One way of deriving (38) is to rewrite the triple product in (25) by means of (35) and

$$\begin{aligned} \mathbf{A}_i{}^k\mathbf{B}_k{}^j &= (\mathbf{A}\cdot\mathbf{B})\delta_i{}^j + i(\mathbf{A}\times\mathbf{B})_i{}^j, \\ \mathbf{A}_k{}^j\mathbf{B}_i{}^k &= (\mathbf{A}\cdot\mathbf{B})\delta_i{}^j - i(\mathbf{A}\times\mathbf{B})_i{}^j, \end{aligned} \tag{39}$$

which are valid for arbitrary traceless $SU(2)$ tensors such as $(\mathbf{V}_k)_i{}^j$ in (35).

One can see that the isovector $\mathbf{A}$ of (38) in the limit of no pseudoscalar singlet does not tend to any one of the combinations given by (36) and (37) which satisfy the $SU(2)\times SU(2)$ algebra. However, one can see that the pion and kaon components of (38) are exactly the same as those in (36) with $b_1=0$ and $b_2=\frac{1}{4}$. Therefore, the isovector (38) satisfies the $SU(2)\times SU(2)$ algebra in the case when η is dropped.

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πN Phase Shifts and Eigenscattering*

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Small variations in eigenphase and mixing-angle behavior are correlated with distinct types of large variations in the elastic phase shift. Applications to recent results for the P_{11} and D_{13} πN phase-shift analyses are discussed.

I. INTRODUCTION

SOME recent πN phase shift analyses[1–5] have had some difficulty in reaching agreement on the behavior of the P_{11} phase shift in the 300–700-MeV pion laboratory kinetic-energy region. To some extent (see Ref. 5) this has also been true of the D_{13} phase shift. In a discussion of the results in Refs. 1–3 for the P_{11} phase shift, Dalitz and Moorhouse[6] have given plausible arguments why the detailed behavior of the phase shift in an energy region of high inelasticity (which is true of the region under consideration) is not critical in deciding the existence of a resonance. The purpose of this paper is to correlate certain *types* of *large* fluctuations in the elastic phase shift with much smaller variations in the parameters describing the eigenscattering of the system. It is then possible that these widely different sets of elastic phase shifts will appear as solutions to very similar data with comparable χ^2 .If this is so, a detailed study of the contrasting forms of these phase shifts in the light of the following discussion may provide a clue to the behavior of the possible underlying eigenparameters. We first present the relevant theory, then discuss in detail manufactured examples of several pertinent situations, and finally apply the ideas developed to a discussion of the existing solutions for the P_{11} and D_{13} partial-wave parameters.

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[3] P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, Phys. Letters **12**, 76 (1964).
[4] P. Bareyre, C. Brickman, A. V. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965).
[5] J. Cence, Phys. Letters **20**, 306 (1966).
[6] R. H. Dalitz and R. G. Moorhouse, Phys. Letters **14**, 159 (1965).

II. THE EIGENPHASE REPRESENTATION

Suppose that in the energy region where the ambiguity occurs, the scattering may be approximately described by a 2×2 symmetric unitary S-matrix diagonal in total I, J and parity. The inelastic channel need not be a two-particle or a quasi-two-particle channel.[7]

$$S=\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}.$$

We may then diagonalize S by a real orthogonal matrix and write

$$S=\begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}\begin{pmatrix} e^{2i\delta_a} & 0 \\ 0 & e^{2i\delta_b} \end{pmatrix}\begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}, \tag{1}$$

where ϵ is the mixing angle, and δ_a and δ_b are the eigenphases. The two eigenstates of S are

$$\Psi_a=\cos\epsilon\,\Psi_1+\sin\epsilon\,\Psi_2,$$
$$\Psi_b=-\sin\epsilon\,\Psi_1+\cos\epsilon\,\Psi_2.$$

From (1),

$$S_{11}=\eta_{11}e^{2i\delta_{11}}=\cos^2\epsilon\, e^{2i\delta_a}+\sin^2\epsilon\, e^{2i\delta_b}. \tag{2}$$

This equation forms the basis for the subsequent discussion.

The matrix element S_{11} given in Eq. (2) may be represented in an Argand plot (Fig. 1) as the complex

[7] In the energy region under consideration, J. Kirz, J. Schwartz, and R. D. Tripp [Phys. Rev. **130**, 2481 (1963)] have suggested that the production of $I=0$ pion pairs and $N^*(1238)$ resonances may be important. Presumably, then, the S-wave production of the former is important in P_{11} (Refs. 3 and 6) and the P-wave production of the latter in D_{13}.

sum of the two phasors

$$z_a = \cos^2\epsilon \, e^{2i\delta_a},$$
$$z_b = \sin^2\epsilon \, e^{2i\delta_b}.$$

We now provide some restriction to the range of values considered for the parameters δ_a, δ_b, and ϵ. From (2), one obtains for the absorption parameter

$$\eta_{11} = \{1 - [\sin 2\epsilon \sin(\delta_a - \delta_b)]^2\}^{1/2}. \quad (3)$$

In most of the analyses under consideration, η_{11} drops rapidly to a value between 0.1 and 0.3, then rises. From (3), one deduces that the quantity $\sin 2\epsilon \sin(\delta_a - \delta_b)$ rises rapidly to a value between 0.995 and 0.954, then falls. This behavior may be produced by various choices of the parameters, but simple arithmetic shows that *at minimum* $\eta_{11} < 0.3$, $36° < \epsilon < 54°$, $73° < (\delta_a - \delta_b) < 107°$. For purposes of illustration, we now make the simplifying but somewhat extreme choice of keeping $\delta_b = 0$, $44° < \epsilon < 46°$, and allowing reasonable variation in δ_a. This choice will produce an unusually small η_{11} at minimum (~ 0.1), but it will soon be obvious that these choices have been made for the purposes of emphasis rather than to cloud any of the essential physics.

At this point, let us consult Fig. 1, where the phasor z_b lies along the ray $\delta_b = 0$. It is easy to see that when the lengths of the phasors z_a and z_b are not too different ($\epsilon \sim 45°$), and δ_a is varying in the vicinity of 90°, the phase of the resultant S_{11} will depend critically on variations in ϵ about 45° which do not noticeably affect the character of the eigenstate δ_a.

By the use of Eq. (2), we have constructed a "pathology catalog". By this we mean a systematic plotting (Figs. 2 and 3) of most of the critical cases of possible interest. This catalog will be used in the discussion of applications. There is little trouble in reconstructing these graphs by conducting the phasors in Fig. 1 through the motions dictated by the choice of behavior of ϵ and δ_a described in the captions (remembering to keep $\delta_b = 0$). We supply some brief comments on the figures which will be useful in the discussion of applications following these comments.

Figures 2(a)–2(c)

This deals with a real resonance (δ_a going rapidly through 90°) in channel a. For $\epsilon < 45°$, δ_{11} displays a resonant behavior [Fig. 2(b)] of apparently narrower width if measured by the slope of the phase shift as it goes through 90°. For $\epsilon > 45°$, δ_{11} reaches a maximum below 45°. This is true for *any* behavior of δ_a, and can be proved easily as follows: From Eq. (2), with $\delta_b = 0$, we derive the formula used in plotting all these curves.

$$\cot 2\delta_{11} = \cot\delta_a + (\tan^2\epsilon - 1)\,\mathrm{cosec}\, 2\delta_a. \quad (4)$$

By differentiating and imposing $\tan\epsilon > 1$, we find that δ_{11} reaches a maximum in the first quadrant at

$$\delta_{11} = \tfrac{1}{2}\cot^{-1}(\tan^4\epsilon - 1)^{1/2}, \quad (5)$$

when

$$\sec^2\delta_a = 2\tan^2\epsilon / (\tan^2\epsilon - 1). \quad (6)$$

The sudden and precipitous drop in δ_{11} seen in Fig. 2(c) will probably not appear in such dramatic form in a phase-shift analysis. It may, however, be detected in such an analysis by a sudden downward shift in the real part of the elastic phase shift from a value in the region below 45°, accompanied by relatively large error bars. In such a case, the position of the break could be indicative of the energy of the resonance in the eigenscattering.[8] An application to the result of Cence[5] for the D_{13} wave will be discussed below.

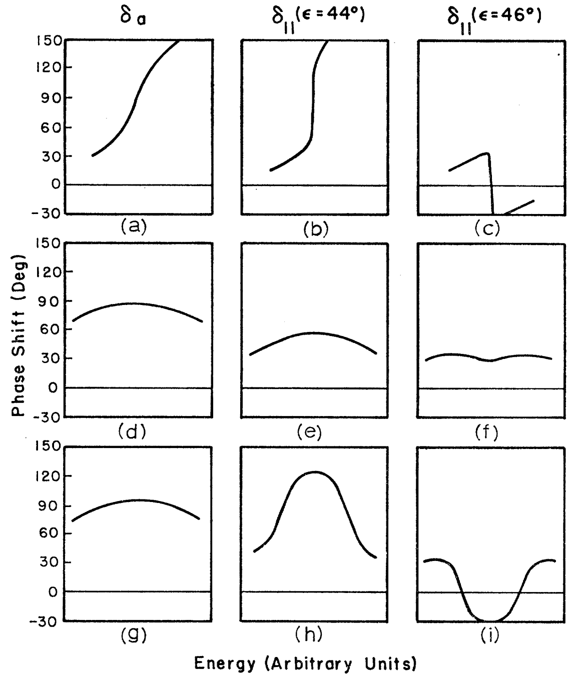

FIG. 2. Plots of dominant eigenphase and resulting elastic phase shift for various cases. Left column reading down: (a) δ_a resonating, (d) δ_a climbing to 86° and falling, (g) δ_a climbing to 94° and falling. Middle column (b), (e), (h): δ_{11} corresponding to (a), (d), (g), respectively, with $\epsilon = 44°$. Right column: same as middle column, with $\epsilon = 46°$.

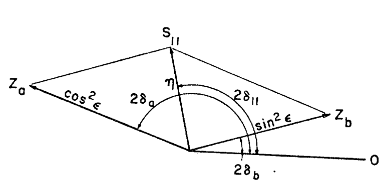

FIG. 1. Argand plot of elastic channel S_{11} as complex sum of z_a and z_b [see Eq. (2) of text]. δ_a and δ_b are the eigenphases, ϵ is the mixing angle.

[8] An example of the δ_{11} behavior in Fig. 2(c) emerging from a specific dynamical model may be seen in an article by P. R. Auvil and J. J. Brehm, Ann. Phys. (N. Y.) **34**, 505 (1965).

Figures 2(d)–2(i)

These figures show that δ_{11} depends critically on whether δ_a reaches a maximum just below or above 90°, with ϵ held constant. For example,

$$\text{in Figs. 2(d) and 2(e)} \quad (\delta_a)_{\max}=86° \rightarrow (\delta_{11})_{\max}= 57°,$$
$$\text{in Figs. 2(g) and 2(h)} \quad (\delta_a)_{\max}=94° \rightarrow (\delta_{11})_{\max}=124°,$$

for $\delta=44°$. Similar comments could be made about Figs. 2(f) and 2(i). There, incidentally, we see a further example of the theorem proved in the last paragraph concerning the behavior of δ_{11} when ϵ stays above 45°. In practice, the variation in all these cases may be somewhat milder but still considerable.

Figure 3

The behavior of δ_a in this case is identical to that in Fig. 2(g). The wild gyrations in δ_{11} are a result of critical dependence on exactly *where* ϵ goes up through 45°. The behavior of ϵ applicable to each of the figures is described in the corresponding figure caption. It is easily seen by reconstructing Fig. 3(c) that it is a general characteristic in such a case for δ_{11} to continue *past 180°*. This remark will be useful in what follows.

III. APPLICATIONS TO P_{11}

As an example we study the results for P_{11} obtained by Auvil *et al.*[3] Bareyre *et al.*,[4] and Cence.[5]

(a). The Auvil analysis has the P_{11} phase shift rising to a maximum near 100° at about 600 MeV, then dropping quite sharply, with large error bars, to about 50° at 700 MeV. The absorption parameter η in this solution (their preferred solution I) drops rapidly to a minimum value of 0.13 ± 0.07, then rises, then seems to fall slightly, then rises again. All these features are compatible with the eigenphase behavior of Fig. 2(g) or 3(a). The vacillations of the absorption parameter are those expected when δ_a goes through 90° (η reaches a minimum), δ_a goes past 90° (η begins to increase), δ_a goes back to 90° (η decreases again), and δ_a decreases from 90° (η increases again.) These are most easily followed with reference to Fig. 1. Finally, the fact that the phase shift goes past 45° is indicative that, at some point, $\epsilon<45°$, i.e., the elastic "width" is $>\frac{1}{2}$.

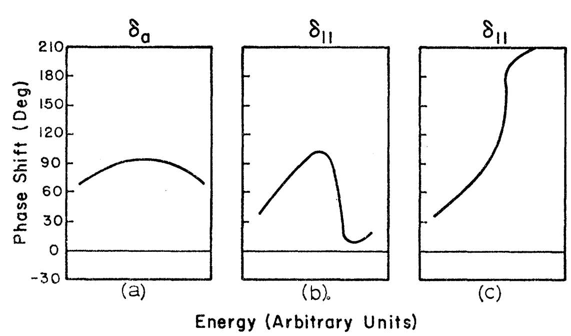

FIG. 3. (a) Same as Fig. 2(g). (b) Corresponding δ_{11}, with ϵ going down through 45° *after* δ_a goes down through 90°. (c) corresponding δ_{11}, with ϵ going down through 45° *before* δ_a goes down through 90°.

(b). The Bareyre analysis has the real part of the phase shift going through 90° at about 600 MeV, after which it shoots up *past 180°* to about 200°. The oscillations of the absorption parameter η greatly resemble those in the Auvil analysis. These two facts, especially the climb past 180°, are surprisingly compatible with Fig. 3(c) and hence with the same eigenphase behavior as in the Auvil case, except that this analysis more definitely suggests that ϵ goes through 45°.

(c). The solution of Cence has the phase shift hovering near 40°, with the absorption parameter η having a value of about 0.5 near 600 MeV, thereafter becoming quite uncertain. There are large uncertainties in this analysis after 600 MeV, but even as it stands it is compatible with the eigenphase δ_a reaching 70° with $\epsilon=44°$. The maximum eigenphase required in the Auvil Bareyre analyses is just over 90°. While this is different from 70°, a small nonzero value of δ_b could conceivably move these fits even closer to a common set of eigenparameters.

IV. APPLICATIONS TO D_{13}

Several analyses show the D_{13} phase shift going quite rapidly through 90°, accompanied by large absorption. These results are all compatible with the behavior of δ_{11} in Fig. 2(b) accompanying a resonant behavior in δ_a with a mixing angle just below 45°. They also show a sharper slope in the elastic phase than is indicated by the width of the resonant cross section; this feature agrees with the behavior illustrated in Fig. 2(b). The dissenting analysis is that of Cence. Here the phase shift reaches a maximum of about 30° *at 600* MeV, where there is a break and a falloff with comparatively large error bars. These results are reminiscent of the behavior of δ_{11} in Fig. 2(c), where δ_a goes through resonance with $\epsilon>45°$. The plunge through zero may be avoided by lifting δ_b off its zero value. This will increase the value of η at resonance [Eq. (3)], which is also a requirement of Cence's analysis.

V. CONCLUSIONS

Some reasonable conclusions to be drawn from the above analysis are that:

(i) There is an eigenchannel for P_{11} scattering which is approximately an equal mixture of πN and something else,[7] and which attains a phase shift of about 90° but does not climb much higher before dropping. This agrees with the conclusions of Ref. 6. The mixing angle is indicated to lie just below 45°. This is only in qualitative agreement with the fit of Bareyre *et al.*[9] to the total

[9] P. Bareyre, C. Bricman, G. Valladas, G. Villet, J. Bizard, and J. Seguinot, Phys. Letters 8, 137 (1964).

$I=\frac{1}{2}$ cross section which yields for the 1400-MeV "bump" a Γ_{el}/Γ_{tot} of 0.65. This number, if set equal to $\cos^2\epsilon$, gives a mixing angle of 36°. (ii) There is an eigenchannel for D_{13} scattering which is resonating. If one believes the Auvil, Bareyre, or other fits (except Cence's), the mixing angle is $<45°$. (Note that one cannot make this statement from a knowledge of η, which is symmetric in ϵ for values about 45°.) The Bareyre *et al.*[9] cross-section fit dictates $\Gamma_{el}>\Gamma_{inel}$, in agreement with the conclusion reached here.

We may throw in an additional point of interest. By following the phasor z_a in Fig. 1, with δ_b held near 0°, it is easy to understand the oft-observed fact that $\eta=|z_a+z_b|$ reaches a minimum. This occurs when δ_a either reverses after reaching a maximum, or goes through 90°.

Some of the points made in this paper will no doubt have been known to some people from their own experience, or as a result of a particular dynamical model. Our purpose has been to provide a simple guide to correlate by eye certain types of widely different phase-shift curves with much narrower variations in a possible set of underlying eigenparameters. The case of a third eigenchannel has not been considered but is not expected to qualitatively change things much if it shows nondescript behavior. The method used here has also allowed us to roughly estimate the inelasticity of a resonance or peak from the phase shift, as has been done for the two cases discussed.

In a later publication dealing with some specific models of the P_{11} enhancement, several of the points treated here in a phenomenological fashion will be seen to emerge from the dynamics.

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Neutrino Pair Production in Bound-Bound Transitions*

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The lifetime of an atomic state against decay induced by the lepton part of the weak interaction is computed. Lifetimes against such decay, in which neutrino-antineutrino pairs are emitted, are extremely long ($\sim 10^{35}-10^{40}$ sec in hydrogen). Despite a very strong dependence on atomic number ($\sim Z^{13}$ for the $2P\rightarrow 1S$ transition) the power radiated in neutrinos is very small ($\sim 10^{-5}$ erg/gm^{-1} sec^{-1} for iron). In the temperature and density range relevant for this process other neutrino processes are quite small and optical luminosity serves as the major energy loss mechanism for a star.

INTRODUCTION

THE universality of the weak interactions predicts the existence of a direct coupling among the leptons. In particular, such a direct coupling between electrons and neutrinos has interesting implications for stellar evolution. One therefore hopes to observe processes which depend on the purely lepton coupling $(\bar{e}\nu)(\bar{\nu}e)$. For this reason it is of interest to compute the lifetime of an atomic state against decay induced by such a coupling.

We may get a rough idea of how this process goes as follows. From simple dimensional considerations we have for the transition rate for a dipole-like transition (e.g., $2P-1S$):

$$\frac{1}{\tau}=\frac{2\pi}{\hbar}|H_w|^2\rho(E)\sim\left(\frac{1}{\hbar}\right)(Gm_p^2)^2(\Delta E)^2(\Delta E)^5,$$

$$\frac{1}{\tau}\sim(Gm_p^2)^2\left(\frac{\Delta E}{m_p}\right)^4\left(\frac{\Delta E r_0}{\hbar c}\right)^2\frac{\Delta E}{\hbar}\sim 10^{-2}(Z\alpha)^{12}\ \mathrm{sec}^{-1},$$

and for the power radiated per gram by the neutrinos:

$$q\sim\frac{\Delta E}{\tau}\times\frac{N_0}{A}\sim\left(\frac{\Delta E}{\tau}\right)\left(\frac{N_0}{2Z}\right)\sim 10^{17}(Z\alpha)^{13}\ \mathrm{erg\ g^{-1}\ sec^{-1}}.$$

In this paper the lifetimes of various states against such decay are calculated. To determine whether this process has astrophysical implications, the power radiated in neutrinos has been estimated. It is seen that the power radiated in such transitions is too small to play any significant role as a mechanism for energy loss during the evolution of a star.

The transitions considered were for the lowest lying states since they have the shortest lifetimes. The usual electromagnetic selection rules no longer apply and the transitions fall into two main categories: (a) The leading term in the matrix element is linear in the momentum transferred to the two neutrinos (e.g.: $2P\rightarrow 1S$). The power radiated per gram has an energy dependence of $(\Delta E)^8$ and consequently a very strong dependence on the atomic number, $\sim Z^{13}$ for these transitions. (b) The

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