Formal Theory of n-Particle Scattering. II*

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Prescriptions are given for constructing S-matrix elements from the general theory of n-particle scattering previously developed. Some simple models based on the separable approximation are discussed and a method for including corrections to this approximation is described.

INTRODUCTION

THIS is the last of a series of papers[1,2] devoted to the formal theory of n-particle scattering. The aim of these investigations is to provide, by suitable generalization of the results obtained by Faddeev[3] and Lovelace,[4] a practical theory for many-particle scattering problems. The first step of this program is of course to solve the four-particle problem; this has been done independently by several authors.[1,5–8] The results were essentially equivalent and can be summarized as follows: For scattering from a given initial state of a four-particle system, there are *six* independent amplitudes which are solutions of coupled integral equations whose kernel is connected[2,9] and does not depend explicitly on the potentials. The six amplitudes correspond to scattering from the (arbitrary but fixed) initial state to a final state which contains two free particles and a two-particle bound state; they are called independent because once they are known all scattering amplitudes from the same initial state to an arbitrary final state (e.g., four free particles or a three-particle bound state plus one free particle, etc. can be calculated. "Connected" means that under suitable conditions[9,10] the kernel is compact[10] and the problem can thus be solved by the usual integral-equations techniques. On the other hand, the kernel depends only on the two- and three-body scattering amplitudes, but these amplitudes are to be known *off* the energy shell. This is the price for not worrying about exact potentials. Beside this generalization of the Faddeev approach there exists the n-particle scattering theory developed by Weinberg[3] who explicitly uses the potentials. In a sense both approaches may seem equivalent since the only rigorous way of determining the two-body off-shell scattering amplitudes is by means of the Lippmann-Schwinger equation which requires explicit knowledge of the exact potentials. On the other hand, the properties of off-shell amplitudes were recently discussed by Noyes[11] who was able to show that there are "model-dependent" as well as "intrinsic" contributions. So the Faddeev equations may lead to the invalidation of some two-particle interaction models but this interesting property requires further investigations. Finally, the off-shell formalism leads naturally to the separable approximation as we shall show below and is certainly easier for practical calculations.

In I, we gave an alternative solution to the four-particle problem, but still in the Faddeev spirit: we showed indeed that *four* scattering amplitudes completely determine the system. These amplitudes which correspond to scattering from a given initial state to a final state made of a three-particle bound state plus one free particle, are, in turn, the solution of coupled integral equations with a connected kernel depending only on off-shell scattering amplitudes for two-or three-particle subsystems.

Both solutions are easily generalized to the n-particle case but while the first one leads to $\frac{1}{2}n(n-1)$ coupled equations, the second one leads only to n equations. Even with this formal improvement, the gap between theory and any practical calculations is still very large. It is the purpose of the present paper to try to fill this gap by suggesting some simple models.

The different practical applications of the three-particle scattering formalism to the $N\pi\pi$,[4,12] the three-neucleon,[13,14] or the three-pion[15] systems are all based on a kind of separable approximation. This means, essentially, that each two-body scattering amplitude is supposed to be dominated by bound states (or resonances). This assumption which seems physically reasonable has been rigorously justified by Lovelace[4] who also showed the connection between this approxi-

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mation and separable potentials.[16] To avoid the purely technical difficulties associated with resonances, we shall, in the following restrict ourselves to bound states. In the three-particle kernel these bound states enter the calculation by means of their "form factors."[4,17] This concept will be generalized in Sec. IV to n-particle bound states. As shown in I, the four-particle kernel can only be solved when all the scattering amplitudes for the different two- and three-particle subsystems are known. We can thus construct two kinds of separable models, at least: one in which only the two-particle systems are supposed to be dominated by bound states and another where this bound-state dominace is supposed to hold for the three-particle as well as for the two-particle subsystems. Both cases will be briefly discussed in Sec. IV.

Section V is devoted to a description of two particular models which are inspired, respectively, by the $N\pi\pi$ system[4] and a possible quark model. In the first case, one may consider the nucleon as a bound state of a nucleon and a certain number of pions; as a first approximation the interactions between the pions may be neglected and so we are led to study a many-particle system with no interactions between some of the particles. In the second model, the nucleon is supposed to be made of three quarks; among the possible models, one may choose one in which quarks interact only via three-body potentials. It is then rather easy to construct for example, a model for the quark-nucleon scattering This will also be done in Sec. V.

In more refined calculations it is of course necessary to include corrections to one of these models or to the separable approximation. The simplest way to include these "perturbations" in the formalism is described in Sec. VI. Since it is completely out of the question, for the moment, to perform any calculation for a system made of a large number of particles, our discussions will generally be restricted to the three-and four-particle problems which are already hard enough. No calculations are presented in this paper; our aim is not to work out a particular example in detail but rather to present a set of assumptions and techniques which, in principle at least, could be applied to most of the practical problems. The reader may wonder why we did not even mention one of the keystones for any application of the many-particle scattering formalism, namely the separation of angular momentum. For three particles, this problem was solved in a democratic[18] way by Omnes[19] and his results have been extended to the general case by Pestieau.[20] A problem which seems much harder, when no *ad hoc* approximations are made, is to include spin in the general formalism. We shall simply ignore this difficulty and neglect spin.

In order to make this paper as self-contained as possible we recall briefly in Sec. II the results of I. At the same time this will fix the notation. Finally Sec. III will be concerned with the construction of the S-matrix elements. Our discussion of unitarity relations will be somewhat more detailed than in I but still remains very simple since we consider only bound-state poles (or cuts) and do not worry about the technical refinements necessary to extend the results to resonances.

What still remains as an intrinsic weak point of this formal theory of n-particle scattering is of course its nonrelativistic character. This makes the theory probably more adequate for nuclear physics than for elementary-particle physics although a relativistic version of the Faddeev equations has already been developed.[21,22]

II. NOTATION

We suppose the Hamiltonian of the n-particle system to be given by

$$H=H_0+\sum_{1\leq i<j\leq n} V_{ij}, \tag{1}$$

H_0 is the kinetic part and $V_{ij}(i\neq j)$ are the potentials between particles i and j. Although most of our considerations remain valid in the presence of many-particle potentials, we shall for simplicity, restrict ourselves in this section to Hamiltonians of the form of Eq. (1). In general, there are many possible asymptotic states. We shall use a Greek index to label any quantity (Hamiltonian, Green function . . .) corresponding to a fixed but arbitrary asymptotic state. As we shall most of the time be concerned with the four-particle problem, we list for this case the asymptotic states as well as some of the corresponding Hamiltonians:

A three-particle bound state plus one free particle; the Hamiltonians will be noted $H_i(1\leq i\leq 4)$ where the index i refers to the free particle, for example,

$$H_1\equiv H_0+V_{23}+V_{24}+V_{34}; \text{ etc.} \tag{2}$$

A two-particle bound state plus two free particles; we shall denote the corresponding Hamiltonians by H_{ij}, where i and j refer to the bound particles

$$H_{ij}=H_0+V_{ij}. \tag{3}$$

Two two-particle bound states; the three possible Hamiltonians will be noted $H_{(ij)}$ if particles i and j are bound together. So

$$H_{(12)}\equiv H_{(34)}=H_0+V_{12}+V_{34}. \tag{4}$$

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[21] V. Alessandrini and R. Omnes, Phys. Rev. **139**, B167 (1965).

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Four free particles; the corresponding Hamiltonian is of course H_0. The complete and asymptotic Green functions are defined, respectively, by

$$G(z)=(H-z)^{-1} \tag{5}$$

and

$$G_\mu(z)=(H_\mu-z)^{-1}, \tag{6}$$

where z is a complex variable. These Green functions satisfy the so-called resolvent equations:

$$\begin{aligned} G(z)-G(z')&=(z-z')G(z)G(z'), \\ G_\mu(z)-G_\mu(z')&=(z-z')G_\mu(z)G_\mu(z'), \end{aligned} \tag{7}$$

and

$$\begin{aligned} G(z)&=G_\mu(z)-G_\mu(z)\bar{V}_\mu G(z) \\ &=G_\mu(z)-G(z)\bar{V}_\mu G_\mu(z), \end{aligned} \tag{8}$$

where $\bar{V}_\mu$ is that part of the interaction potential not contained in H_μ:

$$\bar{V}_\mu \equiv H-H_\mu. \tag{9}$$

In I we introduced two kinds of transition operators

$$X_{\mu\nu}(z)=\bar{V}_\mu-\bar{V}_\mu G(z)\bar{V}_\nu, \tag{10}$$

$$Y_{\mu\nu}(z)=\bar{V}_\nu-\bar{V}_\mu G(z)\bar{V}_\nu. \tag{11}$$

With the help of Eq. (9) it is easy to verify that on the energy shell these operators are equal. For the unitarity relations, both off-shell extensions are needed as we shall see in the following section. The matrix elements of Eqs. (10) and (11) between eigenstates of H_ν and H_μ were shown in I to coincide with the usual transition amplitude for a scattering process. Substituting Eq. (8) in Eqs. (10) and (11), we obtain what we called previously the "basic equations":

$$X_{\mu\nu}(z)=\bar{V}_\mu-\bar{V}_\mu G_\sigma(z)Y_{\sigma\nu}(z)=\bar{V}_\mu-X_{\mu\sigma}(z)G_\sigma(z)\bar{V}_\nu, \tag{12}$$

$$Y_{\mu\nu}(z)=\bar{V}_\nu-\bar{V}_\mu G_\sigma(z)Y_{\sigma\nu}(z)=\bar{V}_\nu-X_{\mu\sigma}(z)G_\sigma(z)\bar{V}_\nu. \tag{13}$$

Repeated indices do not imply a summation. A key point in the construction of the formal theory for n-particle scattering is to note that in each of the basic equations the index σ may be different for each (or some) of the potentials belonging to $\bar{V}_\mu$ or $\bar{V}_\nu$: It is precisely this freedom in the choice of σ which permits one to find integral equations with connected kernels.

Graphically, we may represent[2,9] the system by n horizontal lines and an interaction between particles i and j by a wavy vertical line joining the ith and jth horizontal line. These wavy vertical lines will be called "connections."

We shall adopt the convention that a p-particle bound state can be represented by $(p-1)$ connections (which is the minimum necessary). With this convention in mind, we call a graph disconnected, once-connected, twice-connected, etc., when it contains zero, one, two, connections. The physical meaning of this terminology is of course immediate. A disconnected graph represents n free particles while, for example, a three-times connected graph may refer to a system containing either a four-particle bound state and $(n-4)$ free particles or a three-particle and a two-particle bound state or, finally, three two-particle bound states. In both methods previously developed[1,2] for solving the n-particle problem, we start from an arbitrary but fixed initial state. The first method can be described as follows:

"Once connected graphs $\rightarrow$ twice connected graphs $\rightarrow$ $---\rightarrow(n-2)$ times connected graphs $\rightarrow$ once connected graphs." (14)

Each arrow is the symbol for what we shall call a decomposition. The first arrow, for example, means that a transitiono perator—from the fixed initial state—to a final state made of a two-particle bound state and $(n-2)$ free particles is to be expressed (by a suitable choice of σ in the basic equations) in terms of transition operators to final states which are twice connected. Furthermore, the more-connected states entering the decomposition of a (less-connected) state are supposed to contain it as a subsystem and the potentials in the basic equations are grouped in such a way that they express the virtual desintegration of the more-connected state into the less-connected one. Finally the last decomposition of Eq. (14) associates to each potential V_{ij} the Green function $G_{ij}(z)=(H_0+V_{ij}-z)^{-1}$. If all these conditions are satisfied, we obtain with the help of Eq. (14) (i.e., by substitution of each decomposition in the preceding one) a set of $\frac{1}{2}n(n-1)$ coupled integral equations whose solutions are the transition operators to once connected final states. The prescriptions we have given for each decomposition are necessary to get a kernel which depends only on transition operators for subsystems and no longer on the potentials and whose second power is compact under suitable conditions on the potentials.[9,10]

To illustrate the method, let us take the four-particle problem as an example. Equation (14) then leads to the following decompositions:

$$\begin{aligned} Y_{12\mu}(z)=\bar{V}_\mu&-(V_{13}+V_{23})G_4(z)Y_{4\mu}(z) \\ &-(V_{14}+V_{24})G_3(z)Y_{3\mu}(z) \\ &-V_{34}G_{(12)}(z)Y_{(12)\mu}(z), \end{aligned} \tag{15}$$

$$\begin{aligned} Y_{4\mu}(z)=\bar{V}_\mu&-V_{14}G_{14}(z)Y_{14\mu}(z)-V_{24}G_{24}(z)Y_{24\mu}(z) \\ &-V_{34}G_{34}(z)Y_{34\mu}(z), \end{aligned} \tag{16}$$

$$\begin{aligned} Y_{(12)\mu}(z)=\bar{V}_\mu&-V_{13}G_{13}(z)Y_{13\mu}(z)-V_{14}G_{14}(z)Y_{14\mu}(z) \\ &-V_{23}G_{23}(z)Y_{23\mu}(z)-V_{24}G_{24}(z)Y_{24\mu}(z). \end{aligned} \tag{17}$$

Substituting the equations for $Y_{i\mu}(z)$ and $Y_{(ij)\mu}(z)$ in those for $Y_{ij\mu}(z)$ we obtain *six* coupled integral equations which completely solve the problem with the help of Eqs. (16) and (17) or similar ones. Typical terms in the kernel glven by this method are, for example,

$$(V_{13}+V_{23})G_4(z)V_{14}G_{14}(z)$$

or

$$V_{34}G_{(12)}(z)V_{13}G_{13}(z). \quad (18)$$

Using the second resolvent equations for the asymptotic Green's functions it is then easy[1,2] to remove the explicit dependance on the potentials.

The second method leads to a set of n coupled integral equations for transition operators to a final state made of an $(n-1)$-particle bound state and one free particle. The chain of decompositions is the following[2]:

$$(n-1)+1' \to (n-2)+2' \to (n-3)+3' \to \cdots \to 2 + (n-2)' \to 1+(n-1)' \quad (19)$$

The precise meaning of, for example, the first arrow is that a transition operator to a final state made of a $(n-1)$ particle bound state and one free particle is to be expressed in terms of transition operators to final states consisting of a $(n-2)$-particle bound state *and* a two-particle bound state. The decompositions of Eq. (19) are uniquely determined by the following conditions: For instance, in $(n-2)+2' \to (n-3)+3'$, we require the $(n-3)$ particle bound state to be a subsystem of the $(n-2)$ particle bound state and the three-particle bound state $(3')$ to contain the two-particle bound state $(2')$ as a subsystem. Furthermore, we group the potentials in such a way that they are followed by the Green's function which contains them. With these requirements and by substitution of each decomposition in the preceding one Eq. (19) then leads to a set of n coupled equations whose solutions are the transition operators—from the fixed initial state—to a final state made of $(n-1)$ bound particles and a free one. Once these operators are known the Eqs. (14) and (19) can be used to solve the problem completely. For the four-particle problem, this second method leads to a kernel which contains expressions of the form[2]

$$V_{13}G_{(13)}(z)(V_{14}+V_{34})G_2(z). \quad (20)$$

By using the same techniques as for the first method, the explicit potential dependence can easily be removed.

To end this section, we recall a possible choice of kinematical variables for the four-particle problem. Instead of using the momenta of the particles it is much simpler when the system is made of two two-particle bound states, to take the following combinations:

$$\mathbf{P}=[2(m_1+m_2+m_3+m_4)]^{-1/2}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4), \quad (21)$$

$$\mathbf{q}_{12}=[2m_1m_2(m_1+m_2)]^{-1/2}(m_2\mathbf{p}_1-m_1\mathbf{p}_2), \quad (22)$$

$$\mathbf{q}_{34}=[2m_3m_4(m_3+m_4)]^{-1/2}(m_4\mathbf{p}_3-m_3\mathbf{p}_4), \quad (23)$$

$$\mathbf{q}_{(12)}=[2(m_1+m_2)(m_3+m_4)(m_1+m_2+m_3+m_4)]^{-1/2} \times[(m_3+m_4)(\mathbf{p}_1+\mathbf{p}_2)-(m_1+m_2)(\mathbf{p}_3+\mathbf{p}_4)]. \quad (24)$$

m_i and $\mathbf{p}_i$ are the masses and momenta of the four particles. The normalizations of Eqs. (21) and (24) have been chosen so that

$$H_\sigma=P^2+q_{12}{}^2+q_{34}{}^2+q_{(12)}{}^2.$$

In the following we shall always suppose that we are in the total center-of-mass frame and so we may forget the variable $\mathbf{P}$.

III. UNITARITY CONDITIONS

As is well known[4,23] the complete Green's function of an n-particle system is operator meromorphic in the z plane except on the real axis. As a first simplification to our discussion, we shall forget the possible resonances and restrict ourselves to bound states. The following singularities then occur: (a) The n-particle bound states which give poles; (b) The n-particle branch-point at $z=0$; (c) The branch points at $z=-E_\alpha(E_\alpha>0)$ which correspond to the bound states of the different subsystems (asymptotic channel "α").

The residues at the poles (a) are determined once the bound states wave functions are known. Since we are mostly interested in scattering problems we shall suppose all these bound-state poles to be at the left of *all* the branch points. Finally as a last simplifying assumption we shall suppose that no branch points coincide.

The main problem which must then still be solved is to define correctly the discontinuities through the various cuts. Since the cuts overlap, a possible method to calculate discontinuities is to rotate them by a small angle. This technique was proved to be rigorous for the three-particle problem by Lovelace.[4] The proof requires rather complicated mathematical tools and although it can be extended without any supplementary difficulty to the n-particle problem, we prefer not to use it. We shall instead present more intuitive arguments which lead anyway to the correct result. Let us start with cut (b). Among all asymptotic Green's functions the simplest one which admits a cut of this type, is of course $G_0(z)$, i.e., the Green's function of the free system. For $G_0(z)$ the discontinuity through cut (b) is well defined since $G_0(z)$ has *only* this cut: Let $\Delta_0(z)$ be this discontinuity.[4] It is then easy, and we have done it in I, to calculate the discontinuity $D_0(z)$ of $G(z)$ induced by $\Delta_0(z)$. Using Eq. (8) (for $\mu=0$) we obtain

$$G(z)G(z')=[1-G(z)\bar{V}_0]G_0(z)G_0(z')[1-\bar{V}_0G(z')]. \quad (25)$$

Defining $\Delta_0(E)(E>0)$ by

$$G_0(E+i\epsilon)-G_0(E-i\epsilon)=2\pi i\Delta_0(E). \quad (26)$$

We are then led with the help of Eqs. (7) and (25) to the following identity:

$$D_0G(E)=2\pi i[1-G(E+i\epsilon)\bar{V}_0] \times\Delta_0(E)[1-\bar{V}_0G(E-i\epsilon)]. \quad (27)$$

Similar arguments may be applied to each of the cuts (c). Since the details are slightly different we give one more example. Let us consider, for instance, the cut

[23] T. Kato, Trans. Am. Math. Soc. **70**, 195 (1951).

(c) corresponding to the scattering states made of a bound state of particles 1 and 2 and of $(n-2)$ free particles. The simplest Green function which admits this cut is $G_{12}(z)=(H_0+V_{12}-z)^{-1}$. Let us first look at $G_{12}^0(z)$, i.e., the restriction of $G_{12}(z)$ to the two-particle Hilbert space. The singularities of this Green function are: a pole which for simplification we shall suppose unique at $z=-E_{12}(E_{12}>0)$ corresponding to a bound state and a cut starting form $z=0$ and corresponding to the scattering states. The extension $\hat{G}_{12}(z)$ of $G_{12}^0(z)$ to the Hilbert space of three-particle states then admits two cuts: one starting from $z=0$ and one starting from $z=-E_{12}$ and corresponding to physical systems made of a bound state of particles 1 and 2 while particle 3 remains free. We define $\hat{\Delta}_{12}(E)$ $(E>-E_{12})$, the discontinuity of $\hat{G}_{12}(z)$ through this second cut in the following way:

$$2\pi i\hat{\Delta}_{12}(E)=D\hat{G}_{12}(E)-2\pi iD_0\hat{G}_{12}(E). \quad (28)$$

$D_0\hat{G}_{12}(E)$ is the discontinuity through the first cut of $\hat{G}_{12}(z)$ and can easily be calculated as we just have shown; $D\hat{G}_{12}(E)$ is the total discontinuity of $\hat{G}_{12}(z)$ through the cuts (b) and (c) and it is well defined since $\hat{G}_{12}(z)$ is supposed to be known explicitly. Equation (28) can then finally be extended to the Hilbert space of n-particle states and leads to a unique definition of $\Delta_{12}(E)$, i.e., the discontinuity of $G_{12}(z)$ through the cut corresponding to a bouud state of particles 1 and 2 plus $(n-2)$ free particles. Once $\Delta_{12}(E)$ is known the same reasoning as before leads to the following expression for $D_{12}G(E)$:

$$D_{12}G(E)=2\pi i[1-G(E+i\epsilon)\bar{V}_{12}]\times\Delta_{12}(E)[1-\bar{V}_{12}G(E-i\epsilon)]. \quad (29)$$

So, in general, to calculate $D_\alpha G(E)$, the discontinuity of $G(z)$ through a cut corresponding to a bound state of the subsystem α plus some free particles, we must first calculate $\Delta_\alpha(E)$, i.e., the discontinuity of $G_\alpha(z)$ through this cut. Then, with the help of the resolvent equations, we may write

$$D_\alpha G(E)=2\pi i[1-G(E+i\epsilon)\bar{V}_\alpha]\times\Delta_\alpha(E)[1-\bar{V}_\alpha G(E-i\epsilon)]. \quad (30)$$

For $\alpha=0$ this formula reduces to Eq. (27).

As is well known $\Delta_\alpha(E)$ is the projection operator on the bound state of the subsystem α. For example, if the system is made of four particles and α corresponds to the subsystem of particles 1 and 2, it is easy to verify that

$$\langle\mathbf{q}_{12}',\mathbf{q}_{34}',\mathbf{q}_{(12)}'|\Delta_{12}(E)|\mathbf{q}_{12},\mathbf{q}_{34},\mathbf{q}_{(12)}\rangle=\delta(E+E_{12}-q_{34}{}^2-q_{(12)}{}^2)\langle\mathbf{q}_{12}'|P_{12}|\mathbf{q}_{12}\rangle\times\delta^{(3)}(\mathbf{q}_{34}'-\mathbf{q}_{34})\delta^{(3)}(\mathbf{q}_{(12)}'-\mathbf{q}_{(12)}), \quad (31)$$

where P_{12} may be written as $|\phi_{12}\rangle\langle\phi_{12}|$ with $|\phi_{12}\rangle$ the bound-state wave function. With the general formula (30) and the definitions (10) and (11), we obtain for the discontinuities of the transition operators through the cuts (b) and (c)

$$D_\alpha X_{\mu\nu}(E)=D_\alpha Y_{\mu\nu}(E)=-\bar{V}_\mu D_\alpha G(E)\bar{V}_\nu=2\pi iX_{\mu\alpha}(E+i\epsilon)\Delta_\alpha(E)Y_{\alpha\nu}(E-i\epsilon) \quad (32)$$

which is a generalized unitarity relation off the energy shell. The total discontinuity of the transition operators through the right-hand cuts is then obtained by summation over α (including $\alpha=0$) and we may write

$$X_{\mu\nu}(E+i\epsilon)-X_{\mu\nu}(E-i\epsilon)=Y_{\mu\nu}(E+i\epsilon)-Y_{\mu\nu}(E-i\epsilon]=-2\pi i\textstyle\sum_\alpha X_{\mu\alpha}(E+i\epsilon)\Delta_\alpha(E)Y_{\alpha\nu}(E-i\epsilon). \quad (33)$$

With these relations, the construction of a unitary S matrix becomes trivial. Without writing explicitly the kinematical variables, the S matrix element for the process $\alpha\rightarrow\beta$ may be defined as follows $(E=E_\alpha=E_\beta)$:

$$S_{\beta\alpha}(E)=\delta(\beta-\alpha)-2\pi i\langle\phi_\beta|X_{\beta\alpha}(E)|\phi_\alpha\rangle\delta(E_\alpha-E_\beta)=\delta(\beta-\alpha)-2\pi i\langle\phi_\beta|Y_{\beta\alpha}(E)|\phi_\alpha\rangle\delta(E_\alpha-E_\beta), \quad (34)$$

where $|\phi_\alpha\rangle$ and $\langle\phi_\beta|$ are, respectively, the initial and final bound-state wave functions. In our case, it is easier to introduce a reduced S matrix which will be noted S^R and is defined by

$$S_{\beta\alpha}(E)=S_{\beta\alpha}{}^R(E)\delta(E_\beta-E_\alpha). \quad (35)$$

With the help of Eq. (33) it is then easy to verify that

$$\textstyle\sum_\sigma S_{\beta\sigma}{}^R(E)(S^R)\dagger_{\sigma\alpha}(E)=\delta(\beta-\alpha) \quad \text{or} \quad S^R(S^R)\dagger=1.$$

Similarly Eq. (33) leads also to $(S^R)\dagger S^R=1$ which then proves the unitarity of the one-shell S matrix.

IV. SEPARABLE APPROXIMATIONS

In this section we shall restrict ourselves to the four-particle problem. Aa a first approximation to the scattering operators we may suppose that they are dominated by the four-particle bound-state contributions. This cannot generally be expected to be a good approximation but gives a very simple method to get some information on the scattering amplitudes. To simplify further the discussion let us suppose that our four-particle system admits only one bound state of energy $-E$ whose wave functions is $|\phi\rangle$. The spectral decomposition[4] of $G(z)$ then permits to write in the neighborhood of this bound state.

$$Y_{\beta\alpha}(z)\simeq\bar{V}_\beta|\phi\rangle 1/(z+E)\langle\phi|\bar{V}_\alpha. \quad (36)$$

To get the transition amplitudes we have to calculate the matrix elements of Eq. (36) between eigenstates $\langle\phi_\beta|$ and $|\phi_\alpha\rangle$ of the asymptotic Hamiltonians H_β and H_α. Let E_β and E_α be the corresponding eigenvalues. If, for example, α consists of a bound state of particles 1 and 2 with particles 3 and 4 remaining free, we have

$$E_\alpha=q_{34}{}^2+q_{(12)}{}^2-E_{12},$$

where E_{12} is the binding energy of the given bound

state. In general we may thus write the amplitudes as

$$\langle\phi_\beta|Y_{\beta\alpha}(z)|\phi_\alpha\rangle\cong\langle\phi_\beta|H-H_\beta|\phi\rangle \times 1/(z+E)\langle\phi|H-H_\alpha|\phi_\alpha\rangle \cong(E+E_\beta)\langle\phi_\beta|\phi\rangle 1/(z+E)\langle\phi|\phi_\alpha\rangle(E+E_\alpha). \quad (37)$$

So all transition amplitudes can be calculated once the "overlap functions" $\langle\phi_\beta|\phi\rangle$ are known. It is interesting to remark that these overlap functions are related to the form factors and the "potentials" introduced by Lovelace. Let us recall how he proceeds. Suppose we have a two-body system with a separable potential of the form

$$V_\alpha=|\alpha\rangle\lambda_\alpha\langle\alpha|. \quad (38)$$

Lovelace calls $|\alpha\rangle$ the form factor[4,17] of the bound state. Solving the Schrödinger equation for the potential (38) it is easy to verify that the bound-state wave function can be written as $|\phi_\alpha\rangle=G_0(z)|\alpha\rangle$ for $z\to-E_\alpha$ with E_α the binding energy. The use of the form factors introduces a great simplification in the formulation of the three-particle problem. Lovelace was able to show[4] that if each of the two-body potentials is approximated by sums of separable terms (we shall keep only one term), the three-body problem reduces to multichannel Lippmann-Schwinger equations with "potentials" defined by

$$V_{\beta\alpha}(z)=(1-\delta_{\beta\alpha})\langle\beta|G_0(z)|\alpha\rangle. \quad (39)$$

On the energy shell, i.e., for $z=E_\alpha=E_\beta=E$, the potential becomes

$$V_{\beta\alpha}(E)=(1-\delta_{\beta\alpha})\langle\beta|\phi_\alpha\rangle.$$

Let us remark further that the form factor may also be written as

$$|\alpha\rangle=(H_0+E_\alpha)|\phi_\alpha\rangle=-(H_\alpha-H_0)|\phi_\alpha\rangle. \quad (40)$$

If we suppose now to have a bound state of particles 1, 2, and 3

$$H_4|\phi_4\rangle=-E_4|\phi_4\rangle$$

which can disintegrate, for example, in a bound state $|\phi_{12}\rangle$ of particles 1 and 2, it is rather natural to define a "form factor" of the bound state $|\phi_4\rangle$ relative to the channel 12 by $-(H_4-H_{12})|\phi_4\rangle$. With this convention, the form factors $|\alpha\rangle$ of Lovelace may be called "form factors relative to the free channel"; quite in general a form factor of the bound state $|\phi_\beta\rangle$ relative to the channel α (the bound states of channel α must of course be subsystems of the bound state $|\phi_\beta\rangle$) may then be defined by the expression:

$$-(H_\beta-H_\alpha)|\phi_\beta\rangle\equiv|\beta;\alpha>.$$

Equation (37) then becomes

$$\langle\phi_\beta|Y_{\beta\alpha}(z)|\phi_\alpha\rangle\cong\langle\phi_\beta|;\beta\rangle 1/(z+E)\langle;\alpha|\phi_\alpha\rangle, \quad (41)$$

$\langle\phi_\beta|;\beta\rangle$ is then a "generalized potential" of the type (39). So, in the crudest separable approximation of the four-particle system—the extension to n particles is trivial—all scattering amplitudes can be calculated once the form factors of the four-particle bound states relative to all asymptotic channels are known. As far as the possible experimental determination of relative form factors is concerned, we shall perhaps devote a future work to this problem; anyway, in principle, they can be calculated once the bound-states wave functions are known which reduces the problem to the solving of Schrödinger equations. The method we just sketched is of course of little use for a somewhat detailed study of the scattering amplitudes. What seems a more reliable approximation is to suppose that the two-and three-particle subsystems are dominated by bound states. To simplify as much as possible we shall suppose furthermore that each subsystem is dominated by only *one* bound state. Let us indicate briefly how to proceed for both solutions of the four-particle problem. For each four-particle transition operator, we define reduced scattering amplitudes $T_{\beta\alpha}(z)$ as follows:

$$Y_{\beta\alpha}(z)=|\phi_\beta\rangle T_{\beta\alpha}(z)\langle\phi_\alpha|. \quad (42)$$

$|\phi_\beta\rangle$ and $\phi\langle_\alpha|$ are eigenstates of the asymptotic final and initial Hamiltonians, respectively. Let us look first at Eq. (16). With the help of the resolvent equations it is easy to show that

$$V_{34}G_{34}(z)=T_{34}(z)G_0(z), \quad (43)$$

$$\langle\mathbf{q}_{12}',\mathbf{q}_{34}',\mathbf{q}_{(12)}'|T_{34}(z)|\mathbf{q}_{12},\mathbf{q}_{34},\mathbf{q}_{(12)}\rangle =\delta^{(3)}(\mathbf{q}_{12}-\mathbf{q}_{12}')\delta^{(3)}(\mathbf{q}_{(12)}-\mathbf{q}_{(12)}') \times\langle\mathbf{q}_{34}'|\hat{t}_{34}(z-q_{12}^2-q_{(12)}^2)|\mathbf{q}_{34}\rangle, \quad (44)$$

where $\hat{t}(z)$ is the usual two-body scattering amplitude. Using the separable approximation for each two-body potential, we may write[4]

$$T_{34}(z)\cong|34\rangle t_{34}(z)\langle 34|, \quad (45)$$

$|34\rangle$ is now the form factor of the bound state relative to the free channel and $-t_{34}(z)$ can be interpreted as the bound-state "propagator."[4] Its simplest form is $1/(z+E_{34})$ but, as was shown by Lovelace[4] this leads to a violation of the unitarity conditions. We refer the reader to the papers of Lovelance for a full discussion of these propagators. With Eqs. (42) and (45) we rewrite Eq. (16) in the following way:

$$T_{4\mu}(z)=\langle\phi_4|\bar{V}_\mu|\phi_\mu\rangle\langle\phi_4|14\rangle t_{14}(z)T_{14\mu}(z) -\langle\phi_4|24\rangle t_{24}(z)-\langle\phi_4|34\rangle t_{34}(z)T_{34\mu}(z). \quad (46)$$

The first term can easily be estimated with the help of Eq. (38). It is a remarkable fact that Eq. (46) has exactly the structure of a Lippmann–Schwinger-type equation; beside the first one, each term consists of three parts: a scattering amplitude, a propagator and a generalized potential which is again, as in the three-particle problem, the overlap of a bound-state wave function with a form factor. Equation (17) can be treated in exactly the same way: we may either approxi-

mate directly the three-particle subsystems in a separable way or use only this approximation for the two particle systems. We shall only detail a little bit the first method. Let us take, for example, the expression

$$-(V_{13}+V_{23})G_4(z)Y_{4\mu}(z).$$

If we suppose once more that there exists only one bound-state of particles 1, 2, and 3, it can then be written as

$$-(H_4-H_{12})|\phi_4\rangle(z+E_4)^{-1}\langle\phi_4|\,Y_{4\mu}(z)$$

and its contribution to the scattering amplitude $T_{12_\mu}(z)$ will be given by

$$\langle\phi_{12}|\,4;12\rangle(z+E_4)^{-1}T_{4;\mu}(z). \tag{47}$$

The two last terms of Eq. (15) can be treated in a similar way; so, for example, the last one contributes to $T_{12;\mu}(z)$ by

$$\langle\phi_{12}|\,(12);12\rangle(z+E_{(12)}{}^{-1}T_{(12);\mu}(z). \tag{48}$$

To avoid any confusion, we recall that $E_{(12)}=q_{(12)}{}^2-E_{12}-E_{34}$ where E_{12} and E_{34} are the binding energies of the bound states which make part of the asymptotic channel (12). Equations (47) and (48) are again of the form discussed before but we have used the simplest form of the propagator. More refined expressions for these propagators can easily be constructed in analogy with the two-particle results.[4]

We showed in Sec. II that the two general methods to obtain a set of integral equations with compact kernel for the four-particle problem led to analogous manners of grouping the potentials. So all approximations used in this section can be applied without change to the second kernel. Let us remark finally that the separable approximation scheme can easily be extended to n-particle systems.

V. SPECIAL MODELS

In this section we suggest two particular models for further investigations of four-particle scattering: one where three of the four particles do not interact with each other and one where the interactions of the system are described uniquely with the help of three-body potentials. The first case is inspired by the $N\pi\pi\pi$ system and the second model can be used, for example, for the scattering of quarks[24] by nucleons or also, as an approximation to the scattering of α particles by C^{12}. We start with the second model and suppose it to have a Hamiltonian of the form:

$$H=H_0+V_1+V_2+V_3+V_4. \tag{49}$$

V_1, for example, is a three-body potential which describes the interactions of the subsystem made of particles 2, 3, and 4. As asymptotic states we have only two possibilities: either four free particles or a three-particle bound state plus one free particle. The asymptotic Hamiltonians are H_0 and $H_i=H_0+V_i$. We write the basic equations in the following form:

$$Y_{ji}(z)=\bar{V}_i-\bar{V}_jG_k(z)Y_{ki}(z), \tag{50}$$

where $G_k(z)=(H_k-z)^{-1}$. We choose the index k in such a way that each potential is followed by the Green function which contains it. More explicitly, we have, for example

$$Y_{li(z)}=V_i-V_2G_2(z)Y_{2i}(z)-V_3G_3(z)Y_{3i}(z)-V_4G_4(z)Y_{4i}(z). \tag{51}$$

Before going on, let us first look at the three-particle problem which has for Hamiltonian $H_4=H_0+V_4$. With $G_0(z)=(H_0-z)^{-1}$ and $G_4(z)=(H_4-z)^{-1}$ the free and complete Green functions, it is trivial to repeat the usual formal theory of two-particle scattering and to calculate the transition operator for the only possible transition, namely "3 free particles" $\rightarrow$ 3 free particles." Denoting the corresponding operator by $T_4(z)$, we are led by the basic equations to the following expression:

$$T_4(z)=V_4-V_4G_4(z)V_4. \tag{52}$$

With the help of the second resolvent equations it is easy to verify that

$$G_4(z)V_4=G_0(z)T_4(z);\quad V_4G_4(z)=T_4(z)G_0(z). \tag{53}$$

Substituting Eq. (53) in Eq. (52), we obtain

$$T_4(z)=V_4-V_4G_0(z)T_4(z)$$

which is of course the Lippmann-Schwinger equation, but the important point is that for a three-body potential (with suitable conditions) the kernel is compact and the equation can thus be solved with the usual techniques. If we now substitute Eq. (53) in Eq. (51) we obtain for our particular model a kernel which is excatly the four-body analog of the Fadeev kernel.[3] Introducing furthermore the separable approximation for these three-body potentials, i.e., $V_4=\sum_p|4p\rangle\lambda_{4p}\langle 4p|$ where again $|4p\rangle$ is the form factor (relative to the free channel) of the pth bound state. With this potential, the solution[4] of Eq. (52) may be written as

$$T_4(z)=\sum_p|4p\rangle t_{4p}(z)\langle 4p|, \tag{54}$$

where $t_{4p}(z)$ is defined by $t_{4p}(z)=[1/\lambda_{4p}+r_{4p}(z)]^{-1}$ with $r_{4p}(z)=\langle 4p|G_0(z)|4p\rangle$. Introducing Eq. (54) in Eq. (51), it is possible (and easy) to repeat for this particular four-particle problem the reasoning of Lovelace[4] for the three-particle case and to obtain finally (by introducing the same potentials and symmetrizing the equations in the same way as in Ref. 4) for the four-particle transition operators multichannel Lippmann-Schwinger equations. This result can easily be generalized as follows:

"If the interactions of an n-particle system are described with $(n-1)$-body potentials which may be approximated separably, then the n-particle transition operators are solutions of multichannel two-body Lippmann-Schwinger equations."

[24] M. Gell-Mann, Phys. Letters 8, 214 (1964).

This proposition shows clearly how simple an n-particle problem may become by using only many-body potentials; we believe that this model could be very useful as a first approximation for a lot of processes. For example, in the reactions $C^{12}+\alpha \rightarrow C^{12}+\alpha$ and $C^{12}+\alpha \rightarrow C^{12*}+\alpha$ it would not be unreasonable to approximate the interactions by three-body potentials dominated by C^{12} and C^{12*}. Anyway it should be worthwhile to try the approximation for these particular reactions and we intend to do it in a future work. The first model is, in our opinion, perhaps less interesting but since it reduces the four-particular problem to uncoupled integral equations it may certainly be worthwhile to use it as a first approximation. Small corrections to this approximation can be calculated with the help of the very simple perturbation method which will be developed in the next section. We suppose thus to have a four-particle system described by the following Hamiltonian:

$$H=H_0+V_{12}+V_{13}+V_{14}. \tag{55}$$

So particles 2, 3, and 4 interact only with particle 1 and not with each other. The Green's function is given as usual by $G(z)=(H-z)^{-1}$. The possible asymptotic states are: (a) four free particles; the Hamiltonian is H_0 and the Green function $G_0(z)=(H_0-z)^{-1}$; (b) three-particle bound states plus one free particle; the three possible Hamiltonians will be noted H_i and the Green functions $G_i(z)=(H_i-z)^{-1}(2\leqslant i\leqslant 4)$. More explicitly $H_2=H_0+V_{13}+V_{14}$; $H_3=H_0+V_{12}+V_{14}$; $H_4=H_0+V_{12}+V_{13}$; (c) two-particle bound states plus two free particles; there are again three possibilities. The Hamiltonians are $H_{ij}=H_0+V_{ij}$ and the Green functions $G_{ij}(z)=(H_{ij}-z)^{-1}$.

Experimentally, only channels (b) are useful as initial states. The basic equations are

$$Y_{\alpha i}(z)=\bar{V}_i-\bar{V}_\alpha G_\sigma(z)Y_{\sigma i}(z).$$

To be quite definite, we shall take as initial state an eigenstate of H_4, i.e., a bound state of particles 1, 2, and 3 with particle 4 remaining free. From this initial state seven types of transitions are possible. We have as basic equations, for example,

$$\begin{aligned} Y_{24}(z)&=V_{14}-V_{12}G_\sigma(z)Y_{\sigma 4}(z),\\ Y_{14;\,4}(z)&=V_{14}-(V_{12}+V_{13})G_\sigma(z)Y_{\sigma 4}(z),\\ Y_{04}(z)&=V_{14}-(V_{12}+V_{13}+V_{14})G_\sigma(z)Y_{\sigma 4}(z). \end{aligned}$$

Using the freedom of choice of σ, we rewrite these equasions in the following way:

$$Y_{24}(z)=V_{14}-V_{12}G_3(z)Y_{34}(z), \tag{56a}$$

$$Y_{34}(z)=V_{14}-V_{13}G_4(z)Y_{44}(z), \tag{56b}$$

$$Y_{44}(z)=V_{14}-V_{14}G_2(z)Y_{34}(z), \tag{56c}$$

$$Y_{14;4}(z)=V_{14}-V_{12}G_3(z)Y_{34}(z)-V_{13}G_2(z)Y_{24}(z), \tag{57}$$

$$Y_{04}(z)=V_{14}-V_{12}G_{12}(z)Y_{12;4}(z)-V_{13}G_{13}(z)Y_{13;4}(z)-V_{14}G_{14}(z)Y_{14;4}(z). \tag{58}$$

These equations show that the problem is reduced to solve Eqs. (56). Indeed, once the $Y_{i4}(z)$ are known, we may be Eq. (57) or analogous equations calculate all $Y_{ij;4}(z)$ and then with Eq. (58) $Y_{04}(z)$. Except for the independent terms, it is easy to suppress the explicit potential dependance. We may use either the general method which was detailed in I or proceed as follows: we define

$$V_{12}G_4(z)=X_{12}(z)G_0(z) \quad \text{and} \quad V_{13}G_4=X_{13}(z)G_0(z).$$

The second resolvent equations lead then to

$$X_{12}(z)=T_{12}(z)-T_{12}(z)G_0(z)X_{13}(z), \tag{59}$$

$$X_{13}(z)=T_{13}(z)-T_{13}(z)G_0(z)X_{12}(z). \tag{60}$$

Substituting Eq. (60) in Eq. (59) we get for $X_{12}(z)$ an integral equation with a compact kernel. The interesting property of the model is that Eqs. (56) are easily decoupled: indeed, substitution of Eq. (56b) and (56c) in Eq. (56a) gives

$$\begin{aligned} Y_{24}(z)=V_{14}&-V_{12}G_3(z)V_{14}+V_{12}G_3(z)V_{13}G_4(z)V_{14}\\ &-V_{12}G_3(z)V_{13}G_4(z)V_{14}G_2(z)Y_{24}(z). \end{aligned} \tag{61}$$

It is still possible to simplify the independent term but this does not matter much since we shall not try here to solve Eq. (61) explicitly. Knowing $Y_{24}(z)$, we have solved the problem since Eq. (56c) gives $Y_{44}(z)$ and Eq. (56b) $Y_{34}(z)$. The application of this model to the static $N\pi\pi\pi$ system can be done without any difficulty and leads to even simpler equations since we have three identical particles. The simplifications introduced by identical particles have been discussed in detail by Lovelace[4] and Pestieau[25] and their reasoning can easily be extended to the n-particle problem.

VI. PERTURBATIONS

In this section we develop the simplest possible method to evaluate corrections to the Fadeev-Lovelace equations for small perturbations. Although the method would be valid for n-particle problems we shall restrict ourselves to three particles since this case is already hard enough for practical calculations. Essentially, we may consider three types of perturbations induced, respectively, by a three-body potential, a two-body potential in a channel where the interaction was initially neglected and finally a two-body potential which is added to a separably approximated channel. Let us develop our method for the first case. We suppose the Hamiltonian to be given by

$$H=H_0+V_1+V_2+V_3+V_{123}. \tag{62}$$

$V_1=V_{23}$ is the potential between particles 2 and 3 and V_{123} is the preturbation. The various asymptotic channels together with their Hamiltonians and Green functions have been defined by Lovelace[4] and we shall

[25] J. Pestieau, Mémoire de licence, Louvain, 1965 (unpublished).

keep his notation. We write the basic equations in the following form:

$$Y_{\beta\alpha}(z)=\bar{V}_\alpha'-\bar{V}_\beta' G_\sigma(z) Y_{\sigma\alpha}(z)+V_{123} - V_{123} G_\sigma(z) Y_{\sigma\alpha}(z). \quad (63)$$

We have explicitly separated the three-body part and hence $\bar{V}_\alpha'$ is that part of the interaction which does not belong to H_α and is made only of two-body potentials. The basis of our perturbation method is the following construction: We introduce an additional channel which may be called the perturbation channel and which corresponds to the situation where only the perturbation is acting. In the present case, this channel is characterized by its Hamiltonian $H_{123}=H_0+V_{123}$ and Green function $G_{123}(z)=(H_{123}-z)^{-1}$. The essential point is now that the free index σ in the basic equations needs not to be restricted to observable channels but may also refer to the perturbation channel. So we rewrite Eq. (63) as follows:

$$Y_{\beta\alpha}(z)=\bar{V}_\alpha'-\bar{V}_\beta' G_\sigma(z) Y_{\sigma\alpha}(z)+V_{123} - V_{123} G_{123}(z) Y_{123;\alpha}(z). \quad (64)$$

The problem is now reduced to finding an expression for transitions to the perturbation channel in terms of the physical transitions. The answer is once more given by the basic equations: indeed, we may write

$$Y_{123;\alpha}(z)=\bar{V}_\alpha'-\sum_{\sigma=1}^{3} T_\sigma(z) G_0(z) Y_{\sigma\alpha}(z). \quad (65)$$

It is then evident how to proceed: We start with the unperturbed problem which is supposed solved:

$$Y_{\beta\alpha}^{(0)}(z)=\bar{V}_\alpha'-\bar{V}_\beta' G_\sigma(z) Y_{\sigma\alpha}^{(0)}(z).$$

With the help of the transition operators $Y_{\beta\alpha}^{(0)}(z)$, we define by Eq. (65)

$$Y_{123;\alpha}^{(1)}(z)=\bar{V}_\alpha'-\sum_{\sigma=1}^{3} T_\sigma(z) G_0(z) Y_{\sigma\alpha}^{(0)}(z).$$

Substituting this equation in Eq. (64), we get the first approximation

$$Y_{\beta\alpha}^{(1)}(z)=\bar{V}_\alpha'-\bar{V}_\beta'-G_\sigma(z) Y_{\sigma\alpha}^{(0)}(z)+V_{123} - V_{123} G_{123}(z) Y_{123;\alpha}^{(1)}(z).$$

In general, we get for the nth approximation

$$Y_{123;\alpha}^{(n)}(z)=\bar{V}_\alpha'-\sum_{\sigma=1}^{3} T_\sigma(z) G_0(z) Y_{\sigma\alpha}^{(n-1)}(z),$$

$$Y_{\beta\alpha}^{(n)}(z)=\bar{V}_\alpha'-\bar{V}_\beta' G_\sigma(z) Y_{\sigma\alpha}^{(n-1)}(z)+V_{123} - V_{123} G_{123}(z) Y_{123;\alpha}^{(n)}(z).$$

A problem which we have not solved is the convergence of this method. As far as perturbations induced by two-body potentials are concerned we may repeat without any change the preceding construction. We introduce again a perturbation channel and with the help of the basic equations we express transitions to this channel in terms of the physical transitions. Let us take for example the $N\pi\pi$ system. In a first approximation we may neglect the interaction among the π's and then introduce as a perturbation a π-π interaction which can be assumed to be dominated by the ρ. In this case the perturbation channel is observable ($N\rho$). We may also try to estimate electromagnetic corrections—this gives an interesting model for calculating the n-p mass difference —but these corrections do not lead to an observable perturbation channel.

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