Some Consequences of an Intrinsic Mass Formula*

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The requirements that are placed on a relativistically invariant theory by the assumption that single-particle states with different masses are included in the same multiplet are examined. This assumption implies first that the Lie algegra of the symmetry group which includes the generators of both the Lorentz and internal transformations must also have an infinite number of other elements, and secondly, that the direct product of any two single-particle multiplets is irreducible.

INTRODUCTION

THE remarkable successes of first-order perturbation calculations[1,2] for producing mass formulas in symmetry theories of elementary particles has led to the conjecture that the mass relations are somehow inherent in the symmetries of the particles.[3] Thus, we may speak of a theory with an "intrinsic mass formula." The states of a multiplet in such a theory are not degenerate in mass. The mass formula comes out of the symmetry, rather than out of symmetry-breaking interactions. The inclusion of a mass formula automatically involves us in a discussion of the Poincaré transformation properties of the particles, since their masses are just the invariant lengths of their four-momenta. To distinguish between the relativistic symmetries and those of the particle quantum numbers (such as isospin and hypercharge), we shall call the latter transformations the "internal group." The most favored presently available candidate for an internal group of the hadrons is $SU(3)$, but the conclusions presented here are equally valid for any other group.

Two theorems pertinent to symmetry theories with intrinsic mass formulas are presented, using the transformation properties of particle states to provide proofs. The first is that the complete group of symmetry operations of such a theory must have an infinite number of generators. The second theorem shows that an intrinsic mass formula violates the reducibility of the direct-product (two-particle) states; this leads to such unphysical results as the existence of only a single amplitude to describe baryon-meson scattering.

THEOREM 1

Theorem: An intrinsic mass formula implies a group with an infinite number of generators.

The existence of an intrinsic mass formula requires that the states describing different particles in the same multiplet have different translation properties. That is, the four-momentum of a particle state depends on its internal indices. For example, in the $SU(3)$ theory the π, η, and K mesons are in the same multiplet, called the pseudoscalar meson octet. To distinguish between these particles we need an internal index α that takes on values from one to eight. If these eight mesons or any collection of particles are indeed in the same multiplet, it must be possible to determine the momentum of any of them once the momentum of one of them is known. To indicate this we write the momentum of a particle as $k(\alpha,\mathbf{q})$ where $\mathbf{q}$ is a momentum characteristic of the multiplet. As an example of a rule of determining k for α and $\mathbf{q}$, the possibility that all particles in a multiplet have the same three-momentum may be considered, in which case $k(\alpha,\mathbf{q})$ would be given by

$$k(\alpha,\mathbf{q})=[\mathbf{q},(\mathbf{q}^2+m^2(\alpha))^{1/2}], \tag{1}$$

where $m(\alpha)$ is the mass of the αth particle. There are obviously many other possible rules for determining k from α and $\mathbf{q}$. A knowledge of this rule would tell one how to compare π-nucleon and K-nucleon scattering. The function $k(\alpha,\mathbf{q})$ must satisfy the requirement that

$$k^\mu(\alpha,\mathbf{q})k_\mu(\alpha,\mathbf{q})=m^2(\alpha), \tag{2}$$

and we shall also assume that there is one coordinate system in which all the particles of a multiplet are simultaneously at rest. If the state vectors of the theory are $|k(\alpha,\mathbf{q}),\alpha,A,S,\sigma\rangle$, where A is the name of the multiplet of the internal group (octet, decuplet, singlet, etc.), S, the total spin, and σ, its projection on the z axis, we take the normalization of these states to be

$$\langle k(\alpha',\mathbf{q}),\alpha',A',S',\sigma'|k(\alpha,\mathbf{q}),\alpha,A,S,\sigma'\rangle =\delta[k(\alpha',\mathbf{q}')-k(\alpha,\mathbf{q})]\delta_{AA'}\delta_{\alpha\alpha'}\delta_{SS'}\delta_{\sigma\sigma'}. \tag{3}$$

Under a space-time translation of the coordinate axes through a distance a these states transform according to

$$U(a)|k(\alpha,\mathbf{q}),\alpha,A,S,\sigma\rangle =\exp(ik(\alpha,\mathbf{q})\cdot a)|k(\alpha,\mathbf{q}),\alpha,A,S,\sigma\rangle. \tag{4}$$

Under internal symmetry operations G the states transform according to

$$U(G)|k(\alpha,\mathbf{q}),\alpha,A,S,\sigma\rangle =\textstyle\sum_\beta J^{1/2}(\beta)D_{\beta\alpha}(G)|k(\beta,\mathbf{q}),\beta,A,S,\sigma\rangle J^{-1/2}(\alpha), \tag{5}$$

where $J(\alpha)$ is the Jacobian

$$J(\alpha)=|\partial\mathbf{k}(\alpha,\mathbf{q})/\partial\mathbf{q}|, \tag{6}$$

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).
[3] A. O. Barut, in *Proceedings of the 1964 Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1964), p. 81.

and this factor is required in order that the operation $U(G)$, defined by (5), that represents the internal-group element G, be a unitary operator with the normalization (3). The multiplet transforms invariantly under all possible products of translations, homogeneous Lorentz transformations, and internal operations, as well as under each type of operation separately. The complete set of symmetry operations is therefore the closure, under multiplication, of all products of internal- and Poincaré-type transformations. Accordingly, the infinitesimal generators of the elements of this set must be closed under commutation. We may now show explicitly that the number of generators needed to close the commutation algebra is infinite. This corresponds to the fact that product operators form an infinity of classes O_n where

$$O_n=\{\prod_{i=1}^{n} U(a_i)U(G_i)\} \tag{7}$$

and n may take on any positive integral value; a_i, G_i may be any translations and internal transformations, respectively. The class O_n always contains operators that are not contained in O_k for $k<n$.

Consider the generators of the translations. They are denoted P^μ, and their matrix representation is diagonal in the basis we have chosen:

$$\langle k,\alpha|P^\mu|k',\beta\rangle=\delta_{\alpha\beta}k^\mu(\alpha)\delta^4(k-k'),$$

where we have suppressed the irrelevant indices. Similarly, the generators of the internal operators are

$$\langle k,\alpha|H_i|k',\beta\rangle=J^{1/2}(\alpha)(h_i{}^A)_{\alpha\beta}J^{-1/2}(\beta)\delta^4(k-k'), \tag{8}$$

where $h_i{}^A$ is the matrix of the ith generator in the A representation, and H_i is the corresponding Hermitian operator in Hilbert space. The commutator is therefore

$$\langle k,\alpha|[P_\mu,H_i{}^A]|k',\beta\rangle =\delta(k-k')J^{1/2}(\alpha)(h_i{}^A)_{\alpha\beta}J^{-1/2}(\beta)[k^\mu(\alpha)-k^\mu(\beta)]. \tag{9}$$

Unless $k^\mu(\alpha)=k^\mu(\beta)$ for all α, β which are connected by the generators $h_i{}^A$ (i.e., in general for all α, β), this commutator does not vanish. Its momentum dependence, furthermore, clearly implies that it cannot be expressed as a linear combination of internal generators. In fact, we may write

$$\langle k,\alpha|[C(1)_{i,\mu}]|k',\beta\rangle=J^{1/2}(\alpha)(h_i{}^A)_{\alpha\beta}J^{-1/2}(\beta)\times[k^\mu(\alpha)-k^\mu(\beta)]\delta(k-k'). \tag{10}$$

Then the commutator [Eq. (9)] is just $C(1)_{i,\mu}$.

We may again perform a commutation, defining a $C(2)$ by

$$\langle k,\alpha|[C(2)_{i,\mu,\nu}]|k',\beta\rangle=\langle k,\alpha|[P^\nu,C(1)_{i,\mu}]|k',\beta\rangle.$$

$$\langle k,\alpha|[C(2)_{i,\mu,\nu}]|k',\beta\rangle=\langle k,\alpha|[C(1)_{i,\mu}]|k',\beta\rangle\times[k^\nu(\alpha)-k^\nu(\beta)], \tag{11}$$

which again is a new object, for its momentum dependence is higher than that of $C(1)$. The process may be continued indefinitely, and we always produce new generators $C(n)$ for $n=1, 2, \cdots$. This proves the theorem.

THEOREM 2

Theorem: In a theory with an intrinsic mass formula (for the single-particle states), the direct product of two single-particle multiplets is always irreducible under the internal group.

The internal transformation properties of the one-particle states [Eq. (3)] dictate the decomposition of the direct-product states under the internal group. Thus, one would try to form irreducible submultiplets of direct-product multiplets by summing with appropriate Clebsch-Gordan coefficients of the internal group:

$$|K,\gamma,C\rangle=\sum_{\alpha\beta} C(C,\gamma;A,\alpha,B,\beta)|k,\alpha,A\rangle\otimes|l,\beta,B\rangle. \tag{12}$$

But the translation properties of the state on the left-hand side of this equation will be incompatible with the requirements of Poincaré symmetry. According to Eq. (1), $K=k(\alpha)+l(\beta)$ must be determined by γ alone. Thus, if the coefficient $C(C,\gamma;A,\alpha,B,\beta)$ is nonzero for more than one pair of α and β, the vector K will be ill-defined, unless $k(\alpha)$ and $l(\beta)$ have special forms.

Let α_1, β_1 and α_2, β_2 be two sets of indices for which the coefficient in question is nonzero. Then, if K is to be dependent only on γ, in the rest frame of the A multiplet, we have

$$\mathbf{p}(\beta_1)=\mathbf{p}(\beta_2);\quad [\mathbf{p}^2(\beta_1)+m^2(\beta_1)]^{1/2}+m(\alpha_1) =[\mathbf{p}^2(\beta_2)+m^2(\beta_2)]^{1/2}+m(\alpha_2) \tag{13}$$

while in the rest frame of the B multiplet,

$$\mathbf{q}(\alpha_1)=\mathbf{q}(\alpha_2);\quad [\mathbf{q}^2(\alpha_1)+m^2(\alpha_1)]^{1/2}+m(\beta_1) =[\mathbf{q}^2(\alpha_2)+m^2(\alpha_2)]^{1/2}+m(\beta_2), \tag{14}$$

where we have used the fact that all particles of a multiplet are simultaneously at rest. There must be a Lorentz transformation which transforms the rest frame of the A multiplet into that of the B multiplet. Thus, there must be a direction $\hat{n}$ and an angle θ, such that

$$[\mathbf{q}^2+m^2(\alpha_1)]^{1/2}=m(\alpha_1)\cosh\theta, \tag{15a}$$

$$[\mathbf{q}^2+m^2(\alpha_2)]^{1/2}=m(\alpha_2)\cosh\theta, \tag{15b}$$

and

$$\mathbf{q}=\hat{n}m(\alpha_1)\sinh\theta, \tag{16a}$$

$$\mathbf{q}=\hat{n}m(\alpha_2)\sinh\theta, \tag{16b}$$

where $\mathbf{q}=\mathbf{q}(\alpha_1)=\mathbf{q}(\alpha_2)$, and where we have used the parametric form of the Lorentz transformation. Equations (16) may be solved to give $m(\alpha_1)=m(\alpha_2)$, which contradicts the assertion that there is a single-particle mass formula, unless the Clebsch-Gordan coefficients are nonzero for only a single pair of indices. But this is tantamount to the irreducibility of the direct-product multiplet, which was to have been proven.

CONCLUSION

A study of the physically apparent transformation properties of the single-particle states has led us to two conclusions about the structure of a group that includes both internal and Poincaré symmetries. It has first been shown that only a group with an infinite number of generators can have single-particle multiplets with several different masses. Even if one is willing to accept such a group, it is not possible to construct a physically satisfactory theory. The second theorem shows that no two-particle states are reducible.[4] This implies, for example, that in octet-octet scattering the familiar formula $8\times8=1+8+8+10+\overline{10}+27$ would no longer be valid and only a single scattering amplitude would be required to describe all 64 reactions. The present results are to be contrasted with previous theorems of a similar character.[5,6] The earlier results are obtained by making assumptions about the structure of the Lie algebra of the combined Poincaré and internal groups. These results are confined to groups with a finite number of generators. By treating the states and their transformation properties rather than the operators, we have shown that only groups with an infinity of generators can have an intrinsic mass formula but that these groups are subject to very serious objection.[7]

[4] A special case of the second theorem has been previously discussed. See N. M. Kroll, Phys. Letters 20, 531 (1966).

[5] W. D. McGlinn, Phys. Rev. Letters 12, 467 (1964).

[6] F. Coester, M. Hamermesh, and W. D. McGlinn, Phys. Rev. 135, B451 (1964).

[7] An example of this type of group has been proposed recently by J. Formánek, Nuovo Cimento 43, 741 (1966).

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Some Consequences of an Integral Representation for the Renormalization Function $Z_3(s)$*

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An integral representation for the renormalization function, $Z_3(s)=\Delta_F(s)/\Delta_F'(s)$, and hence for the composite eigenvalue conditions $Z_3(\infty)=0$, $\lim_{s\to\infty} sZ_3(s)=0$ as well, is derived. The representation allows for the possible emergence onto the physical sheet of Castillejo-Dalitz-Dyson (CDD) poles, as the coupling strength increases, by satisfying the requirements of analytic continuation in that parameter. The composite conditions are obtained in a relativistic scalar model (in the elastic approximation) and in a one-pole nonrelativistic model; in confirmation of a surmise due to Ida, these conditions are found manifestly independent of the position of the CDD poles. Applications of the representation to some limit problems in the theory of noncomposite particles, to the Zachariasen model, and to a multichannel generalization of that model are made. The extension of the representation to the nucleon is the basis for a brief numerical analysis of $Z_N(w)$ and related quantities in the one-branch approximation.

I. INTRODUCTION

In a succession of papers[1–3] dealing with the Green's-function approach to some problems in the theory of strong interactions, Ida[2] has been able to show, by paying special attention to propagator zeros, the equivalence of that approach to the ordinary N/D method.[4] Central to his demonstration is the observation that the renormalization function, defined by[5]

$$Z_3(s)\equiv\Delta_F(s)/\Delta_F'(s)\,, \tag{1}$$

is a Herglotz function. It follows that $Z_3(s)$ can be expressed as

$$Z_3(s)=1+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho(s')g^2ds'}{|\mathfrak{D}(s')|^2(s'-\mu^2)^2(s'-s-i\epsilon)}+\sum_n c_n\frac{s-\mu^2}{s_n-s}, \tag{2}$$

where the sum on the right-hand side of Eq. (2) represents the Castillejo-Dalitz-Dyson (CDD) terms[6]; the s_n's are the zeros of the meson propagator $\Delta_F'(s)$. It is assumed that the propagator $\Delta_F'(s)$ satisfies the

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[1] M. Ida, Phys. Rev. 135, B499 (1964); 136, B1767 (1964).

[2] M. Ida, Progr. Theoret. Phys. (Kyoto) 34, 92 (1965). We wish to thank Dr. Ida for a report of this work prior to publication.

[3] M. Ida, Progr. Theoret. Phys. (Kyoto) 34, 990 (1965); 35, 692 (1966).

[4] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960); G. F. Chew, in *Dispersion Relations*, edited by G. R. Screaton (Interscience Publishers, Inc., New York, 1961), p. 167.

[5] Our discussion is limited to the relativistic scalar model considered by Ida (Ref. 2): S-wave scattering (in the elastic approximation) of a scalar baryon (M) and antibaryon with a (scalar) one-meson (μ) intermediate state. For simplicity, we have taken over the notation of Ref. 2 wherever possible.

[6] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101, 453 (1956).