The theorem enables one to calculate from strong or electromagnetic processes the two leading terms in the expansion of the axial-vector vertex in powers of the leptonic four-momentum transfer. Applications to weak pion production, K_{e4} decay, and radiative μ capture are discussed. In particular, we express the radiative μ-capture matrix element, up to and including contributions linear in the leptonic four-momentum transfer and the photon four-momentum, in terms of the elastic weak form factors and pion photoproduction amplitudes.

INTRODUCTION

It is well known[1] that the infrared divergent order k^{-1} term in the matrix element for the radiation of a photon of four-momentum k in any process (the matrix element of the electric current) can be expressed solely in terms of the matrix element for the same process with no current present. Low[2] has shown that current conservation enables one to calculate the electric-current matrix element not only to order k^{-1} but also to order k^0 in terms of the process without the current. In the present work, we derive analogous results for the matrix elements of the axial-vector current. We express each such matrix element in terms of the matrix element for the process with no axial-vector current and the matrix element of the *divergence* of the axial-vector current. The relation is exact to orders k^{-1} and k^0. Under the assumption of a partially conserved axial-vector current (PCAC),[3] we can relate the matrix element of the divergence to the corresponding matrix element of the pion source, which is physically measurable, apart from the usual small off-mass-shell extrapolation.[4] Thus we obtain an expression for the axial-vector matrix element solely in terms of physically measurable quantities. Clearly, this shows that the essential point in Low's derivation is not current conservation, but the fact that the divergence of the current is independently measurable. Results analogous to ours will hold for any current whose divergence is known.

In Sec. I we state two simple lemmas and rederive Low's results from them. In Sec. II we derive the analogous results for the strangeness-conserving weak axial-vector current. We also show how these results are modified when two currents are present, instead of only one. As an application, we treat in Sec. III the following processes: Weak pion production, K_{e4} decay, and radiative μ capture. In particular, we find in the case of radiative μ capture that when terms of order qk and higher are neglected (q=lepton four momentum transfer, k=photon four-momentum), the matrix element can be expressed solely in terms of the elastic weak form factors and pion photoproduction amplitudes. This means that structure effects linear in q or linear in k are determined, giving the leading corrections to the radiative μ capture matrix element previously calculated by Manacher and Wolfenstein[5] and by Opat.[6]

I. LOW'S RESULTS FOR THE ELECTROMAGNETIC CURRENT

We consider the process $a\rightarrow b+\gamma$, where a and b are arbitrary hadron states. The matrix element for the process is given by[7]

$$ {}_{\text{out}}\langle b\gamma|a\rangle_{\text{in}}=ie(2\pi)^4\delta^{(4)}(p_a-p_b-k)\times\frac{1}{(2\pi)^{3/2}(2k_0)^{1/2}}N_aN_b\epsilon_\alpha{}^\star M_\alpha\,, \quad (1)$$

where p_a, p_b, N_a, and N_b are, respectively, the total four-momenta and the normalization factors of the particles in states a and b, ϵ_α is the polarization of the photon, and k is its four-momentum. The quantity M_α is related to the matrix element of the electromagnetic current $J_\alpha{}^{\text{EM}}$ by

$$N_aN_bM_\alpha={}_{\text{out}}\langle b|J_\alpha{}^{\text{EM}}|a\rangle_{\text{in}}\,. \quad (2)$$

Conservation of the electromagnetic current implies that

$$k_\alpha M_\alpha=0\,. \quad (3)$$

We state two simple mathematical lemmas from which Low's results are easily derived. [In the follow-

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[7] Four-vectors have an imaginary fourth component: $p=(\mathbf{p},p_4)=(\mathbf{p},ip_0)$ and $p\cdot q=\mathbf{p}\cdot\mathbf{q}+p_4q_4=\mathbf{p}\cdot\mathbf{q}-p_0q_0$. The quantity $p^\star$ is defined by $\mathbf{p}^\star=\mathbf{p}^*$, $p_4{}^\star=-p_4{}^*$, where * denotes the ordinary complex conjugate. The γ matrices ($\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$) are all Hermitian, and satisfy $\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\delta_{\alpha\beta}$.

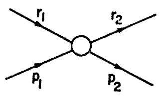

FIG. 1. The nonradiative process.

ing, $O(k^n)$ denotes terms of the nth or higher degree in k.]

Lemma 1: Let $M_\alpha^{\rm II}$ be an arbitrary four-vector function of arbitrary independent variables, which is independent of the four-vector k_α. Then $k_\alpha M_\alpha^{\rm II}=O(k^2)$ implies that $M_\alpha^{\rm II}=0$. Proof: Obvious.

Lemma 2: If $k_\alpha M_\alpha=0$ and $M_\alpha=M_\alpha^{\rm I}+M_\alpha^{\rm II}+O(k)$, where $M_\alpha^{\rm II}$ is independent of k and where $k_\alpha M_\alpha^{\rm I}=0$, then $M_\alpha=M_\alpha^{\rm I}+O(k)$. Proof: $k_\alpha M_\alpha=k_\alpha M_\alpha^{\rm I}=0$ implies $k_\alpha M_\alpha^{\rm II}=O(k^2)$, so by Lemma 1, $M_\alpha^{\rm II}=0$.

Note that "independent of k" is not the same as "zeroth order in k." For example, $k_\alpha/p\cdot k$ is zeroth order in k but is not independent of k.

We now apply the lemmas to the two cases considered by Low. First we discuss scattering of a charged scalar particle from a neutral scalar particle (Fig. 1). We denote the initial and final neutral-particle four-momenta by r_1 and r_2, and the corresponding charged-particle four-momenta by p_1 and p_2. Let $T(s=p_1\cdot r_1+p_2\cdot r_2,\ t=(r_1-r_2)^2,\ \Delta_1=p_1^2+M_1^2,\ \Delta_2=p_2^2+M_2^2)$ be the transition amplitude for the nonradiative process in Fig. 1. We have explicitly indicated the dependence of T on the amount by which the external charged particles are off the mass shell, since the amplitude for the process in which the photon is emitted from one of the external charged particle lines involves the off-mass-shell nonradiative amplitude. The physical nonradiative amplitude is $T(s,t,0,0)$.

The radiative amplitude gets contributions from two types of terms: terms in which the photon is radiated from an external charged particle line [Figs. 2(a) and 2(b); we call these terms $M_\alpha^{\rm ext}$] and terms in which the photon is radiated from an internal line [Fig. 2(c); we call these terms $M_\alpha^{\rm int}$]. The infrared divergent terms come only from $M_\alpha^{\rm ext}$, while $M_\alpha^{\rm int}$ is finite at $k=0$. We write

$$M_\alpha^{\rm int}(k)=M_\alpha^{\rm int}(0)+O(k)\,. \tag{4}$$

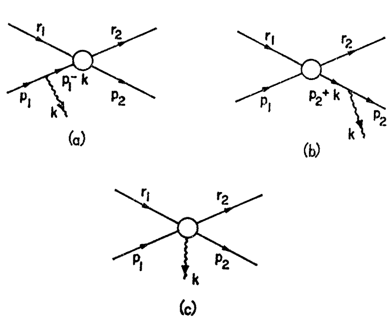

FIG. 2. Contributions to the radiative process.

We can express $M_\alpha^{\rm ext}$ in terms of T,

$$M_\alpha^{\rm ext}=\frac{(2p_2+k)_\alpha}{(p_2+k)^2+M_2^2}T[s+r_2\cdot k,\,t,\,0,\,(p_2+k)^2+M_2^2]
+T[s-r_1\cdot k,\,t,\,(p_1-k)^2+M_1^2,\,0]\frac{(2p_1-k)_\alpha}{(p_1-k)^2+M_1^2}\,. \tag{5}$$

We expand T with respect to k, giving

$$\begin{aligned}M_\alpha^{\rm ext}={}&\frac{(2p_2+k)_\alpha}{(2p_2+k)\cdot k}T[s,t,0,0]-T[s,t,0,0]\frac{(2p_1-k)_\alpha}{(2p_1-k)\cdot k}\\
&+\left(\frac{p_{2\alpha}}{p_2\cdot k}r_2\cdot k+\frac{p_{1\alpha}}{p_1\cdot k}r_1\cdot k\right)\frac{\partial}{\partial s}T[s,t,0,0]\\
&+2p_{2\alpha}\frac{\partial}{\partial\Delta_2}T[s,t,0,\Delta_2]\bigg|_{\Delta_2=0}\\
&+2p_{1\alpha}\frac{\partial}{\partial\Delta_1}T[s,t,\Delta_1,0]\bigg|_{\Delta_1=0}+O(k)\,.\end{aligned} \tag{6}$$

We are now able to rewrite M_α in the form required by Lemma 2,

$$M_\alpha=M_\alpha^{\rm ext}+M_\alpha^{\rm int}=M_\alpha^{\rm I}+M_\alpha^{\rm II}+O(k)\,, \tag{7}$$

$$\begin{aligned}M_\alpha^{\rm I}={}&\frac{(2p_2+k)_\alpha}{(2p_2+k)\cdot k}T[s,t,0,0]-T[s,t,0,0]\frac{(2p_1-k)_\alpha}{(2p_1-k)\cdot k}\\
&+\left(\frac{p_{2\alpha}}{p_2\cdot k}r_2\cdot k+\frac{p_{1\alpha}}{p_1\cdot k}r_1\cdot k-r_{2\alpha}-r_{1\alpha}\right)\\
&\times\frac{\partial}{\partial s}T[s,t,0,0]\,,\end{aligned} \tag{7a}$$

$$\begin{aligned}M_\alpha^{\rm II}={}&(r_{2\alpha}+r_{1\alpha})\frac{\partial}{\partial s}T[s,t,0,0]\\
&+2p_{2\alpha}\frac{\partial}{\partial\Delta_2}T[s,t,0,\Delta_2]\bigg|_{\Delta_2=0}\\
&+2p_{1\alpha}\frac{\partial}{\partial\Delta_1}T[s,t,\Delta_1,0]\bigg|_{\Delta_1=0}+M_\alpha^{\rm int}(0)\,.\end{aligned} \tag{7b}$$

From this we conclude that $M_\alpha=M_\alpha^{\rm I}+O(k)$. In other words, the terms in the radiative amplitude of order k^0 as well as those of order k^{-1} have been determined.

The procedure required by the lemmas may be reduced to a simple recipe: (1) Write down $M_\alpha^{\rm ext}$, the sum of the terms in which the photon is radiated from an external charged particle line. (2) Drop all terms from $M_\alpha^{\rm ext}$ which are explicitly independent of k, giving a truncated amplitude $M_\alpha^{\rm ext\prime}$. (3) Add to $M_\alpha^{\rm ext\prime}$ a ΔM_α independent of k so as to make $k_\alpha(M_\alpha^{\rm ext\prime}+\Delta M_\alpha)=O(k^2)$. Then $M_\alpha^{\rm ext\prime}+\Delta M_\alpha$ is the $M_\alpha^{\rm I}$ required by the lemma.

Let us apply this recipe to the problem considered above. We have computed $M_\alpha{}^{\text{ext}}$ in Eq. (5). In the first term let us expand T with respect to the off-mass-shell variable but not with respect to the energy variable:

$$T[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ =T[s+r_2\cdot k,\, t,\, 0,\, 0]+[(p_2+k)^2+M_2{}^2] \\ \times\left\{\frac{\partial}{\partial\Delta_2}T[s+r_2\cdot k,\, t,\, 0,\, \Delta_2]\bigg|_{\Delta_2=0}+O(k)\right\}. \quad (8)$$

The off-mass-shell derivative term in this expansion, when substituted into Eq. (5), leads only to terms which are either explicitly independent of k or are of first order in k. These terms are dropped in forming the truncated matrix element. We repeat this procedure for the second term in Eq. (5). Thus the truncated matrix element $M_\alpha{}^{\text{ext}\prime}$ is

$$M_\alpha{}^{\text{ext}\prime}=\frac{(2p_2+k)_\alpha}{(2p_2+k)\cdot k}T[s+r_2\cdot k,\, t,\, 0,\, 0] \\ -T[s-r_1\cdot k,\, t,\, 0,\, 0]\frac{(2p_1-k)_\alpha}{(2p_1-k)\cdot k}+O(k). \quad (9)$$

The divergence of $M_\alpha{}^{\text{ext}\prime}$ is

$$k_\alpha M_\alpha{}^{\text{ext}\prime}=T[s+r_2\cdot k,\, t,\, 0,\, 0] \\ -T[s-r_1\cdot k,\, t,\, 0,\, 0]+O(k^2) \\ =(r_2\cdot k+r_1\cdot k)\frac{\partial}{\partial s}T[s,t,0,0]+O(k^2). \quad (10)$$

Hence, ΔM_α is determined to be

$$\Delta M_\alpha=-(r_2+r_1)_\alpha\frac{\partial}{\partial s}T[s,t,0,0]. \quad (11)$$

Clearly, $M_\alpha{}^{\text{ext}\prime}+\Delta M_\alpha$ is identical with the $M_\alpha{}^{\text{I}}$ of Eq. (7a) to order k.

As a second illustration of the procedure, we consider the case when the charged particles have spin $\frac{1}{2}$. This is the simplest photon analog of the axial-vector case, since the axial-vector vertex cannot couple to a spin-zero particle line. As we shall see, the only difference from the preceding case is due to slight complications caused by spin.

We start by writing down $M_\alpha{}^{\text{ext}}$,

$$M_\alpha{}^{\text{ext}}=\bar{u}(p_2)\left\{\left(i\gamma_\alpha+i\frac{\mu}{2M_2}\sigma_{\alpha\beta}k_\beta\right)\frac{1}{i\gamma\cdot(p_2+k)+M_2} \right.\\ \times T[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ +T[s-r_1\cdot k,\, t,\, (p_1-k)^2+M_1{}^2,\, 0] \\ \left.\times\frac{1}{i\gamma\cdot(p_1-k)+M_1}\left(i\gamma_\alpha+i\frac{\mu}{2M_1}\sigma_{\alpha\beta}k_\beta\right)\right\}u(p_1). \quad (12)$$

Let us discuss the first term of Eq. (12). Because the final fermion is off its mass shell, $T[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2]$ contains terms which give a vanishing contribution as $k\to 0$ when multiplied on the left by a spinor $\bar{u}(p_2)$. These terms are not physically measurable in the nonradiative process. It is therefore convenient to write T in the form

$$T[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ =\frac{i\gamma\cdot(p_2+k)+W}{2W}T^N[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ +\frac{-i\gamma\cdot(p_2+k)+W}{2W}T^P[s+r_2\cdot k,\, t,\, 0, \\ \times(p_2+k)^2+M_2{}^2], \quad (13)$$

where W denotes $[-(p_2+k)^2]^{1/2}$. The term $T^P[s,t,0,0]$ is the amplitude measured in the nonradiative process. We rearrange Eq. (13) in the form

$$T[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ =T^P[s+r_2\cdot k,\, t,\, 0,\, 0]+[i\gamma\cdot(p_2+k)+M_2] \\ \times\left\{[-i\gamma\cdot(p_2+k)+M_2]\frac{\partial}{\partial\Delta_2} \right.\\ \times T^P[s+r_2\cdot k,\, t,\, 0,\, \Delta_2]\bigg|_{\Delta_2=0}+O(k) \\ +\frac{1}{2W}\left[1+\frac{i\gamma\cdot(p_2+k)-M_2}{W+M_2}\right] \\ \times\{T^N[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2] \\ \left.-T^P[s+r_2\cdot k,\, t,\, 0,\, (p_2+k)^2+M_2{}^2]\}\right\}. \quad (14)$$

When substituted back into Eq. (12), the term in boldface brackets in Eq. (14) leads to terms either independent of k or of first order in k. Strictly speaking, we should have included in Eq. (12) the negative-frequency terms in the photon–spin-$\frac{1}{2}$–off-mass-shell spin-$\frac{1}{2}$ vertex. By the same argument, these terms do not contribute to the truncated matrix element. Hence, the truncated matrix element is

$$M_\alpha{}^{\text{ext}\prime}=\bar{u}(p_2)\left\{\left(i\gamma_\alpha+i\frac{\mu}{2M_2}\sigma_{\alpha\beta}k_\beta\right)\right.\\ \times\frac{1}{i\gamma\cdot(p_2+k)+M_2}T^P[s+r_2\cdot k,\, t,\, 0,\, 0] \\ +T^P[s-r_1\cdot k,\, t,\, 0,\, 0]\frac{1}{i\gamma\cdot(p_1-k)+M_1} \\ \left.\times\left(i\gamma_\alpha+i\frac{\mu}{2M_1}\sigma_{\alpha\beta}k_\beta\right)\right\}u(p_1)+O(k), \quad (15)$$

which involves only the physically measurable matrix element. Using the identities

$$\bar{u}(p_2)i\gamma\cdot k\frac{1}{i\gamma\cdot(p_2+k)+M_2}=\bar{u}(p_2)\,,$$
$$\frac{1}{i\gamma\cdot(p_1-k)+M_1}i\gamma\cdot k u(p_1)=-u(p_1)\,, \tag{16}$$

we can calculate $k_\alpha M_\alpha{}^{\text{ext}\prime}$,

$$k_\alpha M_\alpha{}^{\text{ext}\prime}=\bar{u}(p_2)\{T^P[s+r_2\cdot k,t,0,0] -T^P[s-r_1\cdot k,t,0,0]\}u(p_1)+O(k^2)\,. \tag{17}$$

The expression between the spinors is identical to Eq. (10) in the spin-zero case. Therefore, ΔM_α is

$$\Delta M_\alpha=-(r_2+r_1)_\alpha\bar{u}(p_2)\frac{\partial}{\partial s}T^P[s,t,0,0]u(p_1)\,, \tag{18}$$

and $M_\alpha{}^{\mathrm{I}}$ is $M_\alpha{}^{\text{ext}\prime}+\Delta M_\alpha$. This is Low's result.

II. AXIAL-VECTOR CURRENT

We now consider the matrix element of the strangeness-conserving weak axial-vector current $J_\alpha{}^{Aj}$ between hadron states a and b,

$$N_aN_bM_\alpha{}^j={}_{\text{out}}\langle b|J_\alpha{}^{Aj}|a\rangle_{\text{in}}\,. \tag{19}$$

The superscript j is an isotopic spin index ($j=1,2,3$). We no longer have the equation $k_\alpha M_\alpha{}^j=0$, since the axial-vector current is not conserved. Let D^j be the matrix element of the divergence of the axial-vector current,

$$N_aN_bD^j=N_aN_bk_\alpha M_\alpha{}^j={}_{\text{out}}\langle b|-i\partial_\alpha J_\alpha{}^{Aj}|a\rangle_{\text{in}}\,. \tag{20}$$

Here, as in Section I, $k=p_a-p_b$. The PCAC hypothesis relates matrix elements of the divergence of the axial-vector current to matrix elements of the pion source,

$${}_{\text{out}}\langle b|\partial_\alpha J_\alpha{}^{Aj}|a\rangle_{\text{in}}=\frac{M_Ng_A}{g_r(0)}\frac{m_\pi{}^2}{k^2+m_\pi{}^2}{}_{\text{out}}\langle b|J_\pi{}^j|a\rangle_{\text{in}}\,, \tag{21}$$

where M_N and m_π are the nucleon and pion masses, $J_\pi{}^j$ is the pion source, $g_A\equiv g_A(0)\approx1.18$ is the weak axial-vector coupling constant, and $g_r(0)$ is the off-mass-shell pion-nucleon coupling constant. The [physical coupling constant is $g_r\equiv g_r(-m_\pi{}^2)$; $g_r{}^2/4\pi\approx14.6$.] We wish to emphasize that the PCAC hypothesis allows one to measure D^j in purely strong interaction experiments.

Since the axial-vector current is not conserved, we will need a slightly modified version of Lemma 2:

Lemma 2′: If $k_\alpha M_\alpha{}^j=D^j$ and $M_\alpha{}^j=M_\alpha{}^{j\mathrm{I}}+M_\alpha{}^{j\mathrm{II}}+O(k)$, where $M_\alpha{}^{j\mathrm{II}}$ is independent of k and where $k_\alpha M_\alpha{}^{j\mathrm{I}}=D^j+O(k^2)$, then $M_\alpha{}^j=M_\alpha{}^{j\mathrm{I}}+O(k)$. This lemma leads to a modification of the recipe stated in Sec. I: (1) Write down $M_\alpha{}^{j\,\text{ext}}$, the sum of terms in which the axial-vector current is coupled to external particle lines. (2) Drop all terms from $M_\alpha{}^{j\,\text{ext}}$ which are explicitly independent of k, giving a truncated amplitude $M_\alpha{}^{j\,\text{ext}\prime}$. (3) Add to $M_\alpha{}^{j\,\text{ext}\prime}$ a $\Delta M_\alpha{}^j$ independent of k so as to make $k_\alpha(M_\alpha{}^{j\,\text{ext}\prime}+\Delta M_\alpha{}^j)=D^j+O(k^2)$. Then $M_\alpha{}^{j\,\text{ext}\prime}+\Delta M_\alpha{}^j$ is the $M_\alpha{}^{\mathrm{I}}$ required by the lemma. We actually will not omit *all* terms of order k, but will consistently retain terms of order k which explicitly contain a pion propagator.

As an illustration of the recipe, we will consider the problem analogous to the second example in Sec. I, scattering of a spin-zero particle from a spin-$\frac{1}{2}$ particle (which we will take to be a nucleon) with an additional coupling of the spin-$\frac{1}{2}$ particle to the axial-vector current. The answer will involve the corresponding matrix element, in which the axial-vector current is replaced by the pion source. We write the pion-emission matrix element in the form

$$\begin{aligned}M_\pi{}^j&={}_{\text{out}}\langle b|J_\pi{}^j|a\rangle_{\text{in}}\,(N_aN_b)^{-1}\\&=\bar{u}(p_2)\bigg\{ig_r(k^2)\tau^j\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}\\&\quad\times T^P[s+r_2\cdot k,t,0,0]+T^P[s-r_1\cdot k,t,0,0]\\&\quad\times\frac{1}{i\gamma\cdot(p_1-k)+M_N}ig_r(k^2)\tau^j\gamma_5+i\bar{T}_\pi{}^j(0)\\&\quad+ik_\lambda\frac{\partial}{\partial k_\lambda}\bar{T}_\pi{}^j(k)\bigg|_{k=0}+O(k^2)\bigg\}u(p_1)\,.\end{aligned} \tag{22}$$

We have explicitly exhibited the Born terms in the form given by dispersion theory, where residues are evaluated at the Born pole and so no nucleon-off-mass-shell terms are present. The way we write the Born terms serves as the definition of the non-Born part $\bar{T}_\pi{}^j(k)$.

We are now ready to write down $M_\alpha{}^{j\,\text{ext}}$,

$$\begin{aligned}M_\alpha{}^{j\,\text{ext}}&=\bar{u}(p_2)\bigg\{ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\frac{1}{i\gamma\cdot(p_2+k)+M_N}\\&\quad\times T[s+r_2\cdot k,t,0,(p_2+k)^2+M_N{}^2]\\&\quad+T[s-r_1\cdot k,t,(p_1-k)^2+M_N{}^2,0]\\&\quad\times\frac{1}{i\gamma\cdot(p_1-k)+M_N}ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\bigg\}u(p_1)\\&\quad+\frac{M_Ng_A}{g_r(0)}\frac{ik_\alpha}{k^2+m_\pi{}^2}M_\pi{}^j\,.\end{aligned} \tag{23}$$

The term in brackets in Eq. (23) is the direct coupling of the axial-vector current to the external nucleon lines. The term proportional to $M_\pi{}^j$ comes from the diagrams shown in Fig. 3; although this term is formally of first order in k, it can be important because of the small mass of the pion.

As we have seen in Sec. I, the truncated matrix element is obtained by dropping the negative frequency part of T and by neglecting off-mass-shell terms. This gives

$$M_\alpha{}^{j\,\mathrm{ext}'}=\bar{u}(p_2)\biggl\{ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\frac{1}{i\gamma\cdot(p_2+k)+M_N}\times T^P[s+r_2\cdot k,\,t,\,0,\,0]+T^P[s-r_1\cdot k,\,t,\,0,\,0]\times\frac{1}{i\gamma\cdot(p_1-k)+M_N}ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\biggr\}u(p_1)+\frac{M_Ng_A}{g_r(0)}\frac{ik_\alpha}{k^2+m_\pi{}^2}M_\pi{}^j+O(k).\quad(24)$$

Using the identities

$$\bar{u}(p_2)i\gamma\cdot k\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}=\bar{u}(p_2)\biggl[-\gamma_5+2M_N\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}\biggr],$$
$$\frac{1}{i\gamma\cdot(p_1-k)+M_N}i\gamma\cdot k\gamma_5u(p_1)=\biggl[-\gamma_5+\frac{1}{i\gamma\cdot(p_1-k)+M_N}2M_N\gamma_5\biggr]u(p_1),\quad(25)$$

we can calculate $k_\alpha M_\alpha{}^{j\,\mathrm{ext}'}$,

$$k_\alpha M_\alpha{}^{j\,\mathrm{ext}'}=\bar{u}(p_2)\biggl\{-\tfrac{1}{2}g_A\tau^j\gamma_5T^P[s+r_2\cdot k,\,t,\,0,\,0]-T^P[s-r_1\cdot k,\,t,\,0,\,0]\tfrac{1}{2}g_A\tau^j\gamma_5+\frac{M_Ng_Am_\pi{}^2}{k^2+m_\pi{}^2}\biggl\{\tau^j\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}\times T^P[s+r_2\cdot k,\,t,\,0,\,0]+T^P[s-r_1\cdot k,\,t,\,0,\,0]\times\frac{1}{i\gamma\cdot(p_1-k)+M_N}\tau^j\gamma_5\biggr\}+O(k^2)\biggr\}u(p_1)-\frac{M_Ng_A}{g_r(0)}\frac{k^2}{k^2+m_\pi{}^2}\bar{u}(p_2)\biggl\{\bar{T}_\pi{}^j(0)+k_\lambda\frac{\partial}{\partial k_\lambda}\times\bar{T}_\pi{}^j(k)\bigg|_{k=0}+O(k^2)\biggr\}u(p_1).\quad(26)$$

In deriving Eq. (26), we have combined the Born terms in $M_\pi{}^j$ with the divergence of the first term in Eq. (24), and have expanded the form factors $g_A(k^2)$ and $g_r(k^2)$ in powers of k^2.

We determine $\Delta M_\alpha{}^j$ by the requirement that

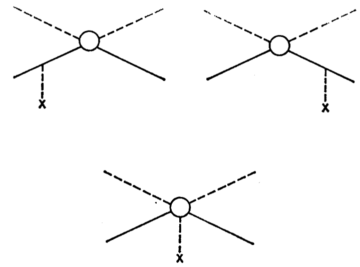

FIG. 3. Pion pole contributions to the axial-vector current matrix element. The axial-vector coupling is denoted by $\times$.

$k_\alpha(M_\alpha{}^{j\,\mathrm{ext}'}+\Delta M_\alpha{}^j)=D^j$, with

$$D^j=-i\frac{M_Ng_A}{g_r(0)}\frac{m_\pi{}^2}{k^2+m_\pi{}^2}\bar{u}(p_2)\times\biggl\{ig_r(0)\tau^j\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}\times T^P[s+r_2\cdot k,\,t,0,\,0]+T^P[s-r_1\cdot k,\,t,\,0,\,0]\times\frac{1}{i\gamma\cdot(p_1-k)+M_N}ig_r(0)\tau^j\gamma_5+i\bar{T}_\pi{}^j(0)+ik_\lambda\frac{\partial}{\partial k_\lambda}\bar{T}_\pi{}^j(k)\bigg|_{k=0}+O(k^2)\biggr\}u(p_1).\quad(27)$$

Comparing Eqs. (26) and (27), we see that $k_\alpha\Delta M_\alpha{}^j$ must satisfy

$$k_\alpha\Delta M_\alpha{}^j=\bar{u}(p_2)\biggl\{\tfrac{1}{2}g_A\tau^j\gamma_5T^P[s+r_2\cdot k,\,t,\,0,\,0]+T^P[s-r_1\cdot k,\,t,\,0,\,0]\tfrac{1}{2}g_A\tau^j\gamma_5+\frac{M_Ng_A}{g_r(0)}\times\biggl[\bar{T}_\pi{}^j(0)+k_\lambda\frac{\partial}{\partial k_\lambda}\bar{T}_\pi{}^j(k)\bigg|_{k=0}\biggr]+O(k^2)\biggr\}u(p_1)\quad(28\mathrm{a})$$

$$=\bar{u}(p_2)\biggl\{\tfrac{1}{2}g_A\tau^j\gamma_5T^P[s,t,0,0]+T^P[s,t,0,0]\tfrac{1}{2}g_A\tau^j\gamma_5+\frac{M_Ng_A}{g_r(0)}\bar{T}_\pi{}^j(0)\biggr\}u(p_1)+k_\alpha\bar{u}(p_2)\biggl\{r_{2\alpha}\tfrac{1}{2}g_A\tau^j\gamma_5\frac{\partial}{\partial s}T^P[s,t,0,0]-\frac{\partial}{\partial s}T^P[s,t,0,0]\tfrac{1}{2}g_A\tau^j\gamma_5r_{1\alpha}+\frac{M_Ng_A}{g_r(0)}\frac{\partial}{\partial k_\alpha}\bar{T}_\pi{}^j(k)\bigg|_{k=0}\biggr\}u(p_1)+O(k^2).\quad(28\mathrm{b})$$

As the reader has undoubtedly noted, the nucleon propagator terms have exactly cancelled between Eq. (26) and Eq. (27), and so do not appear in Eq. (28a). In the term involving $\bar{T}_\pi{}^j$, the pion propagator has dropped out altogether, since

$$\frac{k^2}{k^2+m_\pi^2}+\frac{m_\pi^2}{k^2+m_\pi^2}=1. \tag{29}$$

In going from Eq. (28a) to Eq. (28b), we have simply expanded in powers of k and collected together the terms of zeroth, first, and second order in k.

Since $k_\alpha \Delta M_\alpha{}^j$ is of first order in k, the zeroth-order terms on the right-hand side of Eq. (28b) must vanish identically. This gives

$$\bar{u}(p_2)\bar{T}_\pi{}^j(0)u(p_1)=-\bar{u}(p_2)\left\{\frac{g_r(0)}{2M_N}\tau^j\gamma_5 T^P[s,t,0,0] + T^P[s,t,0,0]\frac{g_r(0)}{2M_N}\tau^j\gamma_5\right\}u(p_1). \tag{30}$$

This formula, which has been obtained previously,[8] expresses the matrix element for the emission of a zero four-momentum pion in terms of the matrix element of the process without the pion. Equation (30) can be used to eliminate $\bar{T}_\pi{}^j(0)$ from the term proportional to $M_\pi{}^j$ in Eq. (24). Comparing the terms of first order in k, we find

$$\Delta M_\alpha{}^j=\bar{u}(p_2)\left\{r_{2\alpha}\tfrac{1}{2}g_A\tau^j\gamma_5\frac{\partial}{\partial s}T^P[s,t,0,0] -\frac{\partial}{\partial s}T^P[s,t,0,0]\tfrac{1}{2}g_A\tau^j\gamma_5 r_{1\alpha} +\frac{M_N g_A}{g_r(0)}\frac{\partial}{\partial k_\alpha}\bar{T}_\pi{}^j(k)\bigg|_{k=0}\right\}u(p_1). \tag{31}$$

Adding this expression to the $M_\alpha{}^{j\,\text{ext}'}$ of Eq. (24) gives the analog of Low's result for the axial-vector case.

A similar method can be applied to the case in which more than one current is acting. As an example, we consider the matrix element[9]

$$M_{\alpha\sigma}{}^j=\int d^4y\, e^{iq\cdot y}\,{}_{\text{out}}\langle b|T[J_\alpha{}^{Aj}(x)J_\sigma(y)]|a\rangle_{\text{in}}. \tag{32a}$$

Calculating $k_\alpha M_{\alpha\sigma}{}^j$, we get

$$\begin{aligned} k_\alpha M_{\alpha\sigma}{}^j &= \int d^4y e^{iq\cdot y}\,{}_{\text{out}}\left\langle b\left|-i\frac{\partial}{\partial x_\alpha}T[J_\alpha{}^{Aj}(x)J_\sigma(y)]\right|a\right\rangle_{\text{in}} \\ &= \int d^4y e^{iq\cdot y}\,{}_{\text{out}}\langle b|-\delta(x_0-y_0)[J_4{}^{Aj}(x)J_\sigma(y)]|a\rangle_{\text{in}} \\ &\quad +\int d^4y e^{iq\cdot y}\,{}_{\text{out}}\langle b|-iT[\partial_\alpha J_\alpha{}^{Aj}(x)J_\sigma(y)]|a\rangle_{\text{in}}. \end{aligned} \tag{32b}$$

The only difference from the case treated above is that the divergence, in addition to having the term with a pion vertex substituted for the axial-vector vertex, also contains an equal-time commutator term. Following the procedure of this section, we can determine $M_{\alpha\sigma}{}^j$, apart from terms of order k and higher. If the divergence of J_σ is also known, we can apply the technique a second time, determining terms of order k which are independent of q. This leaves an error which only involves terms of order qk and higher.[10] We will consider such a case in the next section, when we discuss radiative μ capture.

III. APPLICATIONS

In this section we apply the results of the previous section to several concrete examples. We consider first single-pion production from a nucleon by the axial-vector current. As an illustration of the use of our method in the strangeness-changing case, we discuss K_{e4} decay. We finally discuss the process of radiative μ capture on a proton, an example in which two currents are present.

1. Weak Pion Production

We consider the process

$$\nu(k_\nu)+N(p_1)\to l(k_l)+N(p_2)+\pi^n(q), \tag{33}$$

where the four-momentum of each particle is indicated in parentheses. Let $M_\alpha{}^{jn}$ be the axial-vector matrix element for this process, as defined in Eq. (19), with

$$\begin{aligned} |a\rangle_{\text{in}} &= |N(p_1)\rangle, \\ {}_{\text{out}}\langle b| &= {}_{\text{out}}\langle N(p_2)\pi^n(q)|, \\ k &= k_l-k_\nu=p_1-(p_2+q). \end{aligned} \tag{34}$$

In this case, $T^P[s,t,0,0]$ is the pion-nucleon vertex $ig_r\gamma_5\tau^n$, which has no s dependence. Hence the $\partial/\partial s$ terms in Eq. (31) vanish. Clearly $M_\pi{}^{jn}$, the matrix element with the pion source substituted for the axial-vector current, is just the amplitude for pion-nucleon scattering. We find

$$\begin{aligned} M_\alpha{}^{jn\,\text{ext}'} = \bar{u}(p_2)\Big\{ & ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\frac{1}{i\gamma\cdot(p_2+k)+M_N} \\ & \times ig_r\gamma_5\tau^n+ig_r\gamma_5\tau^n\frac{1}{i\gamma\cdot(p_1-k)+M_N} \\ & \times ig_A(k^2)\gamma_\alpha\gamma_5\frac{\tau^j}{2}\Big\}u(p_1) \\ & +\frac{M_N g_A}{g_r(0)}\frac{ik_\alpha}{k^2+m_\pi^2}M_\pi{}^{jn}, \end{aligned} \tag{35a}$$

[8] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962); S. L. Adler, *ibid.* **139**, B1638 (1965).

[9] In Eq. (32) we have neglected "seagull" terms, which will be included in the calculations of Sec. III.

[10] This method has been applied to the case when only vector currents are present by G. K. Manacher, thesis, Carnegie Institute of Technology Report NYO 9284, 1961 (unpublished).

$$M_{\pi}{}^{jn}=\bar{u}(p_2)\left\{ig_r(k^2)\tau^j\gamma_5\frac{1}{i\gamma\cdot(p_2+k)+M_N}ig_r\gamma_5\tau^n\right.$$
$$+ig_r\gamma_5\tau^n\frac{1}{i\gamma\cdot(p_1-k)+M_N}ig_r(k^2)\tau^j\gamma_5$$
$$\left.+i\bar{T}_{\pi}{}^{jn}(0)+ik_\lambda\frac{\partial}{\partial k_\lambda}\bar{T}_{\pi}{}^{jn}(k)\Big|_{k=0}+O(k^2)\right\}u(p_1). \tag{35b}$$

From Eq. (30), we find that

$$\bar{u}(p_2)i\bar{T}_{\pi}{}^{jn}(0)u(p_1)$$
$$=-i\bar{u}(p_2)\left\{\frac{g_r(0)}{2M_N}\tau^j\gamma_5ig_r\gamma_5\tau^n+ig_r\gamma_5\tau^n\frac{g_r(0)}{2M_N}\tau^j\gamma_5\right\}u(p_1)$$
$$=\bar{u}(p_2)\left\{\frac{g_rg_r(0)}{M_N}\delta^{nj}\right\}u(p_1). \tag{36}$$

From Eq. (31), we have

$$\Delta M_{\alpha}{}^{jn}=\frac{M_Ng_A}{g_r(0)}\bar{u}(p_2)\left\{\frac{\partial}{\partial k_\alpha}\bar{T}_{\pi}{}^{jn}(k)\Big|_{k=0}\right\}u(p_1). \tag{37}$$

From the usual expression for the pion-nucleon scattering amplitude,[11] we find (remembering that $-k$ is the incoming pion four-momentum),

$$\bar{u}(p_2)\left\{\frac{\partial}{\partial k_\alpha}\bar{T}_{\pi}{}^{jn}(k)\Big|_{k=0}\right\}u(p_1)$$
$$=i\frac{\partial}{\partial k_\alpha}\bar{u}(p_2)\left\{\left[-A^{\pi N(+)}\left(\nu=\frac{k\cdot(p_1+p_2)}{2M_N},\ \nu_B=-\frac{q\cdot k}{2M_N},\ k^2\right)-i\gamma\cdot k\bar{B}^{\pi N(+)}(\nu,\nu_B,k^2)\right]\delta^{nj}\right.$$
$$+[-\bar{A}^{\pi N(-)}(\nu,\nu_B,k^2)-i\gamma\cdot k\bar{B}^{\pi N(-)}(\nu,\nu_B,k^2)]$$
$$\left.\times\tfrac{1}{2}[\tau^n,\tau^j]\right\}u(p_1)\,|_{k=0}$$
$$=i\bar{u}(p_2)\left\{\left[\frac{\partial\bar{A}^{\pi N(+)}}{\partial\nu_B}\Big|_{\nu=\nu_B=k^2=0}\frac{q_\alpha}{2M_N}\right]\delta^{nj}\right.$$
$$-\left[\frac{\partial\bar{A}^{\pi N(-)}}{\partial\nu}\Big|_{\nu=\nu_B=k^2=0}\frac{(p_1+p_2)_\alpha}{2M_N}\right.$$
$$\left.\left.+i\gamma_\alpha\bar{B}^{\pi N(-)}\Big|_{\nu=\nu_B=k^2=0}\right]\tfrac{1}{2}[\tau^n,\tau^j]\right\}u(p_1). \tag{38}$$

Other derivative terms vanish at $\nu=0$ because of the well-known[11]) crossing properties of $A^{\pi N}$ and $B^{\pi N}$,

$$\begin{aligned}A^{\pi N(\pm)}(-\nu,\cdots)&=\pm A^{\pi N(\pm)}(\nu,\cdots),\\ B^{\pi N(\pm)}(-\nu,\cdots)&=\mp B^{\pi N(\pm)}(\nu,\cdots).\end{aligned} \tag{39}$$

Since $-k^2$ is the (mass)^2 of the initial pion, Eq. (38) involves the pion-nucleon scattering amplitude extrapolated slightly off mass shell. Note that Eq. (36) is just the consistency condition on πN scattering,[12]

$$\bar{A}^{\pi N(+)}\Big|_{\nu=\nu_B=k^2=0}=\frac{g_rg_r(0)}{M_N}. \tag{40}$$

Equations (35), (37), and (38) give the two leading terms in an expansion of $M_\alpha{}^{jn}$ in powers of k,

$$M_\alpha{}^{jn}=M_\alpha{}^{jn\ \mathrm{ext}'}+\Delta M_\alpha{}^{jn}+O(k). \tag{41}$$

Alternatively, we can use the analog of Eq. (30) to find the leading term in an expansion in powers of q (the soft pion limit). In this case, one would take T^P in Eq. (30) to be the axial-vector vertex. There will be an additional term in Eq. (30) arising from the equal-time commutator of the two axial-vector currents involved. Assuming the commutation relations postulated by Gell-Mann,[13] we find[14]

$$M_\alpha{}^{jn}=M_\alpha{}^{jn\ \mathrm{ext}'}+\Delta M_\alpha{}^{jn\prime}+O(q), \tag{42}$$

with

$$\Delta M_\alpha{}^{jn\prime}=i\frac{g_r(0)}{2M_N}\bar{u}(p_2)\left\{\frac{\mu^V}{g_A}\frac{(p_1+p_2)_\alpha}{2M_N}+i\gamma_\alpha\left[g_A-\frac{1}{g_A}\frac{\mu^V}{g_A}\right]\right\}\tfrac{1}{2}[\tau^n,\tau^j]u(p_1),$$
$$\mu^V=3.70. \tag{43}$$

Clearly, at the point $q=k=0$ we must have $\Delta M_\alpha{}^{jn}=\Delta M_\alpha{}^{jn\prime}$. At this point $p_1=p_2$ and thus $i\gamma_\alpha$ and $(p_1+p_2)_\alpha/(2M_N)$ are equal between spinors. Hence, consistency between Eq. (42) and Eq. (41) demands

$$1-\frac{1}{g_A{}^2}=-\frac{2M^2{}_N}{g_r(0)^2}\left[\frac{\partial\bar{A}^{\pi N(-)}}{\partial\nu}+B^{\pi N(-)}\right]\Big|_{\nu=\nu_B=k^2=q^2=0} \tag{44}$$

which is the sum rule for the axial-vector coupling constant.[15]

[11] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957). Note that, according to Eq. (34), $-k$ is the ingoing pion four-momentum.

[12] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[13] M. Gell-Mann, Physics **1**, 63 (1964).

[14] Y. Nambu and E. Shrauner, Phys. Rev. **128**, 862 (1962); S. L. Adler (to be published); G. Furlan, R. Jengo, and E. Remiddi, Nuovo Cimento **44**, 427 (1966). The diligent reader will actually find that in Eq. (42), and also in Eq. (45), we have dropped certain terms proportional to k_α which are not singular at $k^2=-m_\pi{}^2$. These terms are, of course, determined by our procedure, but they are numerically insignificant in weak pion production because k_α, contracted with the lepton current, becomes proportional to the lepton mass. We have also in Eq. (45) neglected a very small extra term, proportional to δ^{nj}, which appears in Eq. (41) when the pion four-momentum q is taken off mass shell [see W. I. Weisberger, Phys. Rev. **143**, 1302 (1966), Eq. (II.11a)].

[15] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965).

Comparing Eqs. (43) and (38), we may determine the terms linear in either q or k. Our final result is then

$$M_\alpha^{jn}=M_\alpha^{jn\;\mathrm{ext}'}+\Delta M_\alpha^{jn''}+O(qk,q^2,k^2)\,, \tag{45}$$

with

$$\Delta M_\alpha^{jn''}=i\frac{M_N g_A}{g_r(0)}\bar{u}(p_2)\left\{\left[\frac{\partial \bar{A}^{\pi N(+)}}{\partial \nu_B}\bigg|_{\nu=\nu_B=k^2=q^2=0}\frac{q_\alpha}{2M_N}\right]\delta^{nj}\right.$$
$$+\left[\frac{g_r(0)^2}{2M^2_N}\left(1-\frac{1}{g_A^2}\right)\frac{(p_1+p_2)_\alpha}{2M_N}\right.$$
$$-i\frac{\sigma_{\alpha\beta}q_\beta}{2M_N}\bar{B}^{\pi N(-)}\bigg|_{\nu=\nu_B=k^2=q^2=0}+\frac{g_r(0)^2}{2M^2_N}\frac{i\sigma_{\alpha\beta}k_\beta}{2M_N}$$
$$\left.\left.\times\left(1-\frac{1}{g_A^2}-\frac{\mu^V}{g_A^2}\right)\right]\tfrac{1}{2}[\tau^n,\tau^j]\right\}u(p_1)\,. \tag{46}$$

Unfortunately, it is doubtful if Eq. (46) will be of practical use, since there is a strong final-state interaction leading to the (3,3) resonance, which is located only one pion mass away from threshold in energy. This makes it unlikely that k and q will be good expansion parameters. However, we will use the same method of comparing expansions in q and k in dealing with radiative μ capture, where the final-state interaction is negligible and so the expansion may be physically interesting.

2. K_{e4} Decay

Here we consider the process

$$K^+(k^+)\to\pi^+(p^+)+\pi^-(p^-)+\bar{e}(k_e)+\nu(k_\nu)\,. \tag{47}$$

Again the four-momentum of each particle is indicated in parentheses. Let the four-momentum carried away by the lepton pair be k^-,

$$k_e+k_\nu=k^-\,. \tag{48}$$

The most general form of the axial-vector contribution to the decay matrix element is

$$\begin{aligned}M_\alpha&=(2k_0^+2p_0^+2p_0^-)^{1/2}\,{}_{\mathrm{out}}\langle\pi^+\pi^-|J_\alpha^{A,\,\Delta S=-1}|K^+\rangle\\&=\frac{1}{m_K}[F_1(p^++p^-)_\alpha+F_2(p^+-p^-)_\alpha+F_3k_\alpha^-]\,.\end{aligned} \tag{49}$$

The form factors F are functions of the arguments $x=(p^++p^-)\cdot k^-$, $y=(p^+-p^-)\cdot k^-$, and $(k^-)^2$. We define the matrix element for $\pi^+\pi^-\to K^+K^-$ by writing

$$(2k_0^+2p_0^+2p_0^-)^{1/2}\,{}_{\mathrm{out}}\langle\pi^+\pi^-|J_K|K^+\rangle_{\mathrm{in}}=iT_{\pi K}[x,y,(k^-)^2]\,. \tag{50}$$

Then if we assume PCAC in the strangeness-changing case,[16]

$$\partial_\alpha J_\alpha^{A,\,\Delta S=-1}=C_K m_K^2\phi_K\,, \tag{51}$$

we find that

$$\begin{aligned}\frac{1}{m_K}F_1\bigg|_{x=y=(k^-)^2=0}&=C_K\frac{\partial}{\partial x}T_{\pi K}[x,y,(k^-)^2]\bigg|_{x=y=(k^-)^2=0}\,,\\\frac{1}{m_K}F_2\bigg|_{x=y=(k^-)^2=0}&=C_K\frac{\partial}{\partial y}T_{\pi K}[x,y,(k^-)^2]\bigg|_{x=y=(k^-)^2=0}\,.\end{aligned} \tag{52}$$

Hence, the K_{e4} decay amplitudes at a point on the boundary of the Dalitz plot are related to the πK amplitude, with one K meson off mass shell. In terms of the conventional Mandelstam variables, the point $x=y=(k^-)^2=0$ is

$$\begin{aligned}s&=(p^++p^-)^2=-m_K^2\,,\\t&=(k^-+p^+)^2=-m_\pi^2\,,\\u&=(k^--p^-)^2=-m_\pi^2\,.\end{aligned} \tag{53}$$

3. Radiative μ Capture

In this subsection we discuss the process of radiative μ capture by a proton. This is an example of the situation, discussed briefly at the end of Sec. II, in which more than one current is acting. Consider then

$$\mu^-(k_\mu)+p(p_1)\to\nu(k_\nu)+\gamma(k)+n(p_2)\,, \tag{54}$$

and let

$$q=k_\mu-k_\nu \tag{55}$$

be the lepton four-momentum transfer. The matrix element for this process is given by

$$T=\frac{G}{\sqrt{2}}\left\{-\langle n|J_\alpha^W|p\rangle\bar{u}_\nu\gamma_\alpha(1+\gamma_5)\right.$$
$$\times\frac{1}{i\gamma\cdot(k_\mu-k)+m_\mu}ie\gamma_\lambda\epsilon_\lambda^\star u_\mu\frac{1}{(2k_0)^{1/2}}$$
$$\left.+\langle n\gamma|J_\alpha^W|p\rangle\bar{u}_\nu\gamma_\alpha(1+\gamma_5)u_\mu\right\}\,, \tag{56}$$

with ϵ_λ the polarization vector of the photon and G the Fermi constant. The two contributions to T correspond, respectively, to radiation by the muon (which is negatively charged) and to radiation by the hadrons. The matrix element $\langle n|J_\alpha^W|p\rangle$ is given by

$$\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\langle n|J_\alpha^W|p\rangle$$
$$=i\bar{u}(p_2)[F_1^V((q-k)^2)\gamma_\alpha-F_2^V((q-k)^2)\sigma_{\alpha\beta}(q-k)_\beta$$
$$+g_A((q-k)^2)\gamma_\alpha\gamma_5-ih_A((q-k)^2)\gamma_5(q-k)_\alpha]u(p_1)\,. \tag{57}$$

[16] R. P. Feynman, in *Symmetries in Elementary Particle Physics* (Academic Press Inc., New York, 1965), p. 158. The constant C_K is given by $C_K=(M_N+M_\Lambda)g_A^{\Lambda N}/g^{KN\Lambda}(0)$, with $g_A^{\Lambda N}$ the Λ beta-decay coupling constant and $g^{KN\Lambda}$ the KNΛ coupling. For applications of partial conservation of the strangeness-conserving axial-vector current to K_{e4} decays, see C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966) and M. Suzuki, *ibid.* **16**, 212 (1966).

Here, $F_1^V(t)$ and $F_2^V(t)$ are the isovector Dirac and Pauli electromagnetic form factors $[F_1^V(0)=1,\ F_2^V(0)=\mu^V/(2M_N)]$, $g_A(t)$ is the axial-vector form factor, and $h_A(t)$ is the induced pseudoscalar form factor. Applying PCAC to the one-nucleon vertex of the axial-vector current, we find that $h_A(t)$ may be written in the form

$$h_A(t)=\frac{2M_Ng_A[g_r/g_r(0)]}{t+m_\pi^2}+r(t)\,, \tag{58a}$$

$$r(t)=\frac{1}{t}\left[2M_Ng_A(t)-2M_Ng_A\frac{m_\pi^2g_r(t)+tg_r}{g_r(0)(t+m_\pi^2)}\right], \tag{58b}$$
$$r(0)\approx 2M_Ng_A'(0)\,,$$

which explicitly exhibits the one-pion pole part and the remainder $r(t)$.

We write $\langle n\gamma|J_\alpha^W|p\rangle$ in the following form:

$$(2k_0p_{10}p_{20}/M^2_N)^{1/2}\langle n\gamma|J_\alpha^W|p\rangle=e\epsilon_\lambda^\star M_{\lambda\alpha}\,. \tag{59}$$

We wish to use our knowledge of the divergences of the vector and axial-vector currents to calculate $M_{\lambda\alpha}$, up to and including terms linear in q and in k. In order to do this, we have to know the quantities $k_\lambda M_{\lambda\alpha}$ and $q_\alpha M_{\lambda\alpha}$. The first of these may be determined by conservation of the electromagnetic current. When $\epsilon_\lambda^\star$ is replaced by k_λ in Eq. (56), the resulting expression must vanish. This tells us that

$$k_\lambda M_{\lambda\alpha}=-(p_{10}p_{20}/M^2_N)^{1/2}\langle n|J_\alpha^W|p\rangle\,. \tag{60}$$

In order to calculate $q_\alpha M_{\lambda\alpha}$, we made use of our knowledge of the divergences of the vector and the axial-vector parts of the weak current,[17]

$$J_\alpha^W=J_\alpha^V+J_\alpha^A\,, \tag{61a}$$

$$\partial_\alpha J_\alpha^V=ieA_\alpha J_\alpha^V\,,$$
$$\partial_\alpha J_\alpha^A=ieA_\alpha J_\alpha^A+(\sqrt{2}M_Nm_\pi^2g_A/g_r(0))\phi_{\pi^+}\,, \tag{61b}$$

where A_α is the electromagnetic vector potential and ϕ_{π^+} is the field which annihilates a positive pion. Equations (61b) follow from the assumption of minimal electromagnetic coupling and from the divergence equations in the absence of electromagnetism. (The factor $\sqrt{2}$ in the axial-vector equation comes from the definitions of J_α^A and ϕ_{π^+}: $J_\alpha^A=J_\alpha^{A1}-iJ_\alpha^{A2}$ and $\phi_{\pi^+}=(\phi_\pi^1-i\phi_\pi^2)/\sqrt{2}$.) Using Eqs. (61b) to evaluate $\langle n\gamma|\partial_\alpha J_\alpha^W|p\rangle$, we find

$$\epsilon_\lambda^\star q_\alpha M_{\lambda\alpha}=-\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\epsilon_\lambda^\star\langle n|J_\lambda^W|p\rangle+i\frac{\sqrt{2}M_Ng_A}{g_r(0)}\frac{m_\pi^2}{q^2+m_\pi^2}\left(2k_0\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\times e^{-1}\langle n\gamma|J_{\pi^+}|p\rangle\,. \tag{62}$$

[17] S. L. Adler, Phys. Rev. **139**, B1638 (1965).

From Eqs. (60) and (62), we can deduce the gauge condition satisfied by

$$\left(2k_0\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\langle n\gamma|J_{\pi^+}|p\rangle=e\epsilon_\lambda^\star T_{\pi^+\lambda}\,. \tag{63}$$

Replacing $\epsilon_\lambda^\star$ by k_λ in Eq. (62), and multiplying Eq. (60) by q_α, we get

$$\begin{aligned}k_\lambda T_{\pi^+\lambda}&=-\frac{q^2+m_\pi^2}{(q-k)^2+m_\pi^2}\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\langle n|J_{\pi^+}|p\rangle\\&=-\frac{q^2+m_\pi^2}{(q-k)^2+m_\pi^2}\sqrt{2}\bar{u}(p_2)i\gamma_5g_r((q-k)^2)u(p_1)\,.\end{aligned} \tag{64}$$

When $q^2=-m_\pi^2$, Eq. (64) becomes $k_\lambda T_{\pi+\lambda}=0$, the usual gauge condition for on-mass-shell pion photoproduction.

Before stating the results for radiative μ capture, we will discuss the significance of Eqs. (60) and (62). A more conventional way to proceed in calculating $k_\lambda M_{\lambda\alpha}$ and $q_\alpha M_{\lambda\alpha}$ would be to contract the photon in Eq. (59), giving

$$\begin{aligned}eM_{\lambda\alpha}&=\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}i\int d^4x\,e^{-ik\cdot x}(-\Box_x)\times\langle n|T[A_\lambda(x)J_\alpha^W(0)]|p\rangle\\&=\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}i\left[\int d^4x\,e^{-ik\cdot x}\times e\langle n|T[J_\lambda^{EM}(x)J_\alpha^W(0)]|p\rangle+S_{\lambda\alpha}\right],\end{aligned} \tag{65}$$

with

$$S_{\lambda\alpha}=\int d^4x\,e^{-ik\cdot x}\delta(x_0)\times\langle n|[\partial A_\lambda(x)/\partial x_0,J_\alpha^W(0)]|p\rangle\,, \tag{66}$$

where we have assumed that A_λ and J_α^W commute at equal times. The equal time commutator term $S_{\lambda\alpha}$ in Eq. (65), sometimes called a "seagull" or "catastrophic" term, describes the coupling of the weak and electromagnetic currents at the same point (see Fig. 4). It is a reflection of the extent to which A_λ appears in J_α^W. Calculating $k_\lambda M_{\lambda\alpha}$, we now get

$$k_\lambda M_{\lambda\alpha}=\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}\int dx_0e^{ik_0x_0}\delta(x_0)\times\left\langle n\left|\left[\int d^3x\,e^{-i\mathbf{k}\cdot\mathbf{x}}J_0^{EM}(x),J_\alpha^W(0)\right]\right|p\right\rangle+\left(\frac{p_{10}p_{20}}{M^2_N}\right)^{1/2}e^{-1}ik_\lambda S_{\lambda\alpha}\,. \tag{67}$$

FIG. 4. A "seagull" diagram.

The commutator of the currents is

$$\delta(x_0)[J_0^{\mathrm{EM}}(x),J_\alpha{}^W(0)] = -\delta^{(4)}(x)J_\alpha{}^W(0)+[\text{possible gradient terms proportional to } \partial_\alpha\delta^{(3)}(\mathbf{x})]. \quad (68)$$

The first term in Eq. (68) is the one conjectured by Gell-Mann[13]; the possible presence of the gradient terms was pointed out by Schwinger.[18] We see that Eq. (60) implies that the Schwinger terms *exactly cancel* the divergence of the "seagull" terms. This cancellation has been proved by Feynman in a Yang-Mills theory and has been conjectured by him to be a general result.[19] In other words, when calculating the *divergence* of quantities like $M_{\lambda\alpha}$, if one neglects *both* the "seagull" terms and the Schwinger terms, one gets the right result. Note that the "seagull" terms cannot be dropped when calculating the matrix element $M_{\lambda\alpha}$ itself.

In order to state our answer for radiative μ capture, we have to define the amplitudes for pion photoproduction with the pion off-mass-shell. This process is related by crossing symmetry to the matrix element $\langle n\gamma|J_{\pi^+}|p\rangle$ in Eq. (62). We write the photoproduction amplitude in the following form,[20]

$$\begin{aligned}\left(2k_0\frac{p_{10}p_{20}}{M^2{}_N}\right)^{1/2}&\langle N|J_\pi|N\gamma\rangle\\
=e\psi_j{}^*\chi_2{}^*\bar{u}(p_2)\Bigg\{&ig_r(q^2)\tau^j\gamma_5\frac{1}{i\gamma\cdot(p_2+q)+M_N}\\
&\times\tfrac{1}{2}i\left[\gamma_\lambda(1+\tau^3)-\frac{\sigma_{\lambda\xi}k_\xi}{2M_N}(\mu^S+\mu^V\tau^3)\right]\\
&+\tfrac{1}{2}i\left[\gamma_\lambda(1+\tau^3)-\frac{\sigma_{\lambda\xi}k_\xi}{2M_N}(\mu^S+\mu^V\tau^3)\right]\\
&\times\frac{1}{i\gamma\cdot(p_1-q)+M_N}ig_r(q^2)\tau^j\gamma_5\\
&+ig_r(-m_\pi{}^2)[\tau^j,\tau^3]\gamma_5\frac{\tfrac{1}{2}(2q-k)_\lambda}{(q-k)^2+m_\pi{}^2}\\
&+\sum_{s=1}^{4}O_{s\lambda}[\bar{V}_s{}^{(+)}\tfrac{1}{2}\delta^{j3}+\bar{V}_s{}^{(-)}\tfrac{1}{4}[\tau^j,\tau^3]+\bar{V}_s{}^{(0)}\tfrac{1}{2}\tau^j]\\
&-i\gamma_5[\tau^j,\tau^3]q_\lambda\left(1+\frac{q^2}{m_\pi{}^2}\right)\left[g_r'(0)+\frac{g_r(-m_\pi{}^2)-g_r(0)}{m_\pi{}^2}\right]\\
&+\left(1+\frac{q^2}{m_\pi{}^2}\right)O(q^2)\Bigg\}u(p)\chi_1\epsilon_\lambda, \quad (69)\end{aligned}$$

where ψ_j is the isospin wave function of the pion, χ_1 and χ_2 are the nucleon isospinors, k is the ingoing photon four-momentum, and q is the outgoing pion four-momentum. The isoscalar nucleon anomalous magnetic moment has been denoted by $\mu^S[2M_NF_2{}^S(0)=\mu^S=-0.12]$. The four-vectors $O_{s\lambda}$, which satisfy $k_\lambda O_{s\lambda}=0$, are given by

$$\begin{aligned}
O_{1\lambda}&=\tfrac{1}{2}i\gamma_5(\gamma_\lambda\gamma\cdot k-\gamma\cdot k\gamma_\lambda)\,, & \eta_1&=1\\
O_{2\lambda}&=i\gamma_5[(p_1+p_2)_\lambda q\cdot k-(p_1+p_2)\cdot kq_\lambda]\,, & \eta_2&=1\\
O_{3\lambda}&=\gamma_5(\gamma_\lambda q\cdot k-\gamma\cdot kq_\lambda)\,, & \eta_3&=-1 \quad (70)\\
O_{4\lambda}&=\gamma_5[\gamma_\lambda(p_1+p_2)\cdot k-\gamma\cdot k(p_1+p_2)_\lambda]\\
&\quad-iM_N\gamma_5(\gamma_\lambda\gamma\cdot k-\gamma\cdot k\gamma_\lambda)\,. & \eta_4&=1
\end{aligned}$$

The amplitudes $\bar{V}_s$ are functions of the invariants q^2, k^2, ν, and ν_B, with

$$\nu=-k\cdot(p_1+p_2)/2M_N\,,\quad \nu_B=q\cdot k/2M_N\,. \quad (71)$$

The bar on top of the V_s is a reminder that the Born term has been separated off. The numbers η_s specify the crossing properties[20] of the amplitudes $\bar{V}_s$,

$$\bar{V}_s{}^{(\pm,0)}(-\nu,\cdots)=\eta_s(\pm1,1)\bar{V}_s{}^{(\pm,0)}(\nu,\cdots)\,. \quad (72)$$

The terms explicitly proportional to $(1+q^2/m_\pi{}^2)$ in Eq. (69) are necessary to satisfy Eq. (64), the gauge-invariance requirement when the pion is off-mass-shell. Since

$$g_r'(0)+\frac{g_r(-m_\pi{}^2)-g_r(0)}{m_\pi{}^2}\approx\frac{m_\pi{}^2}{2}g_r''(0)\,, \quad (73)$$

the gauge-invariance term is numerically very small. The matrix element $\langle\gamma n|J_{\pi^+}|p\rangle$, which is the one needed in Eq. (62), is obtained from Eq. (69) by the replacements

$$\begin{aligned}\psi_j{}^*\to\psi_j&=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}_j,\\
\epsilon_\lambda&\to\epsilon_\lambda{}^\star,\\
q&\to-q,\\
k&\to-k.\end{aligned} \quad (74)$$

Since the final nucleon is a neutron and the initial one is a proton, we have

$$\chi_2=\begin{pmatrix}0\\1\end{pmatrix},\quad \chi_1=\begin{pmatrix}1\\0\end{pmatrix}. \quad (75)$$

We can now state the result for radiative μ capture:

$$M_{\lambda\alpha}=M_{\lambda\alpha}{}^N+M_{\lambda\alpha}{}^{RPD}+M_{\lambda\alpha}{}^{PPP}+M_{\lambda\alpha}{}^R+O\left(\frac{q^2}{m_R{}^2},\frac{qk}{m_R{}^2}\right). \quad (76)$$

The mass m_R, which characterizes the terms neglected in our calculation, will typically be several pion masses

[18] J. Schwinger, Phys. Rev. Letters 3, 296 (1959) and Phys. Rev. 130, 406 (1963).

[19] R. P. Feynman (private communication).

[20] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957). The amplitudes $(V_1,V_2,V_3,V_4)^{(\pm0)}$, as defined in Eq. (69), are respectively *double* the corresponding amplitudes $(A,B,C,D)^{(\pm0)}$ of CGLN. [The isospin matrix elements in Eq. (69) are one-half those of CGLN.]

or greater in magnitude, since we have explicitly included all pion propagator terms.[21] In Eq. (76), we have

$$M_{\lambda\alpha}{}^{N}=\bar{u}(p_2)\bigg\{[ig_A(q^2)\gamma_\alpha\gamma_5+h_A(q^2)\gamma_5 q_\alpha$$
$$+iF_1{}^V(q^2)\gamma_\alpha-iF_2{}^V(q^2)\sigma_{\alpha\beta}q_\beta]$$
$$\times\frac{1}{i\gamma\cdot(p_2-q)+M_N}i\left[\gamma_\lambda+\mu^p\frac{\sigma_{\lambda\xi}k_\xi}{2M_N}\right]$$
$$+i\left[\mu^n\frac{\sigma_{\lambda\xi}k_\xi}{2M_N}\right]\frac{1}{i\gamma\cdot(p_1+q)+M_N}$$
$$\times[ig_A(q^2)\gamma_\alpha\gamma_5+h_A(q^2)\gamma_5 q_\alpha$$
$$+iF_1{}^V(q^2)\gamma_\alpha-iF_2{}^V(q^2)\sigma_{\alpha\beta}q_\beta]\bigg\}u(p_1)\,, \quad (77)$$

$$M_{\lambda\alpha}{}^{RPD}=\frac{2M_N g_A}{g_r(0)}g_r\frac{\bar{u}(p_2)\gamma_5 u(p_1)}{(q-k)^2+m_\pi{}^2}$$
$$\times\left[\frac{-2q_\lambda q_\alpha}{q^2+m_\pi{}^2}+\delta_{\lambda\alpha}+(q_\lambda k_\alpha-q\cdot k\delta_{\lambda\alpha})S\right], \quad (78)$$

$$M_{\lambda\alpha}{}^{PPP}=-ig_A\frac{q_\alpha}{q^2+m_\pi{}^2}\bar{u}(p_2)$$
$$\times\left[\gamma_5\sigma_{\lambda\xi}k_\xi\frac{\mu^S}{2M_N}+\frac{M_N}{g_r(0)}q_\beta O_{\lambda\beta}\right]u(p_1)\,, \quad (79)$$

$$M_{\lambda\alpha}{}^{R}=i\bar{u}(p_2)\{-\sigma_{\alpha\lambda}(\mu^V/2M_N)$$
$$+2g_A'(0)(q_\lambda\gamma_\alpha+k_\alpha\gamma_\lambda-\delta_{\lambda\alpha}\gamma\cdot k)\gamma_5$$
$$-ir(0)\delta_{\lambda\alpha}\gamma_5+2F_1{}^{V\prime}(0)(q_\lambda\gamma_\alpha+k_\alpha\gamma_\lambda)$$
$$-2F_2{}^{V\prime}(0)(q_\lambda\sigma_{\alpha\beta}q_\beta-k_\alpha\sigma_{\lambda\xi}k_\xi)$$
$$+[M_N g_A/g_r(0)]O_{\lambda\alpha}\}u(p_1)\,, \quad (80)$$

with

$$\mu^p=1.79\,,\quad \mu^n=-1.91\,, \quad (81)$$

and with

$$O_{\lambda\alpha}=\gamma_5\sigma_{\lambda\xi}k_\xi\left[\frac{\partial\bar{V}_1{}^{(0)}}{\partial\nu_B}\bigg|_0\frac{k_\alpha}{2M_N}+\frac{\partial\bar{V}_1{}^{(-)}}{\partial\nu}\bigg|_0\frac{(p_1+p_2)_\alpha}{2M_N}\right]$$
$$+i\gamma_5[(p_1+p_2)_\lambda k_\alpha-(p_1+p_2)\cdot k\sigma_{\lambda\alpha}]\bar{V}_2{}^{(0)}|_0$$
$$+\gamma_5[\gamma_\lambda k_\alpha-\gamma\cdot k\delta_{\lambda\alpha}]\bar{V}_3{}^{(-)}|_0$$
$$+k_\sigma\epsilon_{\lambda\alpha\sigma\mu}\gamma_\mu\bar{V}_4{}^{(0)}|_0\,. \quad (82)$$

In Eq. (82), $|_0$ means evaluation of the $\bar{V}_j$ at the point $\nu=\nu_B=q^2=k^2=0$.

Let us now discuss the various terms in Eq. (76). The nucleon Born term $M_{\lambda\alpha}{}^N$ corresponds to the diagrams of Figs. 5(a)–(d). The term $M_{\lambda\alpha}{}^{\text{RPD}}$ describes radiative

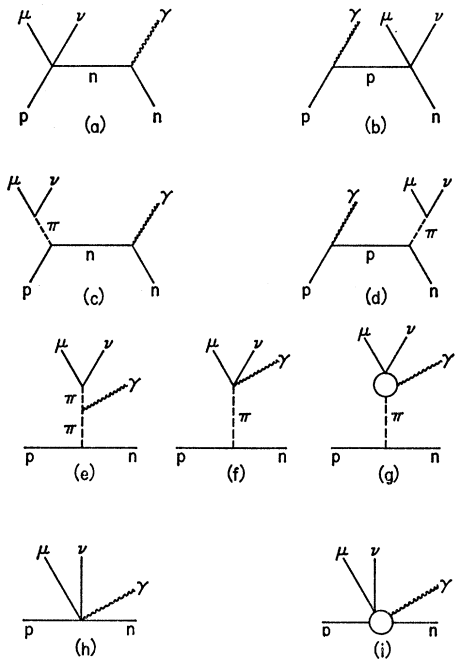

FIG. 5. Contributions to radiative μ capture.

virtual pion decay. The $q_\lambda q_\alpha$, $\delta_{\lambda\alpha}$, and S terms correspond, respectively, to Figs. 5(e)–5(g). The nontrivial structure term S *cannot* be determined by the procedure of this paper, because it is of order qk compared with the $\delta_{\lambda\alpha}$ term. The term $M_{\lambda\alpha}{}^{\text{PPP}}$ describes the structure part of virtual pion photoproduction. The Born part of virtual photoproduction has already been included in $M_{\lambda\alpha}{}^N$ and $M_{\lambda\alpha}{}^{\text{RPD}}$ [see Figs. 5(c)–5(e)]. In writing $M_{\lambda\alpha}{}^{\text{PPP}}$, we have eliminated $\bar{V}_1{}^{(0)}|_0$ by using Eq. (30), which implies

$$\bar{V}_1{}^{(0)}|_0=\frac{g_r(0)}{M_N}F_2{}^S(0)=\frac{g_r(0)\mu^S}{2M^2{}_N}\,. \quad (83)$$

Equation (83) is one of the photoproduction sum rules derived by Fubini, Furlan, and Rossetti.[22]

The remainder term $M_{\lambda\alpha}{}^R$ is necessary to satisfy the divergence equations, Eq. (60) and Eq. (62). The first term, proportional to $\mu^V/2M_N$, has been included in previous calculations. It corresponds to the "seagull" diagram of Fig. 5(h). The remaining terms, linear in q or k, are new. They are represented diagrammatically by Fig. 5(i). We thus see that our procedure has allowed us to determine the leading nontrivial structure effects in radiative μ capture.

ACKNOWLEDGMENTS

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[21] The terms of order q^2 are determined by our procedure, but we have omitted them in writing the answer because they are as small numerically as the undetermined terms of order qk.

[22] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).