tantamount to the irreducibility of the direct-product multiplet, which was to have been proven.

CONCLUSION

A study of the physically apparent transformation properties of the single-particle states has led us to two conclusions about the structure of a group that includes both internal and Poincaré symmetries. It has first been shown that only a group with an infinite number of generators can have single-particle multiplets with several different masses. Even if one is willing to accept such a group, it is not possible to construct a physically satisfactory theory. The second theorem shows that no two-particle states are reducible.[4] This implies, for example, that in octet-octet scattering the familiar formula $8\times 8=1+8+8+10+\overline{10}+27$ would no longer be valid and only a single scattering amplitude would be required to describe all 64 reactions. The present results are to be contrasted with previous theorems of a similar character.[5,6] The earlier results are obtained by making assumptions about the structure of the Lie algebra of the combined Poincaré and internal groups. These results are confined to groups with a finite number of generators. By treating the states and their transformation properties rather than the operators, we have shown that only groups with an infinity of generators can have an intrinsic mass formula but that these groups are subject to very serious objection.[7]

[4] A special case of the second theorem has been previously discussed. See N. M. Kroll, Phys. Letters **20**, 531 (1966).

[5] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964).

[6] F. Coester, M. Hamermesh, and W. D. McGlinn, Phys. Rev. **135**, B451 (1964).

[7] An example of this type of group has been proposed recently by J. Formánek, Nuovo Cimento **43**, 741 (1966).

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Some Consequences of an Integral Representation for the Renormalization Function $Z_3(s)$*

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An integral representation for the renormalization function, $Z_3(s)=\Delta_F(s)/\Delta_F'(s)$, and hence for the composite eigenvalue conditions $Z_3(\infty)=0$, $\lim_{s\to\infty} sZ_3(s)=0$ as well, is derived. The representation allows for the possible emergence onto the physical sheet of Castillejo-Dalitz-Dyson (CDD) poles, as the coupling strength increases, by satisfying the requirements of analytic continuation in that parameter. The composite conditions are obtained in a relativistic scalar model (in the elastic approximation) and in a one-pole nonrelativistic model; in confirmation of a surmise due to Ida, these conditions are found manifestly independent of the position of the CDD poles. Applications of the representation to some limit problems in the theory of noncomposite particles, to the Zachariasen model, and to a multichannel generalization of that model are made. The extension of the representation to the nucleon is the basis for a brief numerical analysis of $Z_N(w)$ and related quantities in the one-branch approximation.

I. INTRODUCTION

In a succession of papers[1–3] dealing with the Green's-function approach to some problems in the theory of strong interactions, Ida[2] has been able to show, by paying special attention to propagator zeros, the equivalence of that approach to the ordinary N/D method.[4] Central to his demonstration is the observation that the renormalization function, defined by[5]

$$Z_3(s)\equiv\Delta_F(s)/\Delta_F'(s)\,, \qquad (1)$$

is a Herglotz function. It follows that $Z_3(s)$ can be expressed as

$$Z_3(s)=1+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')g^2ds'}{|\mathfrak{D}(s')|^2(s'-\mu^2)^2(s'-s-i\epsilon)}+\sum_n c_n\frac{s-\mu^2}{s_n-s}\,, \qquad (2)$$

where the sum on the right-hand side of Eq. (2) represents the Castillejo-Dalitz-Dyson (CDD) terms[6]; the s_n's are the zeros of the meson propagator $\Delta_F'(s)$. It is assumed that the propagator $\Delta_F'(s)$ satisfies the

* Research supported in part by the National Science Foundation and by the Rutgers Research Council.

[1] M. Ida, Phys. Rev. **135**, B499 (1964); **136**, B1767 (1964).

[2] M. Ida, Progr. Theoret. Phys. (Kyoto) **34**, 92 (1965). We wish to thank Dr. Ida for a report of this work prior to publication.

[3] M. Ida, Progr. Theoret. Phys. (Kyoto) **34**, 990 (1965); **35**, 692 (1966).

[4] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); G. F. Chew, in *Dispersion Relations*, edited by G. R. Screaton (Interscience Publishers, Inc., New York, 1961), p. 167.

[5] Our discussion is limited to the relativistic scalar model considered by Ida (Ref. 2): S-wave scattering (in the elastic approximation) of a scalar baryon (M) and antibaryon with a (scalar) one-meson (μ) intermediate state. For simplicity, we have taken over the notation of Ref. 2 wherever possible.

[6] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

Lehmann representation in its unsubtracted form

$$i\Delta_F'(s)=\frac{1}{\mu^2-s}+\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')g^2ds'}{|Z_3(s')\mathfrak{D}(s')|^2(s'-\mu^2)^2(s'-s-i\epsilon)}, \quad (3)$$

or in its once-subtracted form

$$i\Delta_F'(s)=\frac{1}{\mu^2-s}+d+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')g^2ds'}{|Z_3(s')\mathfrak{D}(s')|^2(s'-\mu^2)^2(s'-s-i\epsilon)}. \quad (4)$$

At about the same time, Jin and MacDowell[7] provided a dynamical basis for the appearance on the physical sheet of propagator zeros (as well as for vertex poles) by assuming the possibility of analytically continuing the propagator (as well as related quantities) as a function of the coupling constant. To summarize what in their discussion,[7] is relevant to our present considerations: Jin and MacDowell take for their starting point the Lehmann representation of (2) with the coupling, say g^2, sufficiently weak so that there are no physical propagator zeros, i.e., no CDD poles in $Z_3(s)$. As the coupling is increased, a pole of $Z_3(s)$ on the second sheet moves around the threshold $s=4M^2$, onto the real axis of the physical sheet, below threshold and above the physical pole (of Δ_F'), μ^2, pushing the contour of integration (say C) ahead of it.[8] The pole of $Z_3(s)$ then moves from the right to the left of μ^2 as g^2 passes its critical value.[7] Thus one now finds an additional term in the expression for $Z_3(s)$, on returning to the "original" Lehmann form,[9] which is just the CDD pole term.[8]

Now Ida[1,2] has shown very simply how to determine the constants c_n at these poles s_n in $Z_3(s)$. Because of the division of the scattering amplitude $T(s)$ into its one-meson reducible term[2] and one-meson irreducible term,[2] $U(s)=\mathfrak{N}(s)/\mathfrak{D}(s)$,

$$T(s)=\frac{g^2}{(\mu^2-s)Z_3(s)\{\mathfrak{D}(s)\}^2}+\frac{\mathfrak{N}(s)}{\mathfrak{D}(s)}, \quad (5)$$

poles in $Z_3(s)$ become necessary for the removal of the unwanted poles in $U(s)$, which are, in turn, produced by zeros of $\mathfrak{D}(s)$ [vertex poles[1] of the proper vertex $\Gamma(s)=g/\mathfrak{D}(s)$]. Thus we must have

$$c_n=-\frac{g^2}{(s_n-\mu^2)^2\mathfrak{D}'(s_n)\mathfrak{N}(s_n)}. \quad (6)$$

Adopting Ida's "principle of minimal singularity,"[2] one finds that when $Z_3(\infty)=0$,

$$\frac{1}{g^2}=\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s)ds}{(s-\mu^2)^2|\mathfrak{D}(s)|^2}-\sum_n\frac{1}{(s_n-\mu^2)^2\mathfrak{D}'(s_n)\mathfrak{N}(s_n)}; \quad (7)$$

also, if the propagator requires one subtraction as in Eq. (4), one has in addition

$$0=\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s)ds}{(s-\mu^2)|\mathfrak{D}(s)|^2}-\sum_n\frac{1}{(s_n-\mu^2)\mathfrak{D}'(s_n)\mathfrak{N}(s_n)}. \quad (8)$$

An apparent problem, noted by Ida,[2] remains. In the case of a single pole, say $n=1$, we see that Eqs. (7) and (8) [10] involve s_1 as a "free" parameter. Since s_1 is in a sense a "hidden variable" in the theory, it would be desirable to show explicitly that Eqs. (7) and (8) are, in fact, independent of it. [Our expectation of such a possibility stems from the equivalence of (7) and (8) to their analogs in the N/D method, Eqs. (3.10) and (3.12) of Ref. 2.] It is our purpose in this note to present an alternative integral representation for $Z_3(s)$, suggested by the result (6), which is, indeed, independent of s_1. We will see that the arguments of Jin and MacDowell[7] are more easily illustrated by means of it. Moreover, this representation yields considerable analytic simplicity in calculations based on the Green's-function approach. In Sec. II we derive the integral representation and discuss in terms of it the problem of analytic continuation in coupling strength; the accompanying derivation of the composite eigenvalue conditions is also treated in some detail. In Sec. III we apply the representation to a soluble nonrelativistic model which was earlier scrutinized by Ida.[2] In Sec. IV applications are made to some limit problems in the theory of noncomposite particles, to the Zachariasen model, and to a multichannel generalization of that model. The extension of the representation to the nucleon and a brief numerical analysis of $Z_N(w)$ and related quantities in a "one-branch" approximation are given in Sec. V.

II. INTEGRAL REPRESENTATION OF $Z_3(s)$

We assume the coupling g^2 too weak to produce a zero of $\mathfrak{D}(s)$ on the real axis of the physical sheet below

[7] Y. S. Jin and S. W. MacDowell, Phys. Rev. **137**, B688 (1965).
[8] See Fig. 1 in Ref. 7.
[9] A return to the original (undeformed) contour running from $4M^2$ to ∞ is implied.
[10] In the composite case, these are residue and mass equations, respectively.

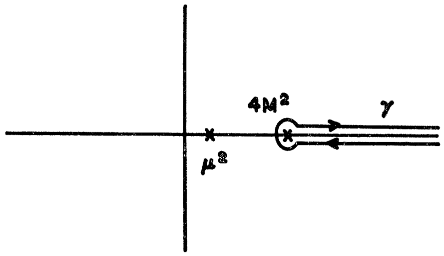

FIG. 1. The complex s plane showing the contour γ in the weak-coupling situation.

threshold; then

$$Z_3(s)=1+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')g^2ds'}{|\mathfrak{D}(s')|^2(s'-\mu^2)^2(s'-s-i\epsilon)} \tag{9}$$

$$=1+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{g^2ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)}\frac{1}{2i\mathfrak{N}(s')}\times\left[\frac{1}{\mathfrak{D}(s'+i0)}-\frac{1}{\mathfrak{D}(s'-i0)}\right], \tag{10}$$

since,

$$\frac{1}{\mathfrak{D}(s+i0)}-\frac{1}{\mathfrak{D}(s-i0)}=\frac{2i}{|\mathfrak{D}(s+i0)|^2}\rho(s+i0)\mathfrak{N}(s+i0). \tag{11}$$

Hence

$$Z_3(s)=1+\frac{s-\mu^2}{2\pi i}\int_\gamma\frac{g^2ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)\mathfrak{N}(s')\mathfrak{D}(s')}, \tag{12}$$

where the contour γ is illustrated in Fig. 1. The problem of analytic continuation in g^2 is then solved by requiring that γ be "pushed" ahead of any pole, say $s_1(g^2)$, as the pole moves along the real axis below threshold with increasing g^2. The situation that obtains when $g^2<g^2_{\text{critical}}$, that is to say, when the pole emerges onto the physical sheet, is illustrated in Fig. 2. We see that the contribution from the CDD pole of $Z_3(s)$ [or zero of $\mathfrak{D}(s)$] is $-g^2\{(s_1-\mu^2)(s_1-s)\mathfrak{N}(s_1)\mathfrak{D}'(s_1)\}^{-1}$. Of course, it is possible to ignore the deformation of γ necessary when zeros of $\mathfrak{D}(s)$ appear on the physical sheet by simply "excluding" them from consideration *when the contour γ is closed.*[11] Thus, if we write

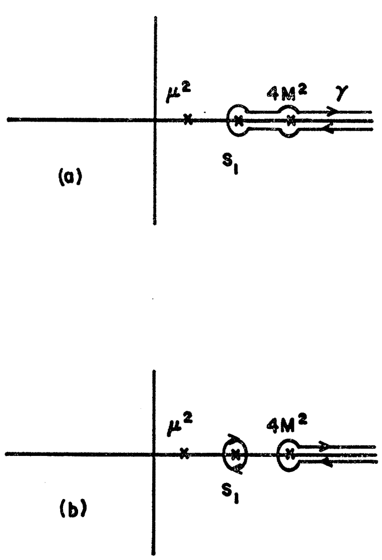

FIG. 2. The complex s plane showing in (a) the contour γ for $g^2<(g^2)_{\text{critical}}$ and in (b) the equivalent contour which exposes the contribution of the propagator zero, s_1 to $Z_3(s)$.

$$\int_\gamma{}'\equiv\int_{\gamma,\text{ ignoring possibility of zeros in }\mathfrak{D}} \tag{13}$$

so that

$$Z_3(s)=1+\frac{s-\mu^2}{2\pi i}\int_\gamma{}'\frac{g^2ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)\mathfrak{N}(s')\mathfrak{D}(s')}, \tag{14}$$

with the composite eigenvalue conditions given by

$$Z_3=0=1-\frac{1}{2\pi i}\int_\gamma{}'\frac{g^2ds'}{(s'-\mu^2)^2\mathfrak{N}(s')\mathfrak{D}(s')}, \tag{15}$$

and

$$\int_\gamma{}'\frac{ds'}{(s'-\mu^2)\mathfrak{N}(s')\mathfrak{D}(s')}=0, \tag{16}$$

then Eqs. (14), (15), and (16) are *manifestly independent of the possible zeros s_n of* $\mathfrak{D}(s)$. [Note that γ naturally excludes pseudoresonances.] On closing the contour γ in the usual way, we find the useful results[12]

$$Z_3(s)=1-\frac{s-\mu^2}{2\pi i}\int_{\gamma_L}\frac{g^2ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)\mathfrak{N}(s')\mathfrak{D}(s')}+\frac{g^2}{(s-\mu^2)}\left[\frac{1}{\mathfrak{N}(s)\mathfrak{D}(s)}-\frac{1}{\mathfrak{N}(\mu^2)\mathfrak{D}(\mu^2)}\right]+\frac{g^2\mathfrak{N}'(\mu^2)}{\{\mathfrak{N}(\mu^2)\}^2}+\frac{g^2\mathfrak{D}'(\mu^2)}{\mathfrak{N}(\mu^2)}, \tag{17}$$

$$Z_3=0=1+\frac{1}{2\pi i}\int_{\gamma_L}\frac{g^2ds'}{(s'-\mu^2)^2\mathfrak{N}(s')\mathfrak{D}(s')}+\frac{g^2\mathfrak{N}'(\mu^2)}{\{\mathfrak{N}(\mu^2)\}^2}+\frac{g^2\mathfrak{D}'(\mu^2)}{\mathfrak{N}(\mu^2)}, \tag{18}$$

and

$$-\frac{1}{2\pi i}\int_{\gamma_L}\frac{ds}{(s-\mu^2)\mathfrak{N}(s)\mathfrak{D}(s)}+\frac{1}{\mathfrak{N}(\mu^2)}=0, \tag{19}$$

[11] The assumption that the contour *can* be closed (that is, that the contribution from the infinite circular arc vansihes) may itself be a useful, albeit weak, constraint on the asymptotic behavior of $\mathfrak{N}(s)$ and $\mathfrak{D}(s)$. Thus, for example, from Eq. (32) we conclude that if $\mathfrak{D}(s)\sim\ln s$ asymptotically (Table I of Ref. 2), then $\mathfrak{N}(s)\sim$constant asymptotically is consistent with the required asymptotic behavior of $|s\mathfrak{N}(s)\mathfrak{D}(s)|$.

[12] We suppose for simplicity that $\mathfrak{N}(s)$ has no isolated zeros. The contour γ_L runs from $s=-\infty$ along the upper side of the left-hand cut to the branch point $s=s_L$, thence back to $s=-\infty$ along the lower side.

where we have used[2]

$$\mathfrak{D}(\mu^2)=1, \tag{20}$$

which follows from[2]

$$\mathfrak{D}(s)=1-\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')\mathfrak{N}(s')ds'}{(s'-\mu^2)(s'-s-i\epsilon)}. \tag{21}$$

If one utilizes Eq. (18) to simplify Eq. (17), without necessarily taking $Z_3=0$, one finds

$$Z_3(s)=Z_3+\frac{g^2}{(s-\mu)^2}\left[\frac{1}{\mathfrak{N}(s)\mathfrak{D}(s)}-\frac{1}{\mathfrak{N}(\mu^2)\mathfrak{D}(\mu^2)}\right] -\frac{1}{2\pi i}\int_{\gamma_L}\frac{g^2ds'}{(s'-\mu^2)\mathfrak{N}(s')\mathfrak{D}(s')(s'-s-i\epsilon)} \tag{22}$$

with the resulting asymptotic behavior for $Z_3(s)$,[2]

$$Z_3(s)\underset{s\to\infty}{\longrightarrow}[s\mathfrak{N}(s)\mathfrak{D}(s)]^{-1}, \tag{23}$$

for particles of intermediate[2] and composite types. {In the case of the pseudoscalar model of Ref. 2, where $\rho(s)=s(1-4M^2/s)^{1/2}$, this behavior is given by

$$Z_\pi(s)\to[\mathfrak{N}_0(s)\mathfrak{D}_0(s)]^{-1}.\}$$

{Note that since

$$(1/2i)[\mathfrak{N}(s+i0)-\mathfrak{N}(s-i0)]=\mathrm{Im}T(s)\mathfrak{D}(s),$$

on the left-hand cut, one also has the further simplification

$$\frac{1}{2\pi i}\int_{\gamma_L}\frac{ds'}{P(s')\mathfrak{N}(s')\mathfrak{D}(s')}=-\int_{-\infty}^{s_L}\frac{ds'\,\mathrm{Im}T(s')}{P(s')|\mathfrak{N}(s')|^2}, \tag{24}$$

where $P(s)$ is a real polynomial in s.}

The eigenvalue condition (18) may be carried somewhat further. For example, parametrically differentiating the condition (19) with respect to the eigenvalue μ^2 one finds

$$\frac{d}{d\mu^2}\frac{1}{\mathfrak{N}(\mu^2)}=\frac{1}{2\pi i}\int_{\gamma_L}\frac{ds}{(s-\mu^2)^2\mathfrak{N}(s)\mathfrak{D}(s)} +\frac{1}{2\pi i}\int_{\gamma_L}\frac{ds}{(s-\mu^2)}\frac{d}{d\mu^2}[\mathfrak{N}(s)\mathfrak{D}(s)]^{-1}; \tag{25}$$

thus[13]

$$\frac{1}{g^2}=\frac{1}{\pi\mathfrak{N}(\mu^2)}\int_{4M^2}^{\infty}\frac{\rho(s')\mathfrak{N}(s')ds'}{(s'-\mu^2)^2}$$

$$+\text{contribution from the left-hand cut.} \tag{26}$$

[13] We have used the result,

$$\mathfrak{D}'(\mu^2)=-\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s)\mathfrak{N}(s)ds}{(s-\mu^2)^2}.$$

Without further simplification of (26), we note its close resemblance to the corresponding condition in S-matrix theory[14]; expression (26) may have some utility in the case of small binding energy.[14]

In the case that the numerator function, $\mathfrak{N}(s)$, is approximated by a sum of poles, as, for example, in a Balázs-type representation of the left-hand cut, the utility of the representation under consideration is immediate; the eigenvalue conditions (15) and (16) then reduce to *algebraic equations*.

III. A SOLUBLE NONRELATIVISTIC MODEL

We consider here briefly the soluble nonrelativistic model used by Ida[2] to illustrate the s_1 independence of Eqs. (7) and (8), but now from the standpoint of our Eqs. (15) and (16). The model is defined by[2]

$$\mathrm{Im}T(s)=-a\pi\delta(s+K_a^2),\quad a>0 \tag{27}$$

with $s=k^2$. One has for $\mathfrak{N}(s)$,

$$\mathfrak{N}(s)=\mathfrak{N}_\infty+a\mathfrak{D}(-K_a^2)/(s+K_a^2), \tag{28}$$

with

$$\mathfrak{D}(s)=1-\mathfrak{N}_\infty(K_B+ik) -\frac{a[1+\mathfrak{N}_\infty(K_a-K_B)]}{1-(a/2K_a)[(K_a-K_B)/(K_a+K_B)]}\times\frac{(K_B+ik)}{(K_a+K_B)(K_a-ik)}, \tag{29}$$

after making use of

$$\mathfrak{D}(-K_a^2)=\frac{1+\mathfrak{N}_\infty(K_a-K_B)}{1-(a/2K_a)[(K_a-K_B)/(K_a+K_B)]}. \tag{30}$$

Then

$$Z_3=0=1-\frac{g^2}{(K_B^2-K_0^2)^2\mathfrak{N}'(-K_0^2)\mathfrak{D}(-K_0^2)} +\frac{g^2\mathfrak{N}'(-K_B^2)}{\{\mathfrak{N}(-K_B^2)\}^2}+g^2\frac{\mathfrak{D}'(-K_B^2)}{\mathfrak{N}(-K_B^2)}, \tag{31}$$

where the second term on the right-hand side of (31) stems from the zero in $\mathfrak{N}(s)$ at $s=-K_0^2=-[\mathfrak{N}_\infty K_a^2+a\mathfrak{D}(-K_a^2)]/\mathfrak{N}_\infty$. Similarly the "mass" equation (19) yields

$$0=\frac{1}{\mathfrak{N}(-K_B^2)}+\frac{1}{(K_B^2-K_0^2)\mathfrak{N}'(-K_0^2)\mathfrak{D}(-K_0^2)}. \tag{32}$$

It is useful to parametrize the functions $\mathfrak{N}$ and $\mathfrak{D}$ in terms of the zero $k=iK_1$ of $\mathfrak{D}(s)$; then as Ida has

[14] M. Nauenberg, Phys. Rev. **124**, 2011 (1961).

shown,[2] the correct K_1-independent results

$$g^2=2K_B\left(\frac{K_a+K_B}{K_a-K_B}\right), \tag{33}$$

$$K_B=K_a\left(\frac{a-2K_a}{a+2K_a}\right), \tag{34}$$

may be conveniently obtained by carefully taking the limit $K_1\to K_a$. It is interesting to note that contrary to an assertion of Ida,[15] we can still use the nonrelativistic version of Eq. (6) to determine c_1, although in the limiting process the propagator zero at $s=-K_1^2$ coincides with the pole in $T(s)$ at $s=-K_a^2$ when $K_1=K_a$. Ida gives the expression appropriate to this situation[16]:

$$a=\left\{\frac{g^2}{(K_B^2-K_a^2)^2\mathfrak{D}'(-K_a^2)c_1}+\mathfrak{N}(-K_a^2)\right\}\frac{1}{\mathfrak{D}'(-K_a^2)}, \tag{35}$$

where, for $K_1=K_a$,

$$\mathfrak{N}(s)=-(K_a-K_B)^{-1}, \tag{36}$$

$$\mathfrak{D}(s)=(K_a+ik)/(K_a-K_B). \tag{37}$$

Instead, let us focus our attention on the nonrelativistic version of (6),[17]

$$c_1=-\frac{g^2}{(K_1^2-K_B^2)^2\mathfrak{D}'(-K_1^2)\mathfrak{N}(-K_1^2)}; \tag{38}$$

we see that the difficulty appears to be with $\lim_{K_1\to K_a}\mathfrak{N}(-K_1^2)$. Observe that since

$$\mathfrak{N}(s)=\mathfrak{N}(\infty)+a\mathfrak{D}(-K_a^2)/(s+K_a^2), \tag{39}$$

we may eliminate $\mathfrak{N}(\infty)$, writing,

$$\mathfrak{N}(s)=\mathfrak{N}(-K_1^2)+a\mathfrak{D}(-K_a^2)\frac{(s+K_1^2)}{(s+K_a^2)(K_1^2-K_a^2)}. \tag{40}$$

For $K_1\approx K_a$, we approximate $\mathfrak{D}(-K_a^2)$ by

$$\mathfrak{D}(-K_a^2)=(K_1^2-K_a^2)\mathfrak{D}'(-K_1^2); \tag{41}$$

then,

$$\mathfrak{N}(-K_1^2)\simeq\mathfrak{N}(s)-a\mathfrak{D}'(-K_1^2)\left(\frac{s+K_1^2}{s+K_a^2}\right) \tag{42}$$

$$\simeq\mathfrak{N}(s)-a\mathfrak{D}'(-K_a^2)\left(\frac{s+K_1^2}{s+K_a^2}\right). \tag{43}$$

Holding s fixed, we take the limit $K_1\to K_a$ on the right-hand side of (43) [$\mathfrak{N}(s)$ is now independent of s]; it follows that

$$\lim_{K_1\to K_a}\mathfrak{N}(-K_1^2)=\mathfrak{N}(-K_a^2)-a\mathfrak{D}'(-K_a^2), \tag{44}$$

where on the right-hand side of (44), $\mathfrak{N}(-K_a^2)$ is given by (36). Thus,

$$c_1=-\frac{g^2}{(K_a^2-K_B^2)^2\mathfrak{D}'(-K_a^2)[\mathfrak{N}(-K_a^2)-a\mathfrak{D}'(-K_a^2)]}, \tag{45}$$

which is, of course, identical with expression (35).

IV. DIVERS APPLICATIONS

In this section we sketch briefly three elementary applications of the representation (12) which serve to illustrate the ease with which results earlier derived by more cumbersome manipulations may now be recovered. As a first example,[3] we show how the transition from intermediate [$Z_3(\infty)=0$] to composite [$Z_3(\infty)=0$ and $\lim_{s\to\infty} sZ_3(s)=0$] particle type may be observed in the scalar theory without explicit reference to the scattering amplitude $T(s)$.[3] One substitutes the representation[18,19]

$$\frac{1}{\mathfrak{D}(s)}=\frac{1}{\mathfrak{D}(\mu^2)}+\frac{(s-\mu^2)}{2\pi i}\int_\gamma{}' \frac{ds'}{(s'-\mu^2)\mathfrak{D}(s')(s'-s-i\epsilon)}$$
$$=\frac{1}{\mathfrak{D}(\mu^2)}+\frac{s-\mu^2}{\pi}\int_{4M^2}^\infty \frac{ds'\rho(s')U^*(s')}{(s'-\mu^2)\mathfrak{D}(s')(s'-s-i\epsilon)}$$
$$+\frac{1}{\mathfrak{D}'(\mu_c^2)}\left(\frac{1}{\mu_c^2-\mu^2}-\frac{1}{\mu_c^2-s}\right) \tag{46}$$

---

[15] See the closing remarks of the Appendix to Ref. 2.

[16] We surmise Eq. (35) was obtained by comparing $T(s)$ in the neighborhood of $s=-K_a^2$ with its equivalent form,

$$U(s)-\Gamma(s)\frac{1}{(s+K_B^2)Z_3(s)}\Gamma(s),$$

in the same neighborhood; thus, taking expressions (36) and (37) for $\mathfrak{N}(s)$ and $\mathfrak{D}(s)$, respectively, we have

$$T\sim\frac{a}{s+K_a^2}$$
$$\sim\frac{g^2}{(K_B^2-K_a^2)^2C_1(K_a^2+s)\{\mathfrak{D}'(-K_a^2)\}^2}+\frac{\mathfrak{N}(-K_a^2)}{(Ka^2+s)\mathfrak{D}'(-K_a^2)}.$$

[17] Note that $\mathfrak{N}(s)$, $\mathfrak{D}(s)$ are the parametrized functions (in terms of K_1).

[18] The representation for $[\mathfrak{D}(s)]^{-1}$ results from the alteration (in the sense of an analytic continuation) in the usual Cauchy representation of the vertex function,

$$\Gamma(s)=\Gamma_c[\mathfrak{D}(\mu^2)/\mathfrak{D}(s)]$$
$$=\Gamma_c+\frac{s-\mu^2}{2\pi i}\int_{\gamma_{\text{elastic}}}\frac{\Gamma(s')ds'}{(s'-\mu^2)(s'-s-i\epsilon)}$$
$$=\Gamma_c+\frac{s-\mu^2}{\pi}\int_{4M^2}^\infty\frac{\operatorname{Im}\Gamma(s')ds'}{(s'-\mu^2)(s'-s-i\epsilon)},$$

to allow for the emergence of a zero of $\mathfrak{D}(s)$ at $s=\mu_c^2$ onto the physical sheet.

[19] Note that

$\lim_{\mu^2\to\mu_c^2}\mathfrak{D}(\mu^2)=0$ and $\lim_{\mu^2\to\mu_c^2}[\mathfrak{D}(s)]^{-1}=[(s-\mu_c^2)\mathfrak{D}'(\mu_c^2)]^{-1}$,

so that $\lim_{\mu^2\to\mu_c^2}\Gamma(s)=0$ for $s\neq\mu^2$, whereas $\lim_{\mu^2\to\mu_c^2}\Gamma(\mu^2)=\Gamma_c$. {Incidentally, we note from (46) that if $[\mathfrak{D}(\infty)]^{-1}$ is finite then $\lim_{\mu^2\to\mu_c^2}[\mathfrak{D}(\infty)]^{-1}$ is also.} Clearly, for s, $\mu^2\approx\mu_c^2$, the vertex function $\Gamma(s)\approx\Gamma_c(\mu^2-\mu_c^2)/(s-\mu_c^2)$. This singular behavior has led to a continuing misapprehension in the literature. For example, Broido and Taylor remark [M. Broido and J. G. Taylor, Phys. Rev. **147**, 993 (1966)] that "the . . . vertex function $\Gamma(E)$ vanishes except at the bound-state energy B, when it takes a finite value. Evidently such a function cannot be treated in a consistent fashion without vanishing effectively everywhere."

into the representation for Z_3,

$$Z_3=0=1-\frac{1}{2\pi i}\int_\gamma' \frac{\Gamma_c^2[\mathfrak{D}(\mu^2)]^2 ds}{(s-\mu^2)^2\mathfrak{N}(s)\mathfrak{D}(s)}$$

$$=1+\Gamma_c^2\frac{[\mathfrak{D}(\mu^2)]^2}{(\mu_c^2-\mu^2)^2\mathfrak{N}(\mu^2)\mathfrak{D}'(\mu_c^2)}$$

$$+\text{terms which vanish as } \mu^2\rightarrow\mu_c^2, \quad (47)$$

and recovers the relation[3]

$$\lim_{\mu^2\rightarrow\mu_c^2} Z_3=1+\Gamma_c^2\mathfrak{D}'(\mu_c^2)/\mathfrak{N}(\mu_c^2)=0. \quad (48)$$

Similarly, one finds that $Z_1=\lim_{\mu^2\rightarrow\mu_c^2}\mathfrak{D}(\mu^2)/\mathfrak{D}(\infty)=0$ together with $Z_3=0$ ensure that

$$\lim_{\mu^2\rightarrow\mu_c^2}\{\lim_{s\rightarrow\infty} sZ_3(s)\}=0.$$

On the other hand, if $Z_3\neq0$, then the pole term in $[\mathfrak{D}(s)]^{-1}$, $[(s-\mu_c^2)\mathfrak{D}'(\mu_c^2)]^{-1}$ furnishes the only nonvanishing contribution to $Z_3(s)$ in the limit $\mu^2\rightarrow\mu_c^2$, and in the neighborhood $s\approx\mu^2\approx\mu_c^2$, one finds[3]

$$Z_3(s)=1+(Z_3-1)\left(\frac{s-\mu^2}{s-\mu_c^2}\right). \quad (49)$$

The solution of Zachariasen's model[20] is too well known to warrant more than brief mention here. We observe merely that in the absence of a left-hand cut, one has

$$\mathfrak{N}(s)=\mathfrak{N}(\infty)=\lambda, \quad (50)$$

where λ is the four-point coupling, so that the representation [Eq. (47)] yields

$$Z_3(s)=Z_3+\frac{g^2[\mathfrak{D}(\mu^2)]^2}{(s-\mu^2)\lambda}\left[\frac{1}{\mathfrak{D}(s)}-\frac{1}{\mathfrak{D}(\mu^2)}\right]$$

$$=Z_3\{Z_3-\lambda[s-\mu^2-(g^2/\lambda)]I(s,\mu^2)\} \quad (51)$$

with

$$Z_3=1-g^2I(\mu^2,\mu^2)=\mathfrak{D}(\mu^2), \quad (52)$$

$$I(s,\mu^2)=\frac{1}{\pi}\int_{4M^2}^\infty \frac{\rho(s')ds'}{(s'-\mu^2)(s'-s-i\epsilon)}, \quad (53)$$

and

$$T(s)=\frac{\lambda}{\mathfrak{D}(s)}+g^2\left[\frac{\mathfrak{D}(\mu^2)}{\mathfrak{D}(s)}\right]^2\frac{1}{Z_3(s)(\mu^2-s)} \quad (54)$$

$$=\frac{\lambda+g^2/(\mu^2-s)}{Z_3-\lambda[s-\mu^2-(g^2/\lambda)]I(s,\mu^2)}. \quad (55)$$

It is no more difficult to obtain a closed form for $Z_3(s)$ in the case of the generalization of the above model to n two-particle channels.[3,21] Thus we are able to transform

$$Z_3(s)=1+\frac{s-\mu^2}{\pi}\int_{s_{\text{th}}}^\infty \frac{ds'\mathbf{\Gamma}^\dagger(s')\boldsymbol{\varrho}(s')\mathbf{\Gamma}(s')}{(s'-\mu^2)^2(s'-s-i\epsilon)}+C\frac{s-\mu^2}{\mu_c^2-s}, \quad (56)$$

where

$$\mathbf{\Gamma}(s)=\mathbf{\mathfrak{D}}^{-1}(s)\mathbf{\mathfrak{D}}(\mu^2)\mathbf{g} \quad (57)$$

with

$$\text{Im}\mathbf{\mathfrak{D}}(s)=-\mathbf{\mathfrak{N}}(s)\boldsymbol{\varrho}(s) \quad (58)$$

and

$$\mathbf{\Gamma}(s)^\dagger\boldsymbol{\varrho}(s)\mathbf{\Gamma}(s)=\text{Im}\mathbf{g}^T\mathbf{\mathfrak{D}}^T(\mu^2)\mathbf{\mathfrak{N}}^{-1T}\mathbf{\Gamma}(s) \quad (59)$$

into

$$Z_3(s)=1+\frac{s-\mu^2}{2\pi i}\int_\gamma' \frac{ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)}\times\mathbf{g}^T\mathbf{\mathfrak{D}}^T(\mu^2)\mathbf{\mathfrak{N}}^{-1T}\mathbf{\Gamma}(s'), \quad (60)$$

and finally, to obtain

$$Z_3(s)=Z_3+(1/(s-\mu^2))\mathbf{g}^T\mathbf{\mathfrak{D}}^T(\mu^2)\times\mathbf{\mathfrak{N}}^{-1T}[\mathbf{\Gamma}(s)-\mathbf{\Gamma}(\mu^2)], \quad (61)$$

with

$$Z_3=1-\mathbf{g}^T\mathbf{\mathfrak{D}}^T(\mu^2)\mathbf{\mathfrak{N}}^{-1T}\{[(d/ds)[\mathbf{\mathfrak{D}}^{-1}\det\mathbf{\mathfrak{D}}]]_{(s=\mu^2)}-\mathbf{\mathfrak{D}}^{-1}(\mu^2)[(d/ds)\det\mathbf{\mathfrak{D}}(s)]_{(s=\mu^2)}\}\times[1/\det\mathbf{\mathfrak{D}}(\mu^2)]\mathbf{\mathfrak{D}}(\mu^2)\mathbf{g}, \quad (62)$$

where

$$\mathbf{\mathfrak{N}}=\boldsymbol{\lambda}. \quad (63)$$

V. PROPERTIES OF THE NUCLEON RENORMALIZATION FUNCTION

In this section the integral representation for the renormalization function is extended to the nucleon and is the basis for a brief numerical analysis of $Z_N(w)$ and related quantities. The usefulness of the representation is readily apparent in the facility with which such an analysis may be carried out.[22] To make life simpler, we confine ourselves to a "one-branch" approximation to the pion-nucleon vertex function.[23] Thus we take

$$\Gamma(w)=gD^{-1}(w), \quad (64)$$

with

$$\text{Im}\Gamma(w)=g\frac{\rho_P(w)N(w)}{|D(w)|^2}, \quad w\geq m+\mu \quad (65)$$

and by steps similar to those of Sec. 2 arrive at an approximate expression for $Z_N(w)$,[24]

$$Z_N(w)\simeq1+\frac{w-m}{2\pi i}\times\int_\gamma' \frac{dw'3g^2(\mu/m)^2}{16\pi(w'-m)^2N(w')D(w')(w'-w-i\epsilon)}, \quad (66)$$

[20] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

[21] Such a closed form is lacking in the lengthy considerations of the latter of Refs. 3.

[22] This subject was studied rather closely by Ida in the second of Refs. 1; we use his notation wherever possible.

[23] This is rather like the approximation made in an S-matrix study of the nucleon as a bound state by L. A. P. Balázs, Phys. Rev. **128**, 1935 (1962).

[24] We have dropped the contribution from negative-energy states which Ida (Ref. 1) estimates as about 10% of the contribution to Z_N from pion-nucleon intermediate states.

with

$$Z_N=Z_N(\infty)$$
$$=1-\frac{1}{2\pi i}\int_\gamma{}' \frac{dw'3g^2(\mu/m)^2}{16\pi(w'-m)^2N(w')D(w')}. \quad (67)$$

Then, in the approximation[1]

$$N(w)=\frac{1}{3}\frac{f^2}{w-m}+\frac{16}{9}\frac{f^{*2}}{w-m_1}D(w_1) \quad (68)$$

with $f^2=(\mu/m)^2(g^2/16\pi)$, $f^{*2}=3f^2/2$, $D(w_1)\simeq1.56$, and $w_1=0.68m$, we find

$$Z_N=1-3f^2\left\{\frac{1}{(w_0-m)^2N'(w_0)D(w_0)}+\frac{3}{f^2}\right\}, \quad (69)$$

where w_0 is the zero of the numerator function $N(w)$ at

$$w_0=m\frac{1+0.68/8D(w_1)}{1+1/8D(w_1)}\simeq0.97m. \quad (70)$$

It is easy to satisfy the requirement $Z_N=0$ even in this crude approximation, as one finds that $Z_N=0$ implies that

$$D(w_0)=1.125[1+1/8D(w_1)]^{-1}\simeq1.04. \quad (71)$$

Since[1] $\mathrm{Re}D(m+2.1\mu)\simeq0$ and $D(m)=1$, a linear fit to $\mathrm{Re}D(w)$ would yield

$$D(w_0)=\frac{w_0-(m+2.1\mu)}{m-(m+2.1\mu)}\simeq1.1, \quad (72)$$

which is reasonably consistent with some curvature.[25] After some formal manipulation of exact relations[1] one finds that

$$\delta m=m-m_0=\lim_{w\to\infty} Z_N^{-1}(w-m)[Z_N(w)-Z_N], \quad (73)$$

and, in the "one-branch" approximation,

$$\delta m^{(0)}=\lim_{w\to\infty}\frac{w}{2\pi i}\times\int_\gamma{}'\frac{dw'3g^2(\mu/m)^2}{16\pi(w'-m)N(w')D(w')(w'-w-i\epsilon)}$$
$$=\lim_{w\to\infty}\left\{3f^2\frac{w}{(w-m)N(w)D(w)}+\text{terms which are asymptotically constant}\right\}, \quad (74)$$

so that $\delta m^{(0)}$ diverges like $\lim_{w\to\infty}(w/\ln w)$.[26] Recently Fried and Truong[27] proposed to determine the sign of the proton-neutron mass difference through considerations based on the relation[28]

$$m-m_0=-\frac{Z_N^{-1}}{\pi}\times\int_{m+\mu}^{\infty}dw[(w-m)\tau_+(w)-(w+m)\tau_-(w)]; \quad (75)$$

however, their assumption that the vertex functions $|\Gamma_\pm|$ are damped out beyond a few nucleon masses would appear to make the nucleonic self-mass finite, otherwise their "more accurate" expression[29] (which includes "implicit e^2 variation of Z_N^{-1}") for $\Delta m=\delta m_p-\delta m_n$ contains a divergent integral, $I(m)$, and is not well defined. Furthermore, the limit $Z_N\to0$ is not a "bootstrap limit" as they say but merely takes an "elementary" nucleon to one of "intermediate" type.[3] Finally, we wish to remark that if one takes the nucleon to be an "intermediate" particle, then it would be natural to consider simply the difference in eigenvalue conditions, $Z_N^{(p)}-Z_N^{(n)}=0$, as providing the necessary relation between $\Delta m=m_p-m_n$ and input discontinuities such as the one-photon exchange graph,[30] for a given f^2.

[25] What is most surprising is the relative lack of dependence on f^2 of this result.

[26] Inclusion of the contribution from the negative-energy state reduces this divergence to a logarithmic one.

[27] H. M. Fried and T. N. Truong, Phys. Rev. Letters **16**, 559 (1966).

[28] This is readily seen to be equivalent to our expression (73).

[29] Equation (15) in Ref. 26.

[30] R. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964).