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Unitarity, Causality, and Fermi Statistics*

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Conventionally, Poincaré-invariant S-matrix elements are constructed from auxiliary field operators, which transform like representations of an auxiliary group. Invariance with respect to the index transformations of this group may be extended to couple spin to internal symmetry properties in a covariant manner, as in $\tilde{U}(12)$ and $SL(6,C)$ theories. It is shown that such an index invariance of the S matrix is compatible with unitarity only if the auxiliary operators are unitary representations of the auxiliary group. It is shown further that local fields transforming as such unitary representations can be made causal only if they satisfy commutation (*not* anticommutation) relations. Thus for index-invariant theories, we establish a direct incompatability between unitarity and causality for Fermi particles.

1. INTRODUCTION

THE success of $SU(6)$ in relating the internal and spin properties of the observed particle multiplets[1] has led to an intensive search for a more general invariance principle, which would incorporate this symmetry in a covariant manner. The direct approach to this problem was shown to be impossible by what we shall call the Poincaré theorem.[2] This states that any extension of the invariance of the theory with respect to a group larger than the outer product of the Poincaré group with the internal symmetry group, which contains this outer product as a subgroup, requires the extension of the energy-momentum vector to more than four components.[3]

The construction of Poincaré invariants is complicated by the fact that under Lorentz transformations the spin suffix of a conventional Fock annihilation or creation operator undergoes a "Wigner rotation," which depends not only on the parameters of the Lorentz transformation, but also on the momentum of the state being transformed. The standard technique for dealing with this problem[4] is to introduce what we have called *auxiliary operators*, which transform as densities under Poincaré transformations, but which are (usually finite, non-unitary) representations of some *auxiliary group*, which contains the homogeneous Lorentz group as a subgroup. These auxiliary operators are specifically constructed so that under the Lorentz transformations the index, which replaces the spin label, transforms independently of the momentum. Just for this reason it is a simple matter to extend the auxiliary group by combining it with any internal symmetry group [say $SU(3)$], and so construct covariant auxiliary operators, which are appropriate to the multiplets of extended symmetries such as $SU(6)$ or $U(6)\otimes U(6)$. This is what was done for example in the theories of $SL(6,C)$[5] and $\tilde{U}(12)$[6] [or $U(6,6)$].

Having formed such operators one may easily construct S operators which are invariant with respect to both Poincaré and internal symmetry groups. By excluding explicit momentum-dependent factors, one may further restrict the S operators to be invariant with respect to the purely index transformations of the auxiliary group (which induce no change in the momentum variables). We refer to this as index invariance. In this way extra restrictions are built into the theory, which couple spin with the internal symmetry properties in a covariant manner. Since it is the auxiliary, not the Poincaré, group which is extended this does not contradict the Poincaré theorem. These results are summarized in very general terms in Sec. 2.

It has been appreciated for some time that the requirement of index invariance is liable to be in contradiction with the unitarity of the S-matrix.[7] In Sec. 3 we establish that this is always the case unless the auxiliary operators transform as unitary representations of the Lorentz group, and there is a one-to-one correspondence between the physical one-particle states and the states of the auxiliary representation. Since these unitary representations are necessarily infinite, index invariance is only consistent with unitarity of the S matrix if the

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[1] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964).

[2] L. O'Raifeartaigh, Phys. Rev. **139**, B1052 (1965); L. Michel *ibid.* **137**, B405 (1965).

[3] For examples of such theories see T. Fulton and J. Wess, Phys. Letters **14**, 57 (1965); **14**, 334 (1965); **15**, 177 (1965); J. M. Charap, P. T. Matthews, and R. Streater, Proc. Roy. Soc. (London) **A290**, 24 (1965).

[4] See S. Weinberg, Phys. Rev. **133**, B1318 (1964); G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **40**, 19 (1966), to be referred to as I. We shall use the notation of this paper and refer to it for a more detailed discussion of the material summarized in Sec. 2.

[5] W. Rühl, Nuovo Cimento **37**, 301 319 (1965); Phys. Letters **14**, 346 (1965); J. M. Charap and P. T. Matthews, Proc. Roy. Soc. (London) **A286**, 300 (1965). Nguyen Van Hien and J. A. Smorodinsky, J. Nucl. Phys. (U. S. S. R.) **3**, 543 (1965).

[6] A. Salam, R. Delbourgo and J. Strathdee. Proc. Roy. Soc. (London) **A284**, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. Letters **14**, 404 (1965); Phys. Rev. **139**, B1355 (1965).

[7] J. M. Cornwall, P. G. O. Freund, and K. T. Mahanthappa, Phys. Rev. Letters **14**, 515 (1965); R. Blanckenbecler, M. L. Goldberger, K. Johnson, and S. B. Treiman, *ibid.* **14**, 518 (1965).

physical particles belong to infinite "towers" of spin multiplets.[8]

So far we have only considered auxiliary operators in momentum space. If we now turn to the question of causality, this is best discussed in terms of local field operators, satisfying causal commutation relations. It is well known that such fields can be formed for finite (nonunitary) representations of the Lorentz group if one assumes the usual relation between spin and statistics (that is commutation relations for integer spin operators; anticommutation relations for half-odd-integer spin operators). We show in Sec. 4 that for index invariant theories, there is a very direct incompatibility for fermions between the unitarity of the S matrix and causality.

2. AUXILIARY OPERATORS

In this section we summarize the procedure for constructing Poincaré invariants. We follow the development of I,[4] but extend the discussion to include the possibility of (infinite) unitary representations of the auxiliary group.

The infinitesimal generators of the Poincaré group are

$$P_\mu, J_{\mu\nu}, \quad (\mu,\nu=0,1,2,3). \tag{2.1}$$

The single particle states belong to the irreducible representations labelled by mass and spin, with components specifying momentum and spin direction. Thus we have

$$|M^2,s,p,s_3\rangle \equiv |p,s\rangle. \tag{2.2}$$

We define

$$K_i = J_{0i}. \tag{2.3}$$

Then the boost operator is the pure Lorentz transformation which transforms rest states to moving states[9]:

$$|p,s\rangle = Ne^{-i\epsilon(p)\cdot\mathbf{K}}|M,s\rangle. \tag{2.4}$$

It is convenient to introduce Fock annihilation and creation operators,[10]

$$a^\dagger(p,s)\rangle_0 = |p,s\rangle, \quad a(p,s)\rangle_0 = 0, \tag{2.5}$$

where $\rangle_0$ is the vacuum state. Under the pure Lorentz transformation which takes p to p', the spin variable s in $a(p,s)$ transforms according to the Wigner rotation which depends not only on the parameters of the Lorentz transformation, but also on the momentum. In order to facilitate the construction of Poincaré invariants involving $a(p,s)$, one introduces an *auxiliary* group, which contains the homogeneous Lorentz group as a subgroup. If we use $|\alpha\rangle$ to denote a representation of the auxiliary group, which contains the spin s in its decomposition, we can define auxiliary operators

$$\begin{aligned} A_\alpha(p) &= \langle\alpha|e^{-i\epsilon\cdot\mathbf{K}}|\beta\rangle\langle\beta|M,s\rangle a(p,s) \\ &\equiv u_\alpha(p)^s a(p,s). \end{aligned} \tag{2.6}$$

Under a Lorentz transformation[11] $U(\eta)$, we have

$$U(\eta)A_\alpha(p)U^{-1}(\boldsymbol{\eta}) = \langle\alpha|e^{i\eta\cdot\mathbf{K}}|\beta\rangle A_\beta(p'), \tag{2.7}$$

$$\equiv S_\alpha{}^\beta A_\beta(p'). \tag{2.8}$$

Thus the advantage of the auxiliary operator over the Fock operator is that under $U(\boldsymbol{\eta})$ the spinor index transformation is decoupled from the momentum.

If we restrict the auxiliary group to be the homogeneous Lorentz group, $\mathfrak{L}$, the states $|\alpha\rangle$ are

$$|\alpha\rangle \equiv |k_0,c;\, j,m\rangle. \tag{2.9}$$

where[12] k_0 and c label an irreducible representation, and j,m (spin and spin direction) determine a particular component. The parameter k_0 is a nonnegative integer or half-integer. For finite (nonunitary) representations $|c|$ differs from k_0 by a positive integer, and[13]

$$k_0 \leqslant j \leqslant |c|-1. \tag{2.10}$$

For unitary (infinite) representations we have: either

$$\text{principal series}\begin{cases} k_0=0,\, \tfrac{1}{2},\, 1,\cdots,(j\geqslant k_0), \\ c \text{ pure imaginary}; \end{cases} \tag{2.11}$$

or

$$\text{supplementary series}\begin{cases} k_0=0, \\ 0\leqslant c<1. \end{cases} \tag{2.12}$$

In a unitary representation, the matrix $\langle\alpha|K|\beta\rangle$ is Hermitian. For a finite representation it is anti-Hermitian.

The role of the constant spinor $\langle\alpha|M,s\rangle$ appearing in (2.6) is best understood in terms of particular examples. If $s=\frac{1}{2}$, we may take for $|\alpha\rangle$ either the "dotted" or "undotted" two-component representations of $\mathfrak{L}$, which in the above notation are labeled

$$(k_0,c) = (\tfrac{1}{2}, \pm\tfrac{3}{2}). \tag{2.13}$$

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[8] C. Fronsdal, in *Proceedings of the 1965 Trieste Seminar on Elementary Particles and High Energy Physics* (International Atomic Energy Agency, Vienna, 1965), p. 655. W. Rühl, Nuovo Cimento **42**, 619 (1966); **43**, 171 (1966); **44**, 572 (1966); **44**, 659 (1966); A. Salam and J. Strathdee, Trieste Report No. IC/66/5 (unpublished). See in particular S. Coleman, Phys. Rev. **138**, B1262 (1965).

[9] The parameters $\epsilon(p)$ satisfy $\cosh|\epsilon| = p_0/m$, $\sinh|\epsilon| = |p/m|$, $\hat{\epsilon}=\hat{p}$. The states $|M,s\rangle$ are eigenstates of the little group of the rest frame, normalized such that $\langle M,s'|M,s\rangle = \delta_{ss'}$.

[10] They satisfy $[a(p,s),a^\dagger(p',s')]_\pm 2\pi\theta(p_0)\delta(p^2-m^2) = (2\pi)^4 \delta^4(p-p')\delta_{ss'}$, where $+$ denotes the commutator and $-$ the anticommutator.

[11] If the velocity of the Lorentz transformation is $\mathbf{v}$, $\cosh|\eta| = \gamma$, $\sinh|\boldsymbol{\eta}| = |v|\gamma$, $\hat{\eta}=\hat{v}$, where $\gamma=(1-v^2)^{-1/2}$.

[12] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., London, 1964); I. M. Gelfand, R. A. Minlos, Z. Ya Shapiro, *Representations of the Rotation and Lorentz Groups* (Pergamon Press, Inc., London, 1963). See also Ref. 8, p. 585.

[13] For the finite nonunitary representations it is more usual to define the $SU(2)$ subgroups with generators $\mathbf{L}=\frac{1}{2}(\mathbf{J}+i\mathbf{K})$, $\mathbf{M}=\frac{1}{2}(\mathbf{J}-i\mathbf{K})$. The corresponding labels of the representations are (l,m), where $l+m=|c|-1$, $l-m=k_0|c|/c$. For both finite and unitary representations if $k_0=0$, the sign of c is not significant and by convention $c>0$.

For either case there is a one-to-one correspondence between the physical states $|M,s\rangle$ and the auxiliary states $|\alpha\rangle$, the components of the latter being specified by $m=\pm\frac{1}{2}$. Thus we may write

$$\langle\alpha|M,s\rangle=\delta_{\alpha}{}^{s},\quad(\alpha,\,s=1,\,2).\tag{2.14}$$

The states $|M,s\rangle$ specify a representation of the little group with generators J_{ij}. Since the parity operator R commutes with J_{ij}, we may also specify the parity of $|M,s\rangle$. On the other hand, for states $|\alpha\rangle$ of $\mathfrak{L}$, we have, for both unitary and nonunitary representations,

$$R|k_0,c,j,m\rangle=\pm|k_0,\,-c,\,j,\,m\rangle.\tag{2.15}$$

Thus, unless $c=0$ (or $k_0=0$), parity can only be included if $|\alpha\rangle$ is the reducible representation which includes states with both $\pm c$. For $s=\frac{1}{2}$ this implies the Dirac representation (combination of "dotted" and "undotted" spinors). Then $|\alpha\rangle$ runs over four values, and the constant spinor for a positive parity physical particle satisfies the restriction

$$(\gamma_0)_{\alpha}{}^{\beta}\langle\beta|M,s\rangle=\langle\alpha|M,s\rangle.\tag{2.16}$$

Alternatively, one may maintain the one-to-one correspondence between the physical states $|M,s\rangle$ and the auxiliary states $|\alpha\rangle$ by doubling the number of physical states to include both parities. Then again

$$\langle\alpha|M,s\rangle=\delta_{\alpha}{}^{s},(\alpha,\,s=1,\cdots,4)\,,\tag{2.17}$$

where now s labels both spin and parity.

In the next section we are led to consider unitary representations $|\alpha\rangle$ in one-to-one correspondence with the states $|M,s\rangle$ of the physical multiplets. This requires that s also runs over an infinite "tower" of spins, as discussed by Fronsdal.[14]

As described in I, it is convenient to introduce two other types of auxiliary operator for the construction of Poincaré invariants:

$$A^{\alpha}(p)=a^{\dagger}(p,s)\langle M,s|\beta\rangle\langle\beta|e^{i\epsilon\cdot\mathbf{K}}|\alpha\rangle\,,\tag{2.18}$$

and

$$\tilde{B}_{\alpha}(p)=\langle\alpha|e^{-i\epsilon\cdot\mathbf{K}}|\beta\rangle\langle\beta|B|M,\bar{s}\rangle b^{\dagger}(p,\bar{s})\,,\quad=\tilde{v}_{\alpha}(p)^{s}b^{\dagger}(p,\bar{s})\,,\tag{2.19}$$

where $\langle M,s|B|M,s'\rangle$ is the spin flip matrix defined in I. From (2.18) we see that $A^{\alpha}(p)$ is defined to transform contravariantly to $A_{\alpha}(p)$ under Lorentz transformations. For unitary representations $|\alpha\rangle$

$$A^{\alpha}(p)=(A_{\alpha}(p))^{\dagger}.\tag{2.20}$$

For nonunitary $|\alpha\rangle$ the relation between A^{α} and $A_{\alpha}{}^{\dagger}$ depends on the particular representation. Thus for the Dirac representation

$$A^{\alpha}(p)=(A_{\beta}(p))^{\dagger}(\gamma_0)_{\beta}{}^{\alpha}\,,\tag{2.21}$$

and in general we define a matrix Γ such that[15]

$$A^{\alpha}(p)=(A_{\beta}(p))^{\dagger}(\Gamma)_{\beta}{}^{\alpha}.\tag{2.22}$$

The operator $\tilde{B}_{\alpha}(p)$, on the other hand, is a creation operator which transforms like $A_{\alpha}(p)$, under Lorentz transformations.

Since the effect of Lorentz transformations on the suffix α is independent of the momentum, it is easy to extend the auxiliary group to contain internal symmetry groups as subgroups such as $U(6,6)$ or $SL(6,C)$. In this case $|\alpha\rangle$ defines a representation of the larger auxiliary group.

In the subsequent sections we restrict the discussion to the auxiliary group being just $\mathfrak{L}$, since this is sufficient to illustrate all the necessary points, and the above extension to include internal symmetries is trivial. Of course the original motivation for introducing index invariance, discussed in the next two sections, was essentially connected with the extended theories.

3. INDEX INVARIANCE AND UNITARITY

We define index transformations to be those under which

$$A_{\alpha}(p)\rightarrow S_{\alpha}{}^{\beta}A_{\beta}(p)\,,\tag{3.1}$$

where $S_{\alpha}{}^{\beta}$ is defined in (2.8). As discussed in the introduction, a restricted class[16] of Poincaré invariant densities (scattering operators) can be defined by requiring them to be index invariant. These can be formed by saturating the indices of the appropriate auxiliary operators. The same procedure of "saturating indices" can be extended to unitary representations as has been shown by Fronsdal.[8]

We now consider under what circumstances this additional requirement of index invariance is consistent

[14] See Ref. 8. We may require the theory to be invariant with respect to $A=\{J_{\mu\nu}\}[\sim](P_{\mu}\otimes S_{\mu\nu})=(J_{\mu\nu}{}^{x}[\sim]P_{\mu})\otimes S_{\mu\nu}$, where $S_{\mu\nu}$ are the generators of an $SL(2,C)$ "spin" group distinct from $J_{\mu\nu}$, and $J_{\mu\nu}{}^{x}=J_{\mu\nu}-S_{\mu\nu}$. Then the little group in the rest frame is $J_{ij}{}^{x}\otimes S_{\mu\nu}$. One may restrict the physical rest frame states to belong to the trivial representation of $J_{ij}{}^{x}$, thus giving for $|M,s\rangle$ the unitary representations of $SL(2,C)$.

[15] For finite irreducible representations $|\alpha\rangle$, this should read $A^{\alpha}=(A_{\beta})^{\dagger}\Gamma_{\beta'}{}^{\alpha}$, since the process of Hermitian conjugation takes the operator out of the representation (for example "dotted" to "undotted"). Alternatively $|\alpha\rangle$ may be taken reducible to include both the required representations [as in the Dirac case (2.21)].

[16] A typical Poincaré invariant density in terms of auxiliary operators which transform as finite dimensional representations (using the "dotted" and "undotted" index notation) is

$$A_{\beta\cdots}{}^{\alpha\cdots}(p)B_{\dot{\lambda}\cdots}{}^{\dot{\kappa}\cdots}(q)p_{\alpha\dot{\kappa}}\cdots C^{\beta\dot{\lambda}\cdots}(r)\delta^4(p+q-r).$$

Since the momentum p is unchanged by the index transformations, index invariants must be constructed by a similar saturating of indices but in expressions which do not depend explicitly on $p_{\alpha\dot{\kappa}}$. Fronsdal shows that scalars can be constructed from unitary representations by an analogous saturation procedure provided an analytic continuation is made in the number of indices attached to the auxiliary operator, and the Hermitian conjugate is correctly defined. [See Fronsdal's equations (6.30)–(6.31) Ref. 12]. This last point is particularly important for the transcription of our argument in Sec. 4 to Fronsdal's notation.

$$|\alpha\rangle=\left|\begin{matrix}\dot{A}_1\cdots\dot{A}_N\\ B_1\cdots B_M\end{matrix}\right\rangle,\quad|M,s\rangle=|M,k_0,c,j,m\rangle,$$

$$A_{\alpha}(0)=\psi_{B_1\cdots B_M}{}^{\dot{A}_1\cdots\dot{A}_N}.$$

with the unitarity of the S matrix. Explicitly the unitarity relation expresses the imaginary part of the T operator in terms of

$$\sum_n T|n\rangle\langle n|T^\dagger, \quad (3.2)$$

where the summation is over a complete set of physical states $|n\rangle$ subject to energy conservation. As usual the states $|n\rangle$ can be expressed as an outer product of Fock creation operators on the vacuum. For consistency the expression (3.2) must be index invariant, if T is assumed to be index invariant. An index-invariant T operator has the form

$$T=\int B^\alpha(q)\cdots A_\alpha(p)f(p\cdot q;\cdots)\times\delta(p\cdots-q\cdots)d^4p\cdots d^4q\cdots \equiv\int t^\alpha A_\alpha(p)d^4p. \quad (3.3)$$

Since $|n\rangle$ is an outer product of creation operators, it is sufficient to consider the contribution of a single particle to a term in the sum (3.2). This leads to an expression of the form,

$$\sum_{p,s} t^\alpha u_\alpha(p)^s(u_\beta(p)^s)^\dagger(\Gamma^\dagger)_\beta{}^\delta t_\delta. \quad (3.4)$$

If we perform an index transformation which, of course, only affects the operators t, this goes to

$$\sum_{p,s} tS^{-1}uu^\dagger\Gamma^\dagger St. \quad (3.5)$$

For this to be index invariant we require[17]

$$\Sigma_s uu^\dagger\Gamma^\dagger=c\mathbf{1}, \quad (3.6)$$

(where c is a constant which we take to be one below). More explicitly (3.6) can be written

$$\langle\alpha|e^{-i\epsilon\cdot\mathbf{K}}|\beta\rangle\langle\beta|M,s\rangle\langle M,s|\gamma\rangle\langle\gamma|e^{i\epsilon\cdot\mathbf{K}\dagger}|\delta\rangle\times\langle\delta|\Gamma^\dagger|\lambda\rangle=\delta_\alpha{}^\lambda. \quad (3.7)$$

(a) For finite nonunitary representations

$$K=-K^\dagger, \quad (3.8)$$

and (3.7) can be written

$$H(p)OH(p)\Gamma^\dagger=\mathbf{1}, \quad (3.9)$$

where

$$H(p)=e^{-i\epsilon\cdot\mathbf{K}}=H^\dagger(p), \quad (3.10)$$

and O is the projection operator

$$O=|M,s\rangle\langle M,s|. \quad (3.11)$$

From (3.9) and the identity

$$O^2=O, \quad (3.12)$$

we get the condition

$$\Gamma^\dagger=H(p)^{-2}, \quad (3.13)$$

for all p, which is clearly impossible. Thus the requirement of index invariance for S-matrix elements is not consistent with unitarity if the auxiliary operators transform as finite representations of the auxiliary group.

(b) For unitary representations, however,

$$K=K^\dagger,\ \Gamma=1, \quad (3.14)$$

and (3.7) can now be written

$$U(p)OU(p)^\dagger=\mathbf{1}, \quad (3.15)$$

where

$$U(p)=e^{-i\epsilon\cdot\mathbf{K}}, \quad (3.16)$$

and

$$U(p)U^\dagger(p)=1. \quad (3.17)$$

The condition (3.15) is satisfied if

$$O=\mathbf{1}, \quad (3.18)$$

which implies

$$\langle\alpha|M,s\rangle=\delta_\alpha{}^s. \quad (3.19)$$

Index invariance is thus consistent with the unitarity of the S matrix provided the auxiliary operators transform as unitary representations of the auxiliary group, and there is a one-to-one correspondence between the physical states and the auxiliary states. This implies infinite "towers" of spin multiplets, all of the same mass.

4. CAUSALITY AND FERMI STATISTICS

We now consider the connection between causality, the auxiliary group, and statistics.[17a] We define causality in terms of a local field, constructed from the auxiliary operators, which is required to satisfy the causal (anti) commutation relation

$$[\psi_\alpha(x),\psi_\beta{}^\dagger(y)]_\pm=O,\ (x-y)^2<O. \quad (4.1)$$

(In (4.1), the + and − refer to commutators and anticommutators, respectively). Here (see *I*),

$$\psi_\alpha(x)=\int[A_\alpha(p)e^{-ipx}+\tilde{B}_\alpha(p)e^{ipx}]\Delta^+(p)d^4p, \quad (4.2)$$

where

$$\Delta^+(p)=(2\pi)^{-3}\theta(p_0)\delta(p^2-m^2). \quad (4.3)$$

[17] For irreducible representations $|\alpha\rangle$ this follows from Schur's lemma. For reducible representations, $\mathbf{1}$ on the right-hand side of (3.6) is replaced by a generalized γ_5, which does not alter the argument in any essential way.

[17a] *Footnote added in proof.* In this paper we discuss the causality conditions in terms of fields constructed from the auxiliary operators defined in (2.18) and (2.19). It is possible to construct more general local fields with simple Poincaré transformation properties in which the components of different spin have different mass and appear with different weight. These will be considered in a separate paper. Their existence does not alter the conclusion regarding the incompatibility of index invariance, unitarity, and Fermi statistics arrived at here.

As has been shown by Weinberg[4] (and in I) for finite representations $|\alpha\rangle$, the condition (4.1) leads to the well known connection between spin and statistics. We therefore only consider unitary representations. If we assume that the particles are either bosons or fermions, that is, that the Fock operators satisfy either commutation or anticommutation relations, we obtain

$$[\psi_\alpha(x),\psi_\beta^\dagger(y)]_\pm=\{\langle\alpha|U(p)OU^\dagger(p)|\beta\rangle e^{-ip(x-y)} \mp\langle\alpha|U(p)\tilde{O}U^\dagger(p)|\beta\rangle e^{ip(x-y)}\}\Delta^+(p)d^4p\,, \quad (4.4)$$

where O and U are defined by (3.11) and (3.16) respectively, and

$$\tilde{O}=B|M,\bar{s}\rangle\langle M,\bar{s}|B^{-1}\,, \quad = |M,\bar{s}\rangle\langle M,\bar{s}|\,. \quad (4.5)$$

Here, $\bar{s}$ in $|M,\bar{s}\rangle$ runs over the same values as s in $|M,s\rangle$, but refer to antiparticle states. We introduce the notation for the projection operators

$$P(p)\equiv U(p)OU^\dagger(p)\,, \quad (4.6)$$

$$\tilde{P}(p)\equiv U(p)\tilde{O}U^\dagger(p)\,. \quad (4.7)$$

To satisfy (4.1) it is necessary to have

$$P(p)=\tilde{P}(-p)\,,\quad \text{Bose statistics,} \quad (4.8)$$

$$P(p)=-\tilde{P}(-p)\,,\quad \text{Fermi statistics,} \quad (4.9)$$

Since $P(p)$ and $\tilde{P}(p)$ are projection operators the condition (4.8) can be satisfied for any unitary "tower", regardless of spin content.[18] On the other hand (4.9) cannot be satisfied.

5. CONCLUSIONS

We have established two theorems[19] which may be considered as extensions of the Poincaré theorem in that they also establish in rather general terms, certain limitations on theories which combine symmetries related to spin, with internal symmetries. These are:

Theorem I: The Unitarity Theorem. Invariance of the S matrix with respect to the index transformations of the auxiliary group is incompatible with the unitarity of the S matrix, unless all the auxiliary operators are unitary representations of the auxiliary group, and there is a one-to-one correspondence between the components of the auxiliary representation and the states of the corresponding physical multiplet.

Theorem II: The Causality Theorem. A local field constructed from those auxiliary operators satisfying Theorem I cannot satisfy a causal anticommutation relation.

We have stated and proved the Unitarity Theorem in terms of the unitarity of the S matrix since consistency with unitarity appears to be a minimum requirement for any satisfactory theory. It is clear that the problem arises in any situation which involves summation over spin, on or off the mass shell. In particular, Feynman propagators corresponding to nonunitary auxiliary operators also destroy index invariance. Thus the theorem applies *a fortiori* to any attempt to construct index-invariant S operators by the Feynman-Dyson technique from an index-invariant interaction Lagrangian built up from such operators.

As we have stated the Causality Theorem, it excludes the possibility of an index invariant causal field theory which includes Fermions. Thus one cannot construct a consistent theory along these lines, even within the rather general framework proposed by Weinberg,[4] and one has lost the attractive features of local fields—the substitution law and CTP invariance. Similar considerations will apply to "S-matrix theories" which introduce causality on the basis of analytic continuation.

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[18] By $P(-p)$ we mean the analytic continuation of $P(p)$ from p_μ to $-p_\mu$. In the case of finite representations, explicit expressions for the corresponding operator $H(p)OH(p)$ are well known (see, e.g., I) and the analytic continuation is trivial. For unitary representations which satisfy the unitarity theorem $O=\mathbf{1}$ and therefore $P(p)=\mathbf{1}$, and the analytic continuation is again trivial. Although the condition (4.6) is necessary in order to satisfy Bose statistics, it is not sufficient, since (4.6) implies

$$[\psi_\alpha(x),\psi_\beta^\dagger(y)]=P_\alpha{}^\beta(i\partial)\int(e^{-ip(x-y)}-e^{ip(x-y)})\Delta^+(p)d^4p,$$

where $\partial\equiv\partial/\partial x_\mu$. This will certainly vanish for $(x-y)^2<0$ if $P(p)$ is a polynomial in p but not necessarily otherwise. However, if we insist on the unitarity of the S matrix, $P(p)=1$ by (3.15), and thus the causality condition is satisfied.

[19] The content of these theorems has appeared previously in the literature in more or less precise terms. Thus it has been appreciated for some time that index invariance with finite representations was inconsistent with unitarity (Ref. 7). The authors who have worked with unitary representations (Ref. 8) appear to have taken it for granted that this would solve the problem of the unitarity of the S matrix, though we have not found any specific discussion of this point. Similarly, it is well known that the usual "proof" of the connection between spin and statistics is only valid for local fields which are finite representations of the auxiliary group. For unitary representations Zumino has made remarks which can be interpreted as implying the content of theorem II. [B. Zumino, in *Proceedings of the 1965 Trieste Seminar on Elementary Particles and High Energy Physics* (International Atomic Energy Agency, Vienna, 1965), p. 657.]