Self-Consistent Determination of Coupling Shifts in Broken $SU(3)$. II*

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In a recent paper, we considered the possibility of dynamical enhancement of $SU(3)$ symmetry breaking in baryon couplings. It was found that certain patterns of symmetry breaking are enhanced and tend to dominate; the results were presented and compared with experiment. In the present companion paper, we explain in detail the methods by which these conclusions were obtained and give a more complete summery of the numerical results.

I. INTRODUCTION

A DETAILED theoretical study has been made by us of $SU(3)$ symmetry breaking in the couplings $g_{BB\Pi}$ and $g_{\Delta B\Pi}$ which connect the $J^P=\frac{1}{2}^+$ baryon octet B, the $\frac{3}{2}^+$ decimet Δ, and the 0^- meson octet Π. The idea was to look for dynamical enhancements in symmetry breaking, the enhancements being associated with instabilities or near-instabilities in the symmetric theory. The main results of this study, and comparisons with experiment, were presented in a recent paper.[1] In the present companion paper, we wish to explain in detail the methods by which these conclusions were obtained and give a more complete summary of the numerical results, with explicit statements where possible of the uncertainties in the model used.

The physical parameters brought into play by $SU(3)$ symmetry breaking include mass shifts δM_i and coupling shifts δg_i. In a bootstrap theory, these depend on other mass and coupling shifts, as well as "driving terms," which include such things as photon exchange (for electromagnetic shifts) and higher order terms. One obtains equations of the form

$$\delta M=A^{MM}\delta M+A^{Mg}\delta g+D^M, \tag{1.1}$$

$$\delta g=A^{gM}\delta M+A^{gg}\delta g+D^g, \tag{1.2}$$

where it is understood that there are many kinds of δM and δg, so that the terms such as A^{MM} are matrices. Now in a previous paper[2] it was argued on dynamical grounds that A^{Mg} is small and can be approximated by zero, leaving (1.1) and (1.2) with solutions of the form

$$\delta M=(1-A^{MM})^{-1}D^M, \tag{1.3}$$

$$\delta g=(1-A^{gg})^{-1}(A^{gM}\delta M+D^g). \tag{1.4}$$

The search for dynamical enhancements in symmetry breaking thus becomes a search for eigenvalues of A^{MM} and A^{gg} near one, which from Eqs. (1.1) and (1.2) are seen to represent nearly-self-supporting instabilities of the dynamical equations. Of course, the identification of the near-instabilities does not provide a complete specification of δM and δg, which also depend on the harder-to-calculate driving terms. Unless the driving terms happen to be nearly orthogonal to the eigenvector corresponding to the instability, however, the pattern of symmetry breaking will tend to follow the instabilities.

The present paper, then, is mainly concerned with the calculation of matrix elements of A^{gg} and A^{gM} that affect $g_{BB\Pi}$ and $g_{\Delta B\Pi}$, and with the eigenvalues of A^{gg}. These complement the previous study[2] of A^{MM} which gave a unique instability followed by δM_B and δM_Δ.

The paper is organized as follows. First, we would like to call attention to two Appendices, which relate to important questions underlying our whole approach. Appendix A deals with the convergence of our dispersion relations. It is shown that the dispersion integrals representing first-order perturbations converge faster than the dispersion integrals representing strong interactions, by one power of the energy W. We believe that this decreased sensitivity to contributions from large W, which are poorly known in practice, is the basic reason why bootstrap calculations of perturbations on the strong interactions[1–4] have achieved better quantitative results than ordinary strong-interaction bootstrap calculations. Appendix B deals with the choice of denominator function. Reasons are given for preferring our choice of denominator function to that recently advocated by Shaw and Wong.[5]

Next we turn to the body of the paper, dealing specifically with coupling shifts. In Sec. II the $SU(3)$-symmetric reciprocal-bootstrap model of B and Δ, which we use a starting point for the study of $SU(3)$-breaking perturbations, is reviewed. In Sec. III the possible types of coupling shift are listed, and the dispersion relations used to calculate elements of the A matrix are written down.

The explicit method for calculating A^{gg} is described in Sec. IV. This is the heart of the paper. A^{gg} splits into a simple "dynamical factor" and a more complicated "group-theory factor." If we represent the symmetry violation by a "spurion" S_σ, the group-theory factor

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[2] R. Dashen and S. Frautschi, Phys. Rev. 137, B1331 (1965).
[3] R. Dashen, S. Frautschi, and D. Sharp, Phys. Rev. Letters 13, 777 (1964).
[4] R. Dashen and S. Frautschi, Phys. Rev. 137, B1318 (1965).
[5] G. Shaw and D. Wong, Phys. Rev. 147, 1028 (1966).

TABLE I. Reciprocal bootstrap model of B and Δ. In $\mathbf{T}^{1/2+}$, the first row and column refer to 8_s, the second row and column to 8_a.

Diagram	Contribution to $\mathbf{T}^{1/2+}$	Contribution to $\mathbf{T}_{10,10}{}^{3/2+}$
B pole in direct channel of ΠB scattering	$-\dfrac{G^2}{W-M^B}\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}$	0
Δ pole in direct channel	0	$-\dfrac{G^{*2}}{W-M^\Delta}$
B exchange pole	$\dfrac{G^2}{10(W-M^B)}\begin{pmatrix}\cos^2\theta+(5/3)\sin^2\theta & 0\\ 0 & (5/3)(\cos^2\theta-\sin^2\theta)\end{pmatrix}$	$\dfrac{4\cos^2\theta[1-(\sqrt{5})\tan\theta]G^2}{15(W-M^B)}$
Δ exchange pole	$\dfrac{G^{*2}}{3(W-2M^B+M^\Delta)}\begin{pmatrix}2 & \sqrt{5}\\ \sqrt{5} & 0\end{pmatrix}$	$\dfrac{G^{*2}}{12(W-2M^B+M^\Delta)}$

involves the overlap between the reaction $\Pi_i+B_j\to\Pi_k+B_l+S_\sigma$ proceeding via the coupling to a particular intermediate state in the direct channel, and the reaction $\bar{\Pi}_k+B_j\to\bar{\Pi}_i+B_l+S_\sigma$ proceeding via the coupling to a particular intermediate state in the crossed channel. Mathematically, the overlap between two different ways of combining five objects (four particles and one spurion) into a singlet is, apart from normalizations and phases, a $9j$ symbol. We have worked out the appropriate expressions and had them evaluated by computer.

We proceed with a discussion of our treatment of the consistency (sometimes called "vertex symmetry"[6,7]) between $BB\Pi$ couplings in the direct and exchange channels in Sec. V. The resulting eigenvalues and eigenvectors of A^{gg} are given in Sec. VI. The same methods permit explicit evaluation of A^{Mg} in Sec. VII, and it is verified that this part of the A matrix is indeed very small, as had been argued earlier.[2] Section VIII deals with the evaluation of A^{gM}. From Eq. (1.4), we see that this allows us to determine which eigenvalues of A^{gg} are most strongly driven by the dominant mass shift δM. Under the assumption that the term $A^{gM}\delta M$, containing the already enhanced mass shift, dominates D^g in (1.4), one then finds that the eigenvalue of A^{gg} lying nearest unity is strongly favored over all other eigenvalues.

Results for the strong $BB\Pi$ and $\Delta B\Pi$ coupling shifts are presented in Sec. IX. Readers who are interested only in the answers rather than in methods of calculation may proceed immediately to Tables XXI and XXII of this section. Section X contains the electromagnetic coupling shifts; the shifts in $BB\Pi$ couplings are conveniently tabulated in Table XXIII. The much more complicated weak nonleptonic couplings are treated in Sec. XI. Here couplings of either charge conjugation and parity are considered. We show that the predictions on parity-violating couplings,[3] which agree particularly well[1] with experiment, are also on especially strong theoretical ground: they satisfy vertex symmetry exactly and are independent of the choice of denominator function in our model.

Section XII contains a comparison of our method with the calculation of Wali and Warnock[8] and with tadpole theory.[9] Finally, in Sec. XIII, the possibility of CP violation is considered.

We do not provide much comparison with experiment in the present paper; for such comparisons and for a bird's-eye view of the results, the reader is referred to our earlier paper.[1]

II. $SU(3)$-SYMMETRIC MODEL

In this section, we review the $SU(3)$-symmetric reciprocal-bootstrap model[2,10] for B and Δ, as a preliminary to the study of perturbations on the model.

The $SU(3)$-symmetric reciprocal-bootstrap model for B and Δ is essentially an $SU(3)$ generalization of the Chew-Low model. One considers pseudoscalar meson-baryon scattering, with B and Δ poles appearing in the direct channel, and B and Δ exchange in the crossed channels. As an approximation, only the nearby "short cuts" from B and Δ exchange are kept in the partial-wave amplitudes. The short cuts are further approximated by "pseudopoles."

We shall define the scattering amplitude for $\Pi B\to\Pi B$ in the $P_{3/2}$ **10** channel by

$$T_{10,10}{}^{3/2+}(W)=M^2(e^{2i\eta_{10}}-1)/2iq^3,\qquad(2.1)$$

where, as usual, W is the center-of-mass energy and q is the center-of-mass momentum. We take M equal to 1 BeV; the factor M^2 is included to make the residues of poles in the amplitude dimensionless.[11] The amplitude

[6] R. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964); K. Lin and R. Cutkosky, *ibid.* **140**, B205 (1965).

[7] F. Ernst, K. Wali, and R. Warnock, Phys. Rev. **141**, 1354 (1966).

[8] K. Wali and R. Warnock, Phys. Rev. **135**, B1358 (1964).

[9] S. Coleman and S. Glashow, Phys. Rev. **134**, B671 (1964).

[10] R. Dashen, Phys. Letters **11**, 89 (1964).

[11] In addition to M^2, (2.1) differs slightly from Eq. (5.1) of Ref. 2 in the choice of kinematic factors. Equation (5.1) avoided some distant kinematic singularities, which are, however, of no importance in an essentially static model such as we are using. The present choice corresponds to the static crossing matrix used in Table I. Actually, the static crossing matrix was also employed in Ref. 2, so we effectively took (2.1) there as well.

for $\Pi B \to \Pi B$ in the $P_{1/2}$ **8** channels is similarly defined by

$$\mathbf{T}^{1/2^+}=(M^2/2iq^3)(\mathbf{S}-\mathbf{1})\,, \qquad (2.2)$$

where, in this case, $\mathbf{T}$ is a 2×2 matrix connecting the channels $\mathbf{8}_s$ and $\mathbf{8}_a$.

The various pole terms in the $B\Pi$ reciprocal bootstrap are listed in Table I. Here, the angle θ is related to the usual F/D ratio λ by[12]

$$\lambda=-(\sqrt{5}/3)\tan\theta. \qquad (2.3)$$

We take θ in the range $\theta\approx -25°$ to $-45°$, corresponding to the value $\lambda\approx\frac{1}{3}$ to $\frac{2}{3}$ which is indicated by several experimental and theoretical arguments.[13–15] G^2 is related to the usual πNN coupling $f_{\pi NN}{}^2\approx 0.08$ as follows:

$$G^2=(20/3)[\cos\theta-(\sqrt{5}/3)\sin\theta]^{-2}(M/M^{\pi})^2 f_{\pi NN}{}^2. \quad (2.4)$$

The reciprocal bootstrap gives

$$G^{*2}=(16/55)\cos^2\theta[1-(\sqrt{5})\tan\theta]G^2\,, \qquad (2.5)$$

which is consistent with the experimental ratio of $N^*N\pi$ and $NN\pi$ couplings.

The residue matrix of the direct-channel baryon pole may be diagonalized by passing from the octet states $|8_s\rangle$ and $|8_a\rangle$ to $|8_\theta\rangle$ and $|8_{\theta*}\rangle$, defined by

$$|8_\theta\rangle=\cos\theta|8_s\rangle+\sin\theta|8_a\rangle\,, \qquad (2.6)$$

$$|8_{\theta*}\rangle=-\sin\theta|8_s\rangle+\cos\theta|8_a\rangle. \qquad (2.7)$$

In the new representation, the direct-channel baryon pole has the form

$$\frac{-G^2}{W-M^B}\begin{array}{c} \begin{array}{cc} 8_\theta & 8_{\theta*} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}. \qquad (2.8)$$

We shall use the 8_θ and $8_{\theta*}$ representations in our study of perturbations.

While it is convenient to make calculations in terms of definite $SU(3)$ representations and residues of poles, we will also wish to express the results in terms of couplings among particles. In the $SU(3)$-symmetric case, the appropriate coupling for $\Pi_i+B_j\to B_k$ is

$$G_{kj}{}^i=G\left[\cos\theta\begin{pmatrix} 8 & 8 & 8_s \\ i & j & k \end{pmatrix}+\sin\theta\begin{pmatrix} 8 & 8 & 8_a \\ i & j & k \end{pmatrix}\right]$$
$$\equiv G\begin{pmatrix} 8 & 8 & 8_\theta \\ i & j & k \end{pmatrix}, \qquad (2.9)$$

where the quantities in brackets are Clebsch-Gordan coefficients as defined by de Swart.[16] Similarly, the $SU(3)$-symmetric coupling for $\Pi_i+B_j\to\Delta_k$ is

$$G_{jk}{}^{*i}=G^*\begin{pmatrix} 8 & 8 & 10 \\ i & j & k \end{pmatrix}. \qquad (2.10)$$

As we have seen, the input parameters of this model are the average masses of the B and Δ supermultiplets, the F/D ratio, and the strong $\Pi BB/\Pi B\Delta$ coupling ratio. The first two quantities are taken from experiment, while the latter two ratios can be taken from the reciprocal-bootstrap theory, which gives a range of values consistent with experiment. These input parameters, as well as the form of the denominator function which is discussed in the next section and Appendix B, will be held fixed in all subsequent perturbations, and no further parameters will be added to the model.

In spite of its crudity, the model just outlined is the best available example of a bootstrap. It correctly predicts strong attraction in the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decimet channels, and repulsion or weaker attraction in the other P-wave channels, in addition to giving the F/D ratio of ΠBB coupling and the ratio of ΠBB to $\Pi B\Delta$ coupling. The reason why such a crude model works so well is not understood. We have nothing to contribute on this topic, but simply take the point of view that the success of the model makes it an especially favorable starting point for the study of $SU(3)$-violating perturbations.

III. SPECIFICATION OF BROKEN-$SU(3)$ MODEL

We now turn to the study of symmetry-breaking perturbations[17] on the reciprocal-bootstrap model of Sec. II. In broken $SU(3)$, the residue matrix for the direct-channel baryon pole will no longer have the simple form of (2.7). Instead, we shall write it as

$$R(N',N)+\delta R_S(N',N)\,, \qquad (3.1)$$

[12] The tangent of θ is the ratio of the coefficient of matrices O_{8_a} and O_{8_s}, each normalized by $\mathrm{Tr}O^2=1$. The usual λ is the ratio of the coefficient of matrices F and D, which are proportional to O_{8_a} and O_{8_s} but have the normalizations $\mathrm{Tr}(F^2)=3$ and $\mathrm{Tr}(D^2)=\frac{5}{3}$. The minus sign in (2.3) arises because we take $\sin\theta$ as the coefficient of $\begin{pmatrix} 8 & 8 & 8_a \\ i & j & k \end{pmatrix}$ in Eq. (2.9), where i refers to the meson in $\Pi_iB_jB_k$ coupling, whereas λ is conventionally proportional to the coefficient $\begin{pmatrix} 8 & 8 & 8_a \\ j & i & k \end{pmatrix}$, and $\begin{pmatrix} 8 & 8 & 8_a \\ i & j & k \end{pmatrix}$ is antisymmetric in each pair of indices.

[13] A. Martin and K. Wali, Phys. Rev. **130**, 2455 (1963).

[14] R. Dalitz, Phys. Letters **5**, 53 (1963).

[15] F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).

[16] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[17] We would like to take this opportunity to list a number of misprints and mistakes in our previous work on perturbations. (i) In Ref. 4, two lines below Eq. (13), there is a misprinted sign and the text should have read $\delta R_{ij}=-f_i\delta f_j-(\delta f_i)f_j$. (ii) In Ref. 4, the entire right side of Eq. (22) should be multiplied by $\frac{1}{2}$. (iii) In Ref. 2, Table X, the $S=27$ term should have read

$$(-13+18\lambda^2-45\lambda^4)/3(5+30\lambda^2-27\lambda^4).$$

This misprint was confined to Table X, and the numerical results quoted in Eq. (5.55) of the text are correct. (iv) In Ref. 2, top of p. 1346, the statement that the $SU(2)$ NN^* reciprocal bootstrap is stable under *all* conditions is incorrect. G. Shaw and D. Wong (Ref. 5) have pointed out to us that A_3 has a unit eigenvalue when D has the straight-line dependence $D=(W-M)$ and the parameter c of the text is taken equal to unity; this is a special case of a general theorem by I. Gerstein and M. Whippman, Ann. Phys. (N. Y.) **34**, 488 (1965). As discussed in Ref. 2, however, there is no reason to believe that these conditions for a unit eigenvalue of A_3 should actually be realized. (v) In Ref. 3, Table I, the entries for **10** and **10*** should each be multiplied by 2. The **10** and **10*** eigenvalues of A remain small, so the discussion of the text is still correct. (vi) In Ref. 2, Table I, the $\mathbf{8}_s\to\mathbf{8}_s$ element should read $(-3+9\lambda^2)/(10+18\lambda^2)$. Again, the misprint was confined to the table, and the resulting eigenvector was correctly stated in the text.

TABLE II. A list of the perturbations on coupling constants which are considered in this paper. $\delta G_S(N)$ refers to the coupling for $B\to B+\Pi$, where the final $B\Pi$ state is in the N representation and the whole coupling transforms like the S representation. $\delta G_S^*(N)$ similarly refers to $\Delta B\Pi$ coupling. For $S=27$, $N=27$ and $27'$ refer to the two ways in which the baryon octet and $B\Pi$ with $N=27$ are combined into $S=27$ by Chilton and McNamee (Ref. 29); for $S=8$, $N=8_\theta(s)$ and $8_\theta(a)$ refers to the two ways in which the baryon octet and $B\Pi$ with $N=8_\theta$ can be combined into $S=8$, and $8_{\theta*}(s)$ and $8_{\theta*}(a)$ have a similar significance.

S	N in δG	N in δG^*
1	8_θ $8_{\theta*}$	10
8	1 $8_\theta(s)$ $8_{\theta*}(s)$ $8_\theta(a)$ $8_{\theta*}(a)$ 10 $\overline{10}$ 27	8_θ $8_{\theta*}$ 10 27
27	8_θ $8_{\theta*}$ 10 $\overline{10}$ 27 $27'$	8_θ $8_{\theta*}$ 10 $\overline{10}$ 27

where N' labels the $SU(3)$ representation of the initial ΠB states ($N'=1$, 8_θ, $8_{\theta*}$, 10, $\overline{10}$, 27), N labels the final ΠB states, and the subscript S on the perturbed residue labels the $SU(3)$ representation which the symmetry violation transforms like.

From (2.7) one sees that $R(N',N)$ has the form

$$R(N',N)=-\delta_{N'8_\theta}\delta_{N8_\theta}G^2. \qquad (3.2)$$

Equation (3.2) exhibits explicitly the general property of factorizability: The residue matrix always factors into the product of two couplings, one connecting the entrance channel to the intermediate baryon state and the other connecting the baryon state to the exit channel. The perturbed residue matrix also has this property. Therefore, in the study of the A matrix where one considers only first-order perturbations, the perturbed residue is a product of an unperturbed coupling $\delta_{N8_\theta}G$ or $\delta_{N'8_\theta}G$ times a perturbed coupling. Thus, the elements of δR we need for a complete specification of perturbed baryon couplings are

$$\delta R_S(8_\theta,N)=-G\delta G_S(N)\,,\quad N\neq 8_\theta\,, \qquad (3.3)$$

$$\delta R_S(8_\theta,8_\theta)=-\delta G_S(8_\theta)G-G\delta G_S(8_\theta)$$
$$=-2G\delta G_S(8_\theta)\,. \qquad (3.4)$$

[For given time-reversal and charge-conjugation properties of the perturbation, $\delta R_S(N,8_\theta)$ can be deduced from $\delta R_S(8_\theta,N)$; these properties are discussed in the next section.]

In a completely analogous fashion, we write for the residue matrix in the $J=\frac{3}{2}^+$ channels

$$R^*(N',N)+\delta R_S^*(N',N)\,, \qquad (3.5)$$

where N', N, and S are as previously defined. Here we have

$$R^*(N',N)=-\delta_{N',10}\delta_{N,10}G^{*2} \qquad (3.6)$$

and we note that $\delta R_S^*(N',N)=0$, unless at least one of N or $N'=10$. The elements of δR^* needed for a specification of perturbed $\Delta B\Pi$ couplings are

$$\delta R^*(10,N)=-G^*\delta G_S^*(N)\,,\quad N\neq 10 \qquad (3.7)$$

$$\delta R^*(10,10)=-2G^*\delta G_S^*(10)\,. \qquad (3.8)$$

Among the various possible values of S (namely, any representation in $\mathbf{8}\times\mathbf{8}\times\mathbf{8}$ or $\mathbf{10}\times\mathbf{8}\times\mathbf{8}$), we shall consider only $S=1$, 8, and 27. For strong perturbations (i.e., $\Delta I=0$, $\Delta Y=0$) the only other possibility is $S=64$, which in practice is not driven by any mass shift and therefore, as we shall find in Sec. IX, probably could not compete with the doubly enhanced $S=8$ term, even if an eigenvalue of A_{64} were near one. $S=1$, 8, and 27 are also the only cases with driving terms D_S in electromagnetic effects of order e^2, and in weak nonleptonic interactions (if a current-current interaction symmetric in the currents is assumed). The various possible $\delta G(N)$ and $\delta G^*(N)$ for $S=1$, 8, and 27 are listed in Table II. Since there are 12 independent coupling perturbations with $S=8$ in our model, the matrix $A_{S=8}{}^{gg}$ which gives their effect on one another will be a 12×12 matrix. Similarly, $A_{S=1}{}^{gg}$ is a 3×3 matrix, $A_{27}{}^{gg}$ is an 11×11 matrix, $A_1{}^{gM}$ is 3×2 (mass shifts with $S=1$ occur once in δM^B and once in δM^Δ), $A_8{}^{gM}$ is 12×3, and $A_{27}{}^{gM}$ is 11×2.

The relation of $\delta G_S(N)$ and $\delta G_S^*(N)$ to couplings among individual particles is as follows. The perturbation on the coupling for $\Pi_i+B_i\to B_k$, for a specific symmetry-breaking transforming like the σ component of representation S, is

$$\delta G_{kj}{}^i=\sum_N Z_N\delta G_S(N)\sum_\nu\begin{pmatrix}8 & S & N\\ k & \sigma & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ i & j & \nu\end{pmatrix}. \qquad (3.9)$$

Equation (3.9) is easily obtained: The second Clebsch-Gordan coefficient represents the projection of Π_iB_j onto representation N, the first Clebsch-Gordan coefficient represents the combination of B_k with the same representation N to form a coupling transforming like S, $\delta G_S(N)$ gives the strength of this coupling, and Z_N is a normalization factor to be specified in Table IV. Similarly, the perturbation on the coupling for $\Pi_i+B_j\to\Delta_k$ is

$$\delta G_{kj}{}^{*i}=\sum_N Z_N^*\delta G_S^*(N)$$
$$\times\sum_\nu\begin{pmatrix}10 & S & N\\ k & \sigma & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ i & j & \nu\end{pmatrix}. \qquad (3.10)$$

Having specified the perturbations to be studied, we now turn to the dispersion relations which will be used to calculate them. The relevant dispersion relations for the S-matrix treatment of perturbations on masses and coupling constants have been developed in Ref. 4. The relations are exact for first-order perturbations. For shifts in $BB\Pi$ couplings they read

$$\delta\mathbf{R}=\frac{d}{dW}\left\{\mathbf{\Delta}^T(W)\left[\frac{1}{2\pi i}\int_C\frac{\mathbf{D}^T\delta\mathbf{T}\mathbf{D}dW'}{W'-W}\right]\mathbf{\Delta}(W)\right\}\Bigg|_{W=M^B}, \tag{3.11}$$

where C is a contour running clockwise around the right- and left-hand cuts of $\mathbf{T}$ (but not around the bound-state pole at M^B),

$$\mathbf{\Delta}=\lim_{W\to M^B}(W-M^B)\mathbf{D}^{-1}(W), \tag{3.12}$$

and the amplitude $\delta\mathbf{T}$ and denominator function $\mathbf{D}$ refer to ΠB scattering in the $P_{1/2}$ state.

Now the unperturbed D function for the $P_{1/2}$ channels has the form

$$D(W)=\begin{bmatrix} D_1 & 0 & 0 & 0 & 0 & 0\\ 0 & D_{8_\theta 8_\theta} & D_{8_\theta 8_{\theta*}} & 0 & 0 & 0\\ 0 & D_{8_{\theta*}8_\theta} & D_{8_{\theta*}8_{\theta*}} & 0 & 0 & 0\\ 0 & 0 & 0 & D_{10} & 0 & 0\\ 0 & 0 & 0 & 0 & D_{\overline{10}} & 0\\ 0 & 0 & 0 & 0 & 0 & D_{27}\end{bmatrix} \tag{3.13}$$

when the matrix elements are taken between states of definite N' and N. In the neighborhood of the baryon pole, we approximate this general form by[18]

$$D(W)\approx\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & D_{8_\theta} & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{3.14}$$

with D_{8_θ} passing through zero at $W=M^B$. Actually, in the simplified model of the present paper, only singularities near the baryon pole are considered in Eq. (3.11), and the approximation (3.14) will be used throughout this paper. The form used for D_{8_θ} is discussed below.

With the approximate form (3.14) for the D matrix, Eq. (3.11) for δR takes the explicit form[4]

$$\delta R(8_\theta,N\neq 8_\theta)=\frac{1}{2\pi i D_{8_\theta}'(M^B)}\int_C\frac{D_{8_\theta}\delta T_{8_\theta,N}dW'}{W'-M^B}, \tag{3.15}$$

[18] Since the physical-coupling shifts and mass shifts are independent of the normalization of the denominator function, there is no loss of generality in setting $D(M^B)=1$ in the nonresonant channels. The approximations in (3.14) are: (i) keeping $D=1$ for W *near* M^B in nonresonant channels; (ii) taking $D_{8_\theta 8_{\theta*}}=D_{8_{\theta*}8_\theta}=0$. The justification for (i) is that the low-energy phase shifts are small in the nonresonant channels and D is slowly varying. As far as (ii) is concerned, $\mathbf{D}_8$ has the form $\mathbf{D}_8(W)=1-\int\varrho\mathbf{N}_8(W'-W)^{-1}dW'$. We can diagonalize $\mathbf{D}_8$ at some energy, such as the energy of the Δ exchange pole. $\mathbf{D}_8$ then remains nearly diagonal over the low-energy region, because the dominant term in $\mathbf{N}_8$ for the $P_{1/2}$ state is Δ exchange which by itself would give an energy-independent F/D ratio (i.e., it would allow an energy-independent diagonalization of $\mathbf{D}_8$).

$$\delta R(8_\theta,8_\theta)=\frac{1}{2\pi i[D_{8_\theta}'(M^B)]^2}\int_C\frac{D_{8_\theta}^2\delta T_{8_\theta,8_\theta}dW'}{(W'-M^B)^2}$$
$$-\frac{D_{8_\theta}''(M^B)}{2\pi i[D_{8_\theta}'(M^B)]^3}\int_C\frac{D_{8_\theta}^2\delta T_{8_\theta,8_\theta}dW'}{W'-M^B}. \tag{3.16}$$

We shall also need the dispersion relation for δM^B, in order to study A^{Mg} in Sec. VII. Here, one is interested in the masses of individual baryons i, $i=1\cdots8$. The exact dispersion relation for the mass shift of the ith baryon is[4]

$$\delta M_B{}^i=\frac{-1}{2\pi iG^2[D_i'(M^B)]^2}\int_C\frac{D_i\delta T_iD_idW'}{W'-M^B}, \tag{3.17}$$

where D_i and δT_i all refer to the channel in which the ith baryon occurs. In the approximation (3.14), D_i is simply D_{8_θ}.

For the $J=\frac{3}{2}^+$ channels, the unperturbed D function again has the form (3.13). We approximate it by

$$\mathbf{D}(W)\approx\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & D_{10} & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, \tag{3.18}$$

where D_{10} has a zero at $W=M^\Delta$. The dispersion relations for $\delta\mathbf{R}^*$ have the same form as (3.11), (3.15), and (3.16), with D and δT now referring to the $P_{3/2}$ channels, D_{8_θ} now replaced by D_{10}, and M^B replaced by M^Δ. Similarly, the dispersion relations for $\delta M_\Delta{}^i$, $i=1\cdots10$, have the same form as (3.17) with G replaced by G^*, M^B by M^Δ, and D_{8_θ} by D_{10}.

We now return to the choice of D_{8_θ} for $J=\frac{1}{2}^+$, and D_{10} for $J=\frac{3}{2}^+$. The form which will be used, for reasons analyzed in Appendix B, is

$$D_{8_\theta}=(W-M^B)(W_0-M^B)/(W_0-W), \tag{3.19}$$

$$D_{10}=(W-M^\Delta)(W_0{}^*-M^\Delta)/(W_0{}^*-W), \tag{3.20}$$

where W_0 and $W_0{}^*$ are additional parameters in the subsequent calculations. The sensitivity of our results to these parameters will be discussed in Secs. VI, VII, and VIII. It is found that A^{gg} is relatively insensitive to W_0 and $W_0{}^*$, especially in the case of $\delta R(8_\theta,N\neq8_\theta)$ and $\delta R^*(10,N\neq10)$, since D_{8_θ} and D_{10} are found only once in the dispersion relation for these quantities [Eq. (3.15)]. The sensitivity of A^{MM} to W_0 and $W_0{}^*$ has been studied in previous papers[2,19] and is somewhat greater. We shall find in Sec. VIII that A^{gM} is *extremely* sensitive to the exact form of the D function, so that we cannot calculate the over-all magnitude of A^{gM} reliably.

In the above discussion, we have restricted ourselves to δG's which do not violate parity. Since we will also be interested in the parity-odd violations of $SU(3)$ induced by the weak interactions, we now discuss the

[19] R. Dashen, Phys. Rev. **135**, B1196 (1964).

changes that must be made in the above formulas for nonparity conserving ($P=-1$) δG's, To study the ($P=-1$) δG's, we consider the residue matrix of the direct-channel baryon pole in the amplitude for ΠB ($J=\frac{1}{2}$ P wave) $\to \Pi B$ ($J=\frac{1}{2}$ S wave). The unperturbed residue vanishes in this case and since the parity violation occurs in the coupling of the final state to the pole, we can write the residue as

$$\delta R_S(N,N')=-\delta_{N8_\theta}G\delta G_{SN'} \quad (P=-1) \quad (3.21)$$

for $N=1$, 8_θ, $8_{\theta*}$, 10, $\overline{10}$, and 27. Note that for $N'=8_\theta$, the relation between δR and δG does not contain the factor of 2 which is present in the $P=+1$ case [cf., Eqs. (3.3) and (3.4)]. The relation between $\delta G_S(N)$ and $\delta G_{kj}{}^i$ is still given by (3.9), but on account of the above-mentioned factor of 2, we use the Z_N's given in Sec. XI [following Eq. (11.3)] rather than those of Table IV.

To treat the ($P=-1$) δG^*'s, we look at ΠB ($J=\frac{3}{2}$ P wave) $\to$ (ΠB) ($J=\frac{3}{2}$ D wave). In analogy to the δG's, the relation between $\delta G_{kj}{}^{*i}$ and $\delta G_S{}^*(N)$ is given by (3.10), where the $Z_N{}^*$'s are to be taken from Sec. XI.

Next we must specify the S- and D-wave denominator functions. We assume that in the low-energy region under consideration, the $\frac{1}{2}^-$ and $\frac{3}{2}^-$ denominator functions can be set equal to unity in all $SU(3)$ channels. Equation (3.11) then gives

$$\delta R(8_\theta,N)=\frac{1}{2\pi i D_{8_\theta}'(M^B)}\int_C \frac{D_{8_\theta}\delta T_{8_\theta,N}dW'}{W'-M^B} \quad (P=-1) \quad (3.22)$$

and an analogous equation for δR^*, where δT is, of course, the parity-violating amplitude for

$$J=\tfrac{1}{2}^+ \to J=\tfrac{1}{2}^-.$$

Note that we no longer have to write a special equation for $N=8_\theta$.

To conclude this discussion of the general method used in this paper, we comment briefly on two of the most flagrant omissions in our treatment: vector meson exchange, and effects on $A^{gM}\delta M$ of shifting the external pseudoscalar-meson mass. These omitted terms, while not negligible, are expected to be somewhat smaller than the B- and Δ-exchange terms for the following reasons:

(i) Vector-meson exchange is a rather short-range effect and should therefore be less important for perturbations than it is in the strong interactions.

(ii) The effect of shifts in external Π mass can be studied explicitly. On the right cut in Eqs. (3.15)–(3.17), the kinematic factor q^{-3} in the definition (2.1), (2.2) of T is modified in a calculable way. On the left cut, the "short cuts" for B and Δ exchange are modified in a calculable way. (One has to leave the pseudopole approximation and go back to the short cuts to study this effect.) In each case, the effect of δM^Π is multiplied[2] by M^Π/M^B and the numerical results are rather small.[20]

(iii) In any case, an omitted term such as $A^{g(BB\Pi)M^\Pi(\mathrm{ext})}$ is an "off-diagonal" part of the A matrix. As such, it can influence eigenvalues of A only through the combination $A^{g(BB\Pi)M^\Pi}A^{M^\Pi g(BB\Pi)}$. Thus the eigenvalues of A studied in this paper are not sensitive to pseudoscalar mass effects unless δM^Π strongly influences baryon properties *and* baryon properties also strongly influence δM^Π. The same statement applies to vector-meson effects.

IV. EQUATIONS FOR A^{gg}

The purpose of this section is to provide the specific equations needed to calculate A^{gg}.

The elements of A^{gg} to be considered were described in Sec. III; they include the effects of shifts in B-exchange and Δ-exchange couplings on the $BB\Pi$ and $\Delta B\Pi$ couplings in the direct channel. The interactions which are involved, written in terms of fields, are the $BB\Pi$ interaction

$$H_{BB\Pi}=\sum_{\alpha\beta k} G_{\alpha\beta}{}^k\bar{\psi}_\alpha\gamma_5\psi_\beta\phi_k \quad (4.1)$$

(it is understood that one takes the commutator of $\bar{\psi}$ and ψ to avoid infinities), and the $\Delta B\Pi$ interaction

$$H_{\Delta B\Pi}=\sum_{\alpha\beta k} G_{\alpha\beta}{}^{*k}\bar{\psi}_{\Delta_{\alpha\mu}}\psi_{B\beta}\partial_\mu\phi_k + \sum_{\alpha\beta k} G_{\alpha\beta}{}'^{*k}\bar{\psi}_{B\beta}\psi_{\Delta_{\alpha\mu}}\partial_\mu\phi_{\bar{k}}. \quad (4.2)$$

$G_{\alpha\beta}{}^k$ is the sum of the $SU(3)$-symmetric coupling (2.9) and the perturbation (3.9); $G_{\alpha\beta}{}^{*k}$ is the sum of the $SU(3)$-symmetric coupling (2.10) and the perturbation (3.10). By standard methods,[21] one finds that Hermiticity of the interaction Hamiltonian implies the conditions

$$G_{\alpha\beta}{}^k=(-1)^{Q_k}\bar{G}_{\beta\alpha}{}^{\bar{k}}, \quad (4.3)$$

$$G_{\alpha\beta}{}^{*k}=(-1)^{Q_k}\bar{G}_{\alpha\beta}{}'^{*k}, \quad (4.4)$$

where the bar over G denotes complex conjugation and Q_k refers to the electric charge of particle k.

In studying the weak interactions, we shall be interested in couplings with various properties under C, T, and P. For $P=+$, one finds that

$$G_{\alpha\beta}{}^k=C(-1)^{Q_k}G_{\beta\alpha}{}^{\bar{k}}, \quad (4.5)$$

$$G_{\beta\alpha}{}^{*k}=C(-1)^{Q_k}G_{\alpha\beta}{}'^{*k}, \quad (4.6)$$

and that

$$\left.\begin{array}{l} G, G^*, G'^* \text{ real}\}T=+, P=+, \\ G, G^*, G'^* \text{ imaginary}\}T=-, P=+, \end{array}\right. \quad (4.7)$$

where $C=\pm$ and $T=\pm$ refer to the behavior of the interaction Hamiltonian under charge conjugation and

[20] F. Henyey (private communication).
[21] See, for example, S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961).

time reversal, respectively:

$$CHC^{-1}=\pm H\,,$$
$$TH(t)T^{-1}=\pm H(-t)\,.$$

If parity is violated, one finds by studying the relevant analogs of Eqs. (4.1) and (4.2) that the Hermiticity and charge-conjugation conditions (4.3)–(4.6) are unchanged, whereas the time-reversal condition becomes

$$G,\,G^*,\,G'^*\ \text{imaginary}\}\,T=+,\,P=-\,,\qquad(4.8)$$

$$G,\,G^*,\,G'^*\ \text{real}\}\,T=-,\,P=-\,.\qquad(4.9)$$

We now describe in some detail how to calculate a typical element of A^{gg}, namely, the effect of $\delta R(8_\theta,X)$ and $\delta R(X,8_\theta)$ shifts in baryon exchange on the direct-channel residue $\delta R(8_\theta,X')$ (Fig. 1). X and X' can take on any values in Table II. If $X'\neq 8_\theta$, for example, the dispersion integral to be evaluated is

$$\delta R(8_\theta,X')=\frac{1}{2\pi i D_{8\theta}'(M^B)}\int_C \frac{D_{8\theta}(W')\delta T_{8_\theta,X'}dW'}{W'-M^B}\,.\qquad(3.15)$$

The main job is to calculate $\delta T_{8_\theta,X'}$. First we shall calculate $\delta T_{\nu_1\nu_2,\nu_3\nu_4}$, and then project out the contribution to $\delta T_{8_\theta,X'}$. To obtain the contribution to $\delta T_{\nu_1\nu_2,\nu_3\nu_4}$ from coupling shifts in baryon exchange, one evaluates the Feynman diagrams, Figs. 1(a) and 1(b), by standard methods. As usual,[2,4] δT divides into a factor involving

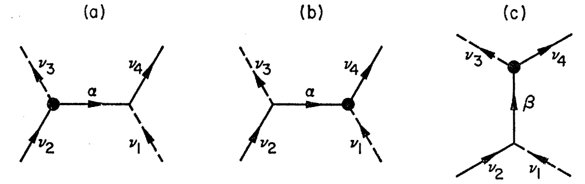

FIG. 1. Diagrams representing residue shifts (a) $\delta R(X,8_\theta)$ and (b) $\delta R(8_\theta,X)$ in baryon exchange, and the residue shift (c) $\delta R(8_\theta,X')$ in the direct-channel baryon pole term. The coupling shift occurs at the vertex with the blob. Baryons are represented by solid lines and pseudoscalar mesons by broken lines.

products of $SU(3)$ Clebsch-Gordan coefficients, and a dynamical factor which is the same for all $SU(3)$ states. (It depends on the nucleon-exchange pole and ordinary spin crossing.)

We evaluate the $SU(3)$ factor first. For $\Pi_{\nu_1}+B_{\nu_2}\to\Pi_{\nu_3}+B_{\nu_4}$, the contribution to δT from exchange diagrams 1(a) and 1(b) is

$$\delta T_{\nu_1\nu_2,\nu_3\nu_4}=\sum_\alpha(T_a\delta G_{\alpha\nu_2}{}^{\bar\nu_3}G_{\nu_4\alpha}{}^{\nu_1}+T_b\delta G_{\alpha\nu_2}{}^{\bar\nu_3}G_{\nu_4\alpha}{}^{\nu_1})\,.\qquad(4.10)$$

Here, T_a and T_b are the $SU(3)$-independent "dynamical factors" for Figs. 1(a) and 1(b), respectively. To obtain the correct $SU(3)$ labels on the couplings, one notes that $\bar\psi$ in Eq. (4.1) creates particles, ψ destroys particles, and ϕ_k destroys k and creates $\bar k$. By use of condition (4.5), Eq. (4.10) can be re-expressed as

$$\delta T_{\nu_1\nu_2,\nu_3\nu_4}=\sum_\alpha(T_a\delta G_{\alpha\nu_2}{}^{\bar\nu_3}G_{\alpha\nu_4}{}^{\bar\nu_1}(-1)^{Q_{\nu_1}}+T_bCG_{\alpha\nu_2}{}^{\bar\nu_3}\delta G_{\alpha\nu_4}{}^{\bar\nu_1}(-1)^{Q_{\nu_1}})\qquad(4.11)$$

(we take $C=+$ automatically for G but leave both possibilities, $C=+$ and $C=-$, open for δG). It is convenient to express G and δG in terms of $SU(3)$ Clebsch-Gordan coefficients by means of Eqs. (2.9) and (3.9). Equation (4.11) then becomes

$$\delta T_{\nu_1\nu_2,\nu_3\nu_4}=G\delta G_S(X)Z_X(-1)^{Q_{\nu_1}}\sum_{\alpha,\nu}\begin{pmatrix}8&S&X\\ \alpha&\sigma&\nu\end{pmatrix}$$
$$\times\left[T_a\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&8_\theta\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}+T_bC\begin{pmatrix}8&8&8_\theta\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right]\qquad(4.12)$$

for perturbations that transform under $SU(3)$ rotations like the σ component of representation S. Next we need to perform a suitably weighted sum over ν_1, ν_2, ν_3, ν_4 to project out $\delta T_{8_\theta,X'}$. In this connection, note that the amplitude for Fig. 1(c) is

$$-(\sum_\beta G_{\beta\nu_2}{}^{\nu_1}\delta G_{\nu_4\beta}{}^{\bar\nu_3})(W-M^B)^{-1}.$$

It can be re-expressed with the help of Eqs. (4.5), (2.9), and (3.9) as

$$-(\sum_\beta G_{\beta\nu_2}{}^{\nu_1}\delta G_{\nu_4\beta}{}^{\bar\nu_3})(W-M^B)^{-1}=-(\sum_\beta G_{\beta\nu_2}{}^{\nu_1}\delta G_{\beta\nu_4}{}^{\nu_3}(-1)^{Q_{\nu_3}}C)(W-M^B)^{-1}$$
$$=\frac{-G\delta G_S(X')Z_{X'}(-1)^{Q_{\nu_3}}C}{W-M^B}\sum_{\beta\nu'}\begin{pmatrix}8&S&X'\\ \beta&\sigma&\nu'\end{pmatrix}\begin{pmatrix}8&8&8_\theta\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}.\qquad(4.13)$$

Here we have Fig. 1(c), which is pure $8_\theta\to X'$, written in terms of individual particle states. Evidently the factors depending on ν_1, ν_2, ν_3, and ν_4 in (4.13) can be used as a projection operator for $8_\theta\to X'$. Indeed, letting N_S and N_N denote the dimensions of the S and N representations, respectively, multiplying (4.13) by $N_SN_{X'}{}^{-1}Z_{X'}{}^{-2}$ times the coefficient of $-G\delta G_S(X')(W-M^B)^{-1}$ in (4.13), summing over all indices and using the orthonormality relations[16]

$$\sum_{\nu_1\nu_2}\begin{pmatrix}8 & 8 & X\\ \nu_1 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & X'\\ \nu_1 & \nu_2 & \nu'\end{pmatrix}=\delta_{XX'}\delta_{\nu\nu'}\,, \tag{4.14}$$

$$\sum_{\beta\nu}\begin{pmatrix}8 & S & X'\\ \beta & \sigma & \nu'\end{pmatrix}^2=\frac{N_{X'}}{N_S}\sum_{\beta\nu}\begin{pmatrix}8 & X' & S\\ \beta & \nu' & \sigma\end{pmatrix}^2=\frac{N_{X'}}{N_S}\,, \tag{4.15}$$

one obtains $-G\delta G_S(X')(W-M^B)^{-1}$, which is precisely the contribution of Fig. 1(c) to $\delta T_{8_\theta,X'}$. Multiplication of (4.13) by an operator of the same form but with $(X,\nu)\neq(X',\nu')$, or $(X'',\beta')\neq(8_\theta,\beta)$, or $(S',\sigma')\neq(S,\sigma)$, gives zero because of the orthogonality relation (4.14). Thus we have found the suitably normalized projection operator

$$\delta T_{8_\theta,X'}=N_SN_{X'}^{-1}Z_{X'}^{-1}C\sum_{\beta\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}(-1)^{Q_{\nu_3}}\begin{pmatrix}8 & S & X'\\ \beta & \sigma & \nu'\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & \beta\end{pmatrix}\begin{pmatrix}8 & 8 & X'\\ \nu_3 & \nu_4 & \nu\end{pmatrix}\delta T_{\nu_1\nu_2,\nu_3\nu_4}. \tag{4.16}$$

Applying this projection to the exchange terms (4.12), one obtains

$$\delta T_{8_\theta,X'}=\frac{G\delta G_S(X)Z_XN_S}{Z_{X'}N_{X'}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & \beta\end{pmatrix}\begin{pmatrix}8 & 8 & X'\\ \nu_3 & \nu_4 & \nu'\end{pmatrix}\Bigg[T_aC\begin{pmatrix}8 & 8 & X\\ \bar\nu_3 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \bar\nu_1 & \nu_4 & \alpha\end{pmatrix}$$
$$+T_b\begin{pmatrix}8 & 8 & 8_\theta\\ \bar\nu_3 & \nu_2 & \alpha\end{pmatrix}\begin{pmatrix}8 & 8 & X\\ \bar\nu_1 & \nu_4 & \nu\end{pmatrix}\Bigg](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}8 & S & X\\ \alpha & \sigma & \nu\end{pmatrix}\begin{pmatrix}8 & S & X'\\ \beta & \sigma & \nu'\end{pmatrix}. \tag{4.17}$$

The only factors in (4.17) which depend on the energy W are the "dynamical factors" T_a and T_b. Therefore, the result of plugging (4.17) into the dispersion relation (3.15) for $X'\neq8_\theta$ is

$$\delta R_S(8_\theta,X')=\frac{-G\delta G_S(X)Z_XN_S}{Z_{X'}N_{X'}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & \beta\end{pmatrix}\begin{pmatrix}8 & 8 & X'\\ \nu_3 & \nu_4 & \nu'\end{pmatrix}\Bigg[\eta_a^{BB}C\begin{pmatrix}8 & 8 & X\\ \bar\nu_3 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \bar\nu_1 & \nu_4 & \alpha\end{pmatrix}$$
$$+\eta_b^{BB}\begin{pmatrix}8 & 8 & 8_\theta\\ \bar\nu_3 & \nu_2 & \alpha\end{pmatrix}\begin{pmatrix}8 & 8 & X\\ \bar\nu_1 & \nu_4 & \nu\end{pmatrix}\Bigg](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}8 & S & X\\ \alpha & \sigma & \nu\end{pmatrix}\begin{pmatrix}8 & S & X'\\ \beta & \sigma & \nu'\end{pmatrix}, \tag{4.18}$$

where

$$\eta_a^{BB}=\frac{-1}{2\pi iD_{8_\theta}'(M^B)}\int_C\frac{D_{8_\theta}(W')T_a(W')dW'}{(W-M^B)} \tag{4.19}$$

and η_b^{BB} is similarly defined.

For parity-conserving couplings, the dynamical factors T_a and T_b are evaluated as follows. Evaluation of the Feynman diagrams, Figs. 1(a) and 1(b), gives the coupling factors of Eq. (4.10) times the projection of the nucleon-exchange pole onto the $P_{1/2}$ state. The projected nucleon-exchange pole gives the usual "short cut" around $W=M^B$ and the long cut along the imaginary W axis. Our approximations involve keeping only the "short cut" and replacing it by an equivalent pseudopole $(W-M^B)^{-1}$. The pseudopole is also multiplied by the usual static-spin crossing factor $-\frac{1}{3}$ for crossing the $P_{1/2}$ state into the $P_{1/2}$ state. Thus, for parity-conserving couplings,

$$T_a=T_b\approx-1/3(W-M^B). \tag{4.20}$$

In the present case, the contour C in (4.19), which generally encloses the left- and right-hand singularities (but not the bound-state pole), shrinks to a clockwise circuit of the exchange pseudopole (which we displace from the direct-channel pole by ϵ for this purpose). Since at this pole $D_{8_\theta}(W')(W'-M^B)^{-1}$ is just $D_{8_\theta}'(M^B)$, we obtain

$$\eta_a^{BB}=\eta_b^{BB}=-\tfrac{1}{3}. \tag{4.21}$$

The foregoing results applied to $X'\neq8_\theta$. For $X'=8_\theta$, all relations are unchanged through Eq. (4.17), which now (in the parity-conserving case) must be plugged into (3.16) rather than (3.15). One again obtains (4.18), but η_a^{BB} is now given by

$$\eta_a^{BB}=\frac{-1}{2\pi i[D_{8_\theta}'(M^B)]^2}\int_C\frac{D_{8_\theta}^2T_adW'}{(W'-M^B)^2}+\frac{D_{8_\theta}''(M^B)}{2\pi i[D_{8_\theta}'(M^B)]^2}\int_C\frac{D_{8_\theta}^2T_adW'}{W'-M^B} \tag{4.22}$$

and η_b^{BB} satisfies a similar equation. Inserting (4.20) into (4.22), one obtains the same value as (4.21) for η^{BB} [the second term in (4.22) contributes nothing when the zero of D overlaps the exchange pole].

TABLE III. Values of the quantity η_a for couplings with $P=+$. K is defined by $K(W)=D(W)[(W-M)D'(M)]^{-1}$. The quantity η_a is the same for all C and T. For $P=+$, $\eta_b=\eta_a$.

	η_a for general D	η_a for linear-D approximation
$\eta_a^{BB}(X\neq 8_\theta)$	$-\frac{1}{3}$	$-\frac{1}{3}$
$\eta_a^{BB}(X=8_\theta)$	$-\frac{1}{3}$	$-\frac{1}{3}$
$\eta_a^{B\Delta}(X\neq 8_\theta)$	$\frac{4}{3}K_{8\theta}(2M^B-M^\Delta)$	$\frac{4}{3}$
$\eta_a^{B\Delta}(X=8_\theta)$	$\frac{4}{3}K_{8\theta}(2M^B-M^\Delta)\{K_{8\theta}(2M^B-M^\Delta)-D_{8\theta}(2M^B-M^\Delta)D_{8\theta}''(M^B)[D_{8\theta}'(M^B)]^{-2}\}$	$\frac{4}{3}$
$\eta_a^{\Delta B}(X\neq 10)$	$\frac{2}{3}K_{10}(M^B)$	$\frac{2}{3}$
$\eta_a^{\Delta B}(X=10)$	$\frac{2}{3}K_{10}(M^B)\{K_{10}(M^B)-D_{10}(M^B)D_{10}''(M^\Delta)[D_{10}'(M^\Delta)]^{-2}\}$	$\frac{2}{3}$
$\eta_a^{\Delta\Delta}(X\neq 10)$	$\frac{1}{3}K_{10}(2M^B-M^\Delta)$	$\frac{1}{3}$
$\eta_a^{\Delta\Delta}(X=10)$	$\frac{1}{3}K_{10}(2M^B-M^\Delta)\{K_{10}(2M^B-M^\Delta)-D_{10}(2M^B-M^\Delta)D_{10}''(M^\Delta)[D_{10}'(M^\Delta)]^{-2}\}$	$\frac{1}{3}$

It is straightforward to derive the effects of δR^* shifts in Δ exchange on the direct-channel residues δR and δR^*, and of δR shifts on δR^*, by the same methods used above for the effect of δR shifts on δR. One finds, for the effects of Δ exchange,

$$\delta R_S(8_\theta,X')=\frac{-G^*\delta G_S^*(X)Z_X^*N_S}{Z_{X'}N_{X'}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&8_\theta\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{B\Delta}\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&10\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{B\Delta}C\begin{pmatrix}8&8&10\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}10&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}8&S&X'\\ \beta&\sigma&\nu'\end{pmatrix}\quad(4.23)$$

and

$$\delta R_S^*(10,X')=\frac{-G^*\delta G_S^*(X)Z_X^*N_S}{Z_{X'}^*N_{X'}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&10\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{\Delta\Delta}C\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&10\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{\Delta\Delta}\begin{pmatrix}8&8&10\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}10&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}10&S&X'\\ \beta&\sigma&\nu'\end{pmatrix}\quad(4.24)$$

and, for the effect of B exchange on the Δ residue,

$$\delta R_S^*(10,X')=\frac{-G\delta G_S(X)Z_XN_S}{Z_{X'}^*N_{X'}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&10\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{\Delta B}\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&8_\theta\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{\Delta B}C\begin{pmatrix}8&8&8_\theta\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}8&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}10&S&X'\\ \beta&\sigma&\nu'\end{pmatrix}.\quad(4.25)$$

The coefficients η_a for the various cases of positive-parity couplings are listed in Table III. These coefficients are independent of T and C, which affect only the coupling factors. The coefficient η_b equals η_a in each case as we found for η^{BB}. The η coefficients for negative parity couplings are quite different and will be discussed in Sec. XI.

We are now finally in a position to evaluate A^{gg}, which is essentially given by the coefficients of $G\,\delta G$ on the right side of Eqs. (4.18), (4.23), (4.24), and (4.25). Of course, we are free to evaluate A^{gg} in terms of any convenient set of basis states we like; for example, either δG's or δR's may be used and their normalization factors Z and Z^* are at our disposal. We choose Z and Z^* and the states connected by A in such a way as to make A symmetric.[22]

A^{gg} can be symmetrized exactly only when the linear-D approximation is made in η, so let us consider that case first. $A^{G(X')G(X)}$, for example, will be deduced from Eq. (4.18). The sum over Clebsch-Gordan coefficients in (4.18) is symmetric between X and X', and the factor in front also becomes symmetric if we take $Z_X=(N_X)^{-1/2}$ and $Z_{X'}=(N_{X'})^{-1/2}$. Similarly, Eq. (4.24), from which $A^{G^*(X')G^*(X)}$ will be deduced, becomes symmetric if Z_X^* is taken proportional to $(N_X)^{-1/2}$. Next we look at (4.23) $[A^{G(X')G^*(X)}]$ and (4.25) $(A^{G^*(X')G(X)})$. Once again, the sum over Clebsch-Gordan coefficients is symmetric—that is, the sum in (4.25) equals the sum in (4.23) with X and X' interchanged. The inequity in the coefficients, $\eta^{B\Delta}=2\eta^{\Delta B}$, can be offset by dividing Z_X^* by an overall factor $\sqrt{2}$ relative to Z_X. If δG and δG^* were used as the basis, the factors G and G^* would provide another asymmetry; this is

[22] In this way we are guaranteed that the eigenvectors of A form a basis in our space of states. Also the numerical procedure is simplified. (For example, the Caltech Computer Center subroutine for determining the eigenvalues and eigenvectors of a matrix happened to work only for symmetric matrices.)

avoided by using $\delta R(8_\theta,X)=-G\delta G(X)$ and $\delta R^*(10,X)=-G^*\delta G^*(X)$ as the basis. In other words, we shall actually calculate a matrix A^{RR} which differs from A^{gg} by a change of basis. The eigenvalues, which are the solutions of $\det(A-\lambda I)=0$, are unaffected by this change of basis. Symmetry would now prevail, were it not for the fact that $\delta R(8_\theta,8_\theta)=-2G\delta G(8_\theta)$ and $\delta R^*(10,10)=-2G^*\delta G^*(10)$ [Eqs. (3.4) and (3.8)]. This last asymmetry is overcome by letting $Z_{8_\theta}=\sqrt{2}(8)^{-1/2}$ rather than $(8)^{-1/2}$, and by letting $Z_{10}^*=\sqrt{2}(10)^{-1/2}$. Our final choice of Z's and Z^*'s is summarized in Table IV.

With this choice of Z's, the expressions for the matrix A^{RR} come out with a factor $N_S(N_XN_{X'})^{-1/2}$ in front, modified by various square roots of two. In order to absorb most of these $\sqrt{2}$'s and give the results in a more unified form, we define n_X and n_X^* to have the values in Table IV, and replace N_X or $N_{X'}$ by n_X when they refer to $BB\Pi$ couplings, and by n_X^* when they refer to $\Delta B\Pi$ couplings. The elements of A^{RR} are now given by the coefficients of δR on the right side of Eqs. (4.18), (4.23), (4.24), and (4.25). They have the values

$$A^{R(8_\theta,X')R(8_\theta,X)}=\frac{N_S}{(n_Xn_{X'})^{1/2}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&8_\theta\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{BB}C\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&8_\theta\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{BB}\begin{pmatrix}8&8&8_\theta\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}8&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}8&S&X'\\ \beta&\sigma&\nu'\end{pmatrix},\quad(4.26)$$

$$A^{R(8_\theta,X')R^*(10,X)}=\frac{N_S}{(2n_X^*n_{X'})^{1/2}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&8_\theta\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{B\Delta}\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&10\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{B\Delta}C\begin{pmatrix}8&8&10\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}10&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}8&S&X'\\ \beta&\sigma&\nu'\end{pmatrix},\quad(4.27)$$

$$A^{R^*(10,X')R(8_\theta,X)}=\frac{N_S\sqrt{2}}{(n_Xn_{X'}^*)^{1/2}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&10\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{\Delta B}\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&8_\theta\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{\Delta B}C\begin{pmatrix}8&8&8_\theta\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}8&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}10&S&X'\\ \beta&\sigma&\nu'\end{pmatrix},\quad(4.28)$$

$$A^{R^*(10,X')R^*(10,X)}=\frac{N_S}{(n_X^*n_{X'}^*)^{1/2}}\sum_{\alpha\beta\nu\nu'}\sum_{\nu_1\nu_2\nu_3\nu_4}\begin{pmatrix}8&8&10\\ \nu_1&\nu_2&\beta\end{pmatrix}\begin{pmatrix}8&8&X'\\ \nu_3&\nu_4&\nu'\end{pmatrix}\left[\eta_a^{\Delta\Delta}C\begin{pmatrix}8&8&X\\ \bar\nu_3&\nu_2&\nu\end{pmatrix}\begin{pmatrix}8&8&10\\ \bar\nu_1&\nu_4&\alpha\end{pmatrix}\right.$$
$$\left.+\eta_b^{\Delta\Delta}\begin{pmatrix}8&8&10\\ \bar\nu_3&\nu_2&\alpha\end{pmatrix}\begin{pmatrix}8&8&X\\ \bar\nu_1&\nu_4&\nu\end{pmatrix}\right](-1)^{Q_{\nu_1}+Q_{\nu_3}}\begin{pmatrix}10&S&X\\ \alpha&\sigma&\nu\end{pmatrix}\begin{pmatrix}10&S&X'\\ \beta&\sigma&\nu'\end{pmatrix}.\quad(4.29)$$

As we have already said, the eigenvalues of A^{RR} are the same as the eigenvalues of A^{gg} which we started out to calculate. A little work is needed, however, to obtain the physically interesting ratios of coupling shifts $\delta G_{kj}{}^i$ and $\delta G_{kj}{}^{*i}$ that correspond to a given eigenvector of A^{RR}. Each eigenvector resulting from diagonalization of A^{RR} as defined by (4.26)–(4.29) is a set of numbers $\delta R(8_\theta,X)$, $\delta R^*(10,X)$ with X running over the values listed in Table II. The corresponding coupling shifts, in the same basis, are

$$\delta G(X\neq 8_\theta)=-\delta R(8_\theta,X\neq 8_\theta)/G,\quad(3.3)$$

$$\delta G(8_\theta)=-\delta R(8_\theta,8_\theta)/2G,\quad(3.4)$$

$$\delta G^*(X\neq 10)=-\delta R^*(10,X\neq 10)/G^*,\quad(3.7)$$

$$\delta G^*(10)=-\delta R^*(10,10)/2G^*.\quad(3.8)$$

The individual-particle coupling shifts $\delta G_{kj}{}^i$ and $\delta G_{kj}{}^{*i}$ are given in terms of δG, Z, δG^*, and Z^* by Eqs. (3.9) and (3.10). Using Eqs. (3.3), (3.4), (3.7), (3.8), and the specification of Z and Z^* in Table IV, one obtains

$$\delta G_{kj}{}^i=-G^{-1}\sum_X(n_X)^{-1/2}\delta R_S(8_\theta,X)$$
$$\times\sum_\nu\begin{pmatrix}8&S&X\\ k&\sigma&\nu\end{pmatrix}\begin{pmatrix}8&8&X\\ i&j&\nu\end{pmatrix}\quad(4.30)$$

TABLE IV. Values of the normalization factors Z_X and Z_X^* and the factors n_X and n_X^*, used in defining A^{RR} for $P=+$ couplings.

X	Z_X	Z_X^*	n_X	n_X^*
1	1		1	
8_θ, $8_\theta(s)$, $8_\theta(a)$	$\frac{1}{2}$	$\frac{1}{4}$	16	8
8_{θ^*}, $8_{\theta^*}(s)$, $8_{\theta^*}(a)$	$(8)^{-1/2}$	$\frac{1}{4}$	8	8
10	$(10)^{-1/2}$	$(10)^{-1/2}$	10	20
$\overline{10}$	$(10)^{-1/2}$	$(20)^{-1/2}$	10	10
27, 27′	$(27)^{-1/2}$	$(54)^{-1/2}$	27	27

and

$$\delta G_{kj}{}^{*i} = -\frac{G^{*-1}}{\sqrt{2}} \sum_X (n_X{}^*)^{-1/2} \delta R_S{}^*(10,X) \times \sum_\nu \begin{pmatrix} 10 & S & X \\ k & \sigma & \nu \end{pmatrix} \begin{pmatrix} 8 & 8 & X \\ i & j & \nu \end{pmatrix} \quad (4.31)$$

for symmetry breaking transforming like the σ component of representation S. Note that G^{-1} is numerically about equal to $G^{*-1}/\sqrt{2}$ [Eq. (2.5)].

The basic equations we have just derived all hold for any D function. The convenient property that A^{RR} as defined by (4.26)–(4.29) is symmetric, however, holds only for the linear-D approximation because otherwise the differences between $\eta_a{}^{B\Delta}(X\neq 8_\theta)$ and $\eta_a{}^{\Delta B}(X=8_\theta)$, etc., in Table III introduce asymmetric terms into A. For example, setting $W_0=7M_B/3$ in D_{8_θ} (3.19), one obtains $\eta_a{}^{B\Delta}(X\neq 8_\theta)=0.8\times\frac{4}{3}$ in place of the linear value $\frac{4}{3}$. By taking $W_0{}^*\approx 8M_B/3$ in D_{10} (3.20), one obtains the same reduction,[23] $\eta_a{}^{\Delta B}(X\neq 10)=0.8\times\frac{2}{3}$ in place of $\frac{2}{3}$. The symmetry between A^{RR^*} and A^{R^*R} can thus be maintained readily enough, except for $\eta_a{}^{B\Delta}(8_\theta)$ and $\eta_a{}^{\Delta B}(10)$ which come out $\approx 0.95\times\frac{4}{3}$ and $0.95\times\frac{2}{3}$, respectively. Similarly, for the value of $W_0{}^*$ considered above, $\eta_a{}^{\Delta\Delta}(X\neq 10)\approx 0.7\times\frac{1}{3}$ in place of the linear value $\frac{1}{3}$, but $\eta_a{}^{\Delta\Delta}(10)\approx 0.9\times\frac{1}{3}$, introducing an asymmetry. In studying nonlinear D, we ignored the asymmetry from this source by using, for example, the 0.8 rather than the 0.95 reduction through A^{RR^*} and A^{R^*R}. Evidently, this approximation could introduce errors of order 15% into the results for nonlinear D. The errors in $\eta^{\Delta\Delta}$ are less important because $\eta^{\Delta\Delta}$ is a small term to start with (this can be traced back to the crossing matrix for ordinary spin, which is responsible for making $\eta^{B\Delta}$ and $\eta^{\Delta B}$ the biggest terms in Table III).

The basic formulas (4.26)–(4.31) define the eigenvalues and eigenvectors of A^{gg} for all P, C, and T. The C dependence is explicit in (4.26)–(4.29). The P dependence is contained in the factors η, n, and n^*. (The values of these factors for $P=-$ are given in Sec. XI.) A^{gg} is independent of T, which affects only the reality properties of the couplings connected by A^{gg}.

In practice, both the evaluation of the elements of A^{RR} (4.26)–(4.29) and the diagonalization to determine its eigenvalues and eigenvectors were performed by computer. The following checks were made on the computer results:

(i) The elements of A^{RR} giving the effect of $\delta R_S(8_\theta,8_\theta)$ and $\delta R_S{}^*(10,10)$ exchange terms on $\delta R_S(8_\theta,8_\theta)$ and $\delta R_S{}^*(10,10)$ possess certain simplifying features. Ordinary conservation laws ensure that only $\alpha=\nu$ and $\beta=\nu'$ contribute to the sums over products of Clebsch-Gordan coefficients in Eqs. (4.26)–(4.29). Furthermore, since for a given S there is only one δR and one δR^* term of this type [except for $\delta R_S(8_\theta,8_\theta)$, where the two terms that arise, $X=8_\theta(s)$ and $X=8_\theta(a)$, are easily distinguished by their symmetry properties], each of these elements of A can be evaluated by considering a single value of β. With these simplifications, it is easy enough to evaluate these elements of A^{RR} by hand. The results are already available in Ref. 2, where precisely the same sums over products of Clebsch-Gordan coefficients[24] were needed to calculate $A^{MM\,(\text{exchange})}$. Thus, we were able to check the computer results for these elements of A^{RR} against results obtained earlier by hand.

(ii) The elements of A^{RR} connecting *parity-violating* couplings also possess simplifying features, especially when expressed in terms of a different basis (see Ref. 3 and Sec. XI of the present paper). These simplifications made it possible to determine by hand the eigenvalues of A^{RR} for parity-violating couplings in Ref. 3, and these eigenvalues provided another check on the computer results.

V. VERTEX SYMMETRY

In a fully satisfactory calculation, the couplings obtained would naturally satisfy the Hermiticity condition

$$G_{\alpha\beta}{}^k = (-1)^{Q_k} \bar{G}_{\beta\alpha}{}^{\bar{k}}. \quad (4.3)$$

This condition says, for instance, that the coupling of the Λ bound state to the $\Sigma^-\pi^+$ channel should equal the coupling of the Σ^- bound state to the $\Lambda\pi^-$ channel, up to a known phase factor. Our approximate calculation fails to ensure this result, however, because it fails to enforce unitarity of the S matrix in all channels fully, and one knows from

$$S = e^{i\int H\,dt} \quad (5.1)$$

that the unitarity of S is related to the Hermiticity of H.

This is a well-known problem.[6,7] Approximations which automatically possess correct Hermiticity properties have been constructed, but always at the cost of some other desirable property which the theory should also have.

We handled the problem as follows:

(i) The couplings with the wrong Hermiticity property [a minus sign in Eq. (4.3)] were projected out of A^{RR}. All eigenvalues and eigenvectors of A^{RR} listed in the rest of the present paper have been so treated,[25] except for Table VIII, which is presented for comparison.

(ii) The eigenvalues and eigenvectors obtained after the projection were compared with those obtained before projection in order to see how serious the incon-

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[23] We take this value of $W_0{}^*$ only by way of illustration. If a different $W_0{}^*$ is used, the reduction of $\eta^{B\Delta}$ and $\eta^{\Delta B}$ can still be kept symmetric by another change in the basis vectors.

[24] A similar connection between the Clebsch-Gordan coefficients in elements of A^{RR} and A^{MR} will be worked out in detail in Sec. VII.

[25] Couplings with the wrong Hermiticity should also be projected out of A^{Mg} and A^{gM}. This was not necessary in the present paper, however, because all we do with A^{Mg} is to show it is very small (Sec. VII), and we use A^{gM} only to estimate the effect of mass shifts on several of the enhanced eigenvectors for coupling shifts (out of which, terms of the wrong Hermiticity have already been projected).

TABLE V. Couplings with $P=+$, $T=+$, $C=-$. Each coupling is expressed as a vector with components $\delta R_S(8_\theta,X)$. The vectors listed here are not orthonormalized.

	$S=8$				$S=27$	
	V_1^8	V_2^8	V_3^8		V_1^{27}	V_2^{27}
$\delta R(8_\theta,1)$	-1	0	-1	$\delta R(8_\theta,8_\theta)$	$-2\sqrt{2}\cos\theta$	$-\sqrt{2}(\cos\theta+\sqrt{5}\sin\theta)$
$\delta R(8_\theta,8_\theta(s))$	$\frac{4}{5}\cos\theta$	$-\sin\theta/\sqrt{2}$	$-\cos\theta/10$	$\delta R(8_\theta,8_{\theta^*})$	$4\sin\theta$	$2(\sin\theta-\sqrt{5}\cos\theta)$
$\delta R(8_\theta,8_{\theta^*}(s))$	$-\frac{4}{5}\sqrt{2}\sin\theta$	$-\cos\theta$	$\sin\theta/5\sqrt{2}$	$\delta R(8_\theta,10)$	$-\sqrt{5}$	$-\sqrt{5}$
$\delta R(8_\theta,8_\theta(a))$	0	$\cos\theta/\sqrt{2}$	$\sin\theta/2$	$\delta R(8_\theta,\overline{10})$	$\sqrt{5}$	0
$\delta R(8_\theta,8_{\theta^*}(a))$	0	$-\sin\theta$	$\cos\theta/\sqrt{2}$	$\delta R(8_\theta,27)$	$-(14)^{1/2}$	$-(7/2)^{1/2}$
$\delta R(8_\theta,10)$	$2/(10)^{1/2}$	1	$1/(10)^{1/2}$	$\delta R(8_\theta,27')$	0	$-(15/2)^{1/2}$
$\delta R(8_\theta,\overline{10})$	$2/(10)^{1/2}$	-1	$1/(10)^{1/2}$	$\delta R^*(10,8_\theta)$	0	0
$\delta R(8_\theta,27)$	$\sqrt{3}/5$	0	$3\sqrt{3}/5$	$\delta R^*(10,8_{\theta^*})$	0	0
$\delta R^*(10,8_\theta)$	0	0	0	$\delta R^*(10,10)$	0	0
$\delta R^*(10,8_{\theta^*})$	0	0	0	$\delta R^*(10,\overline{10})$	0	0
$\delta R^*(10,10)$	0	0	0	$\delta R^*(10,27)$	0	0
$\delta R^*(10,27)$	0	0	0			

sistencies of the model are. Many eigenvalues changed by 0.1 to 0.2 and the eigenvectors changed even more. The leading eigenvalue $A_{S=8}{}^{gg}=0.93$, which is what we use exclusively to determine strong and electromagnetic coupling shifts, changed only from 0.96 to 0.93, however, and the associated eigenvector only by 5%. (By this we mean that the inner product of the eigenvector before projection with the eigenvector after projection is 0.95.) Other eigenvalues which play a leading role in weak couplings are also quite stable under projection. The leading eigenvalues for $P=+$, $\mathcal{C}=-$,[26] $S=8$ couplings change from 0.92 to 0.89 and 0.85 to 0.82, with the associated eigenvectors changing by 7% and by 5%, respectively, and the leading eigenvalue and eigenvector for $P=-$, $S=8$ couplings are unchanged by the projection for reasons mentioned in Sec. XI. Thus the lack of Hermiticity in our model does not appear to have any serious effect on our main conclusions.

In the rest of this section, we give the technical details of the projection. Note that, since $\Delta\Pi$ external states were not considered, the problem of consistency between couplings for $\Delta_\alpha \to B_\beta \Pi_k$ and $B_\beta \to \Delta_\alpha \Pi_{\bar{k}}$ did not arise in our model. Thus we can confine our attention to the consistency between $B_\alpha \to B_\beta \Pi_k$ and $B_\beta \to B_\alpha \Pi_{\bar{k}}$.

For parity-conserving couplings with $T=+$ (i.e., real couplings), condition (4.3) for Hermiticity is the same as condition (4.5) for $C=+$.[27] Thus, the spurious terms to be projected out in this case are the $C=-$ couplings. The number of such terms for each S can be determined by considering some $SU(3)$–charge-conjugation properties of $\bar{B}B\Pi$ couplings. The $\bar{B}B$ states $Y=\mathbf{1}$, $\mathbf{8}_a$, $\mathbf{8}_s$, and $\mathbf{27}$ with $J=0$ all have $\mathcal{C}=+$,[26] and the states $Y=\mathbf{10}+\overline{\mathbf{10}}$ and $\mathbf{10}-\overline{\mathbf{10}}$ provide one $\mathcal{C}=+$ and one $\mathcal{C}=-$ combination. The $J=0$ Π state has $\mathcal{C}=+$. The total $\mathcal{C}$ of the $BB\Pi$ interaction is $\mathcal{C}(\bar{B}B)\times\mathcal{C}(\Pi)\times\Big[$phase factor under interchange of Y, i, and 8, k in $\begin{pmatrix} 8 & Y & S \\ k & i & \sigma \end{pmatrix}\Big]$. One finds that both $S=1$ couplings ($Y=\mathbf{8}_a$ and $\mathbf{8}_s$, combined with the octet of Π's to make $S=1$) are $\mathcal{C}=+$ since $\mathbf{1}$ is one of the symmetric terms in $\mathbf{8}\times\mathbf{8}$. This is the reason why $SU(3)$-symmetric bootstraps automatically produce Hermitian couplings: there is no $\mathcal{C}=-$, $S=1$ coupling. Three of the eight $S=8$ couplings have $\mathcal{C}=-$, however, and two of the six $S=27$ couplings have $\mathcal{C}=-$.[28] For strong perturbations transforming like the $Y=0$, $I=0$ members of representation S, C is equal to $\mathcal{C}$, so we need to project out three $S=8$ couplings and two $S=27$ couplings. The same result holds for electromagnetic perturbations ($Y=0$, $I_Z=0$, $I=0$, 1, 2 members of representation S).

Now A^{RR} is calculated in terms of couplings labelled by the representation of the ΠB state, rather than the $B\bar{B}$ state [the ΠB labeling is most convenient because

TABLE VI. The nonzero elements of $\begin{pmatrix} 10 & 27 & 27 \\ \alpha & \sigma & \nu \end{pmatrix}$ for $\sigma=$ ($Y=0$, $I=0$ member of $\mathbf{27}$). For this value of σ, the α and ν of nonzero elements have the same (Y,I,I_3). The choice of over-all sign is arbitrary.

Y	I	I_3 for α and ν	$\begin{pmatrix} 10 & 27 & 27 \\ \alpha & \sigma & \nu \end{pmatrix}$
1	$\frac{3}{2}$	$\frac{3}{2}$	$(1/14)^{1/2}$
1	$\frac{3}{2}$	$\frac{1}{2}$	$(1/14)^{1/2}$
1	$\frac{3}{2}$	$-\frac{1}{2}$	$(1/14)^{1/2}$
1	$\frac{3}{2}$	$-\frac{3}{2}$	$(1/14)^{1/2}$
0	1	1	$-(3/35)^{1/2}$
0	1	0	$-(3/35)^{1/2}$
0	1	-1	$-(3/35)^{1/2}$
-1	$\frac{1}{2}$	$\frac{1}{2}$	$-(8/35)^{1/2}$
-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-(8/35)^{1/2}$

[26] For a self-charge conjugate representation, $\mathcal{C}$ is equal to the charge-conjugation parity of the $I=0$, $Y=0$ member of the representation. M. Gell-Mann, Phys. Rev. Letters **12**, 155 (1964).

[27] The effects of T, C, P, and Hermitian conjugation are linked through the TCP theorem.

[28] For $S=8$, the three $\mathcal{C}=-$couplings are the antisymmetric octet formed from $Y=\mathbf{8}_s$ and Π, the antisymmetric octet formed from $Y=\mathbf{8}_a$ and Π, and the octet formed from Π and one of the combinations of $Y=\mathbf{10}$ and $\overline{\mathbf{10}}$. For $S=27$, the two $\mathcal{C}=-$ couplings are one of the $\mathbf{27}$'s formed from Π and $Y=27$, and one of the $\mathbf{27}$'s formed from Π and a combination of $Y=\mathbf{10}$ and $\overline{\mathbf{10}}$.

the D functions in Eqs. (3.15)–(3.17) refer to definite representations of IIB]. To obtain the $\mathcal{C}=-$ couplings in the IIB labeling, we write the condition for $C=-$ coupling

$$\delta G_{\alpha\beta}{}^{k}=-(-1)^{Q_k}\delta G_{\beta\alpha}{}^{\bar{k}} \tag{4.5}$$

and express both sides in terms of δR's by means of Eq. (4.30):

$$\sum_X (n_X)^{-1/2}\delta R_S(8_\theta,X)\sum_\nu \begin{pmatrix} 8 & S & X \\ \alpha & \sigma & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & X \\ k & \beta & \nu \end{pmatrix}$$
$$=-(-1)^{Q_k}\sum_{X'}(n_{X'})^{-1/2}\delta R_S(8_\theta,X')$$
$$\times\sum_{\nu'}\begin{pmatrix} 8 & S & X' \\ \beta & \sigma & \nu' \end{pmatrix}\begin{pmatrix} 8 & 8 & X' \\ \bar{k} & \alpha & \nu' \end{pmatrix}. \tag{5.2}$$

We take for σ the $I=0$, $Y=0$ component of S so that $\mathcal{C}$ will have the same value as C. Then we simply apply (5.2) to various values of α, β, k until the required number of linearly independent $\mathcal{C}=-$ couplings is obtained. The $P=+$, $T=+$, $\mathcal{C}=-$ couplings obtained in this way for $S=8$ and $S=27$ are listed in Table V. To project out spurious $\mathcal{C}=-$ couplings from $A_{S=8}{}^{RR}$, for example, one first orthonormalizes the V^8's in Table V (denote the resulting vectors by $\tilde{V}^8$), and then forms

$$(A_{S=8}{}'^{RR})_{ij}=\sum_{kl}P_{ik}(A_{S=8}{}^{RR})_{kl}P_{lj}, \tag{5.3}$$

$$P_{lj}=\delta_{lj}-\sum_{r=1}^{3}(V_r{}^{S=8})_l(V_r{}^{S=8})_j. \tag{5.4}$$

It is the eigenvalues and eigenvectors of A'^{RR} which we will actually apply to strong and electromagnetic and (the $P=+$, $T=+$, $\mathcal{C}=+$ part of) weak interactions.

In Sec. XI, in connection with weak interactions, we will give the alternative procedure to be followed when one is *interested* in $\mathcal{C}=-$ couplings, and wishes to eliminate spurious $\mathcal{C}=+$ couplings which got mixed into $A^{RR(\mathcal{C}=-)}$ as a result of approximations.

VI. RESULTS FOR A^{RR}

We are now ready to determine the eigenvalues and eigenvectors of A^{RR} by calculating the elements in Eqs. (4.26)–(4.29), "purifying" with the projection of Sec. V, and then diagonalizing the resulting matrix. The values of n_X and η needed to calculate (4.28)–(4.29) are given in Tables III and IV. The requisite $SU(3)$ Clebsch-Gordan coefficients are given explicitly in convenient tables by McNamee and Chilton,[29] who followed the conventions of de Swart.[16] The only coefficients that could not be obtained directly from these references were

$$\begin{pmatrix} 10 & S=27 & X=27 \\ \alpha & \sigma & \nu \end{pmatrix},$$

[29] P. McNamee and F. Chilton, Rev. Mod. Phys. **36**, 1005 (1964).

TABLE VII. The matrix A^{RR} for $P=+$, $C=+$, $T=+$, $S=8$ couplings calculated with $\theta=-28°$ and linear D functions, *before* the projection of Sec. V is made.

	$\delta R(8_\theta,1)$	$\delta R(8_\theta,8_\theta(s))$	$\delta R(8_\theta,8_{\theta^*}(s))$	$\delta R(8_\theta,8_\theta(a))$	$\delta T(8_\theta,8_{\theta^*}(a))$	$\delta R(8_\theta,10)$	$\delta R(8_\theta,\overline{10})$	$\delta R(8_\theta,27)$	$\delta R^*(10,8_\theta)$	$\delta R^*(10,8_{\theta^*})$	$\delta R^*(10,10)$	$\delta R^*(10,27)$
$\delta R(8_\theta,1)$	−0.064	0.050	−0.017	0.021	0.045	−0.048	0.048	−0.067	−0.138	0.259	0.027	0.000
$\delta R(8_\theta,8_\theta(s))$	0.050	−0.078	0.019	−0.086	−0.064	0.044	0.037	−0.138	0.126	−0.237	−0.456	0.000
$\delta R(8_\theta,8_{\theta^*}(s))$	−0.017	0.019	0.073	0.082	0.061	−0.078	−0.000	−0.161	−0.220	0.415	0.117	−0.000
$\delta R(8_\theta,8_\theta(a))$	0.021	−0.086	0.082	−0.000	0.000	0.070	0.083	0.006	0.199	0.105	0.000	0.744
$\delta R(8_\theta,8_{\theta^*}(a))$	0.045	−0.064	0.061	0.000	0.000	−0.000	−0.146	0.090	−0.001	−0.000	0.000	−0.161
$\delta R(8_\theta,10)$	−0.048	0.044	−0.078	0.070	−0.000	−0.139	−0.077	0.043	−0.395	0.000	−0.389	0.284
$\delta R(8_\theta,\overline{10})$	0.048	0.037	−0.000	0.083	−0.146	−0.077	−0.016	0.009	−0.220	−0.328	0.389	0.284
$\delta R(8_\theta,27)$	−0.067	−0.138	−0.161	0.006	0.090	0.043	0.009	−0.111	0.123	−0.232	0.275	0.000
$\delta R^*(10,8_\theta)$	−0.138	0.126	−0.220	0.199	−0.001	−0.395	−0.220	0.123	0.139	−0.000	0.137	−0.100
$\delta R^*(10,8_{\theta^*})$	0.259	−0.237	0.415	0.105	−0.000	0.000	−0.328	−0.232	−0.000	−0.039	−0.021	0.015
$\delta R^*(10,10)$	0.027	−0.456	0.117	0.000	0.000	−0.389	0.389	0.275	0.137	−0.021	0.000	0.060
$\delta R^*(10,27)$	0.000	0.000	−0.000	0.744	−0.161	0.284	0.284	0.000	−0.100	0.015	0.060	0.066

which are needed for $S=27$ perturbations on the $\Delta B\Pi$ couplings. Actually, since the A matrix is independent of the "direction" the perturbation takes in $SU(3)$ space, we only need the coefficients for one value of σ. With the aid of the Casimir operators, we calculated by hand the coefficients for $\sigma=(Y=0,\ I=0$ member of representation $S=27)$. The coefficients are given in Table VI (note that the overall sign is arbitrary).

As an example, we give A^{RR} for $P=+, C=+, T=+$, $S=8$ couplings in some detail. First consider $\theta=-28°$ $(F/D\approx 0.4)$ and linear D functions. The 12×12 matrix $A_{S=8}{}^{RR}$ *before* the projection of Sec. V is given in Table VII.[30] The eigenvalues and eigenvectors obtained by diagonalizing it are given in Table VIII. This table is included mainly for comparison with Table IX, where we give the eigenvalues and eigenvectors of the matrix $A_{S=8}{}^{RR}$ that remains *after* the projection of Sec. V. It is this latter table which is used in the ensuing discussions of strong and electromagnetic perturbations. Note that, as mentioned in the previous section, the leading eigenvalue and eigenvector of Table IX are almost unchanged from Table VIII, although other eigenvalues and eigenvectors change considerably. The reasons for this fortunate behavior are not understood.

Next we present two tables which illustrate the degree of sensitivity of A^{RR} to the parameters of our model. Table X gives the eigenvalues and eigenvectors of A^{RR} for the same conditions as Table IX, except with $\theta=-28°$ changed to $\theta=-40°$. This corresponds to changing F/D from ≈ 0.4 to ≈ 0.6. One sees in particular that the leading eigenvalue and eigenvector are quite insensitive to this change.

Table XI gives the eigenvalues and eigenvectors of A^{RR} for the same conditions as Table IX, except with some effects of nonlinear D taken into account. Specifically, the D functions discussed following Eq. (4.31) are used. The factors η are taken to be $\eta^{BB}=-\frac{1}{3}$, $\eta^{B\Delta}=0.8\times\frac{4}{3}$, $\eta^{\Delta B}=0.8\times\frac{2}{3}$, and $\eta^{\Delta\Delta}=0.67\times\frac{1}{3}$. The asymmetries introduced by nonlinear D are not taken into account here; their effect was estimated in Sec. IV. The striking result obtained in Table XI is that the eigenvalues are uniformly reduced by about 20%, while the eigenvectors are essentially unchanged. This reflects the fact that A is dominated by the off-diagonal terms A^{RR^*} and A^{R^*R}, which are uniformly reduced by 20% when nonlinear D is taken into account. The dominance of A^{RR^*} and A^{R^*R} can be traced to the spin crossing matrix contained in the factor η, which has the values $\frac{4}{3}$ for $\eta^{B\Delta}$, $\frac{2}{3}$ for $\eta^{\Delta B}$, but only $-\frac{1}{3}$ for η^{BB}, and $\frac{1}{3}$ for $\eta^{\Delta\Delta}$. The dominance can be seen clearly in Table VII, where all the individual elements of A are displayed.

To summarize the results of varying the parameters of our model: The leading eigenvector of $A_{S=8}{}^{RR}$, which controls the ratios of strong coupling perturbations and electromagnetic coupling perturbations, is not very sensitive to reasonable variations of the F/D ratio and the form of the D functions.

The calculation of elements of A^{RR} connecting other types of coupling proceeds in a similar way. The results for weak couplings will be given in Sec. XI. The results for $P=+$, $C=+$, $T=+$, $S=27$ couplings, calculated with linear D functions are given in Tables XII (eigenvalues and eigenvectors of the $A_{27}{}^{RR}$ obtained *before* the projection of Sec. V) and XIII (eigenvalues and eigenvectors of $A_{27}{}^{RR}$ *after* the projection of Sec. V). The point to note here is that $A_{27}{}^{RR}$ does have an eigenvalue near one, although the precise values of the eigenvalue and eigenvector are changed substantially by the projection of Sec. V and are therefore rather unreliable.

VII. CALCULATION OF A^{MR}

The dispersion relation for first-order mass shifts has been given in Sec. III:

$$\delta M_i=\frac{-1}{2\pi i G^2[D_i'(M)]^2}\int_C \frac{D_i\delta T_i D_i}{W'-M}dW'. \quad (3.17)$$

In studying the effects of coupling shifts on mass shifts, one considers expressions of the form

$$\delta T_i \approx C_{ij}\delta R_j/(W-M^x), \quad (7.1)$$

where M^x is the position of the exchange singularity and C_{ij} is a numerical coefficient. Inserting (7.1) into (3.17), one finds

$$\frac{\delta M_i}{M}=\frac{C_{ij}\delta R_j}{G^2}\frac{[D_i(M^x)]^2}{[D_i'(M)]^2 M(M^x-M)}. \quad (7.2)$$

For linear D, the factors to the right of δR reduce to $(M^x-M)/M$, and $A^{M_iR_i}$, which is the coefficient of δR_j, takes the form

$$A^{M_iR_j}=C_{ij}(M^x-M)/G^2M. \quad (7.3)$$

In Ref. 2 it was observed that A^{MR} is small in the static model because $(M^x-M)/M$ is a small factor (e.g., for the baryon-exchange contribution to M_B, $M^x=M_B=M$ and the factor vanishes; for the Δ-exchange contribution to M_B, $M^x=2M_B-M_\Delta$ and the factor is about $-\frac{1}{3}$). As mentioned in the Introduction, we took advantage of this fact and ignored A^{MR}. This allowed us to study mass shifts before coupling shifts were studied, and then made possible solutions of the form (1.3) and (1.4) for coupling shifts.

In the present section we shall estimate A^{MR} more carefully and show that it is indeed very small, its elements not exceeding about 0.1.

The method for calculating C_{ij} in Eq. (7.1) has already been described in detail in Sec. IV. For example, we can take over Eq. (4.17) for the effect of a coupling shift on the amplitude $\delta T(8_\theta\rightarrow X')$ with only the following minor changes: (i) For baryon mass shifts, the final state is specialized to $X'=8_\theta$, since in our formalism a mass

[30] We are greatly indebted to Barbara Zimmerman, who programmed and performed the computer calculations of all elements, eigenvalues, and eigenvectors of A^{RR} given in this paper.

TABLE VIII. The eigenvalues and eigenvectors obtained by diagonalizing the *unprojected* matrix A^{RR} of Table VII. The components of each eigenvector are listed in the column under the corresponding eigenvalue. The eigenvectors are normalized to 1.

	Eigenvalues											
	0.962	0.839	0.691	0.478	0.199	−0.010	−0.040	−0.305	−0.617	−0.726	−0.751	−0.890
	Eigenvectors											
$\delta R(8_\theta,1)$	0.031	0.117	0.206	−0.085	−0.122	−0.859	0.100	0.234	−0.109	0.028	0.287	−0.143
$\delta R(8_\theta,8_\theta(s))$	−0.110	0.230	−0.361	0.120	−0.464	−0.013	0.406	0.349	0.133	−0.293	−0.178	0.387
$\delta R(8_\theta,8_{\theta^*}(s))$	0.081	0.205	0.521	−0.108	−0.213	0.484	0.318	0.354	−0.089	0.090	0.296	−0.236
$\delta R(8_\theta,8_\theta(a))$	0.557	0.032	0.036	0.410	0.064	−0.014	0.140	−0.079	−0.335	0.363	0.098	0.482
$\delta R(8_\theta,8_{\theta^*}(a))$	−0.148	−0.004	0.129	0.044	0.477	−0.089	0.767	−0.308	0.149	−0.068	−0.099	−0.040
$\delta R(8_\theta,10)$	0.163	0.398	−0.245	−0.129	0.388	0.048	−0.118	0.167	0.554	0.091	0.451	0.149
$\delta R(8_\theta,\overline{10})$	0.341	−0.261	−0.122	−0.438	−0.336	−0.053	0.206	−0.112	0.370	0.492	−0.216	−0.098
$\delta R(8_\theta,27)$	0.024	−0.342	−0.120	−0.015	0.441	−0.000	0.039	0.738	−0.109	0.150	−0.296	−0.060
$\delta R^*(10,8_\theta)$	−0.129	−0.365	−0.032	0.700	−0.177	−0.005	0.023	0.058	0.405	0.145	0.252	−0.270
$\delta R^*(10,8_{\theta^*})$	0.016	0.277	0.573	0.194	0.018	−0.110	−0.232	0.071	0.402	0.060	−0.538	0.171
$\delta R^*(10,10)$	0.166	−0.571	0.324	−0.196	−0.001	−0.000	−0.024	0.036	0.204	−0.455	0.241	0.438
$\delta R^*(10,27)$	0.676	0.091	−0.106	0.139	0.032	−0.010	0.015	−0.008	0.048	−0.512	−0.168	−0.456

TABLE IX. The eigenvalues and eigenvectors obtained by diagonalizing the matrix A^{RR} *after* the projection of Sec. V. Here, A^{RR} again refers to couplings with $P=+$, $C=+$, $T=+$, $S=8$, and $\theta=-28°$, and linear D functions were used. The three eigenvectors with zero eigenvalue refer to $C=-$ couplings.

	Eigenvalues											
	0.931	0.655	0.489	0.000	0.000	−0.000	−0.018	−0.062	−0.259	−0.493	−0.706	−0.761
	Eigenvectors											
$\delta R(8_\theta,1)$	0.070	0.076	−0.199	−0.441	−0.082	−0.504	0.415	−0.145	0.500	−0.042	−0.178	0.122
$\delta R(8_\theta,8_\theta(s))$	−0.251	−0.268	0.030	0.562	−0.270	−0.158	0.280	−0.294	0.258	0.033	0.441	−0.146
$\delta R(8_\theta,8_{\theta^*}(s))$	−0.024	0.507	0.282	0.068	−0.550	+0.238	0.129	−0.066	0.041	0.436	−0.169	0.232
$\delta R(8_\theta,8_\theta(a))$	0.564	0.011	0.271	0.239	0.064	−0.318	0.020	−0.166	−0.107	0.192	−0.317	−0.517
$\delta R(8_\theta,8_{\theta^*}(a))$	−0.086	0.255	0.099	0.032	0.482	0.173	0.571	−0.431	−0.338	−0.117	0.009	0.112
$\delta R(8_\theta,10)$	0.048	0.224	−0.284	0.651	0.187	−0.160	−0.019	0.247	0.143	−0.162	−0.357	0.380
$\delta R(8_\theta,\overline{10})$	0.298	−0.186	−0.307	0.024	−0.425	0.472	0.240	−0.097	−0.032	−0.471	−0.254	−0.144
$\delta R(8_\theta,27)$	0.119	−0.079	0.010	0.025	0.404	0.532	0.026	0.006	0.650	0.296	−0.036	−0.135
$\delta R^*(10,8_\theta)$	−0.021	−0.446	0.681	0.000	0.000	0.000	−0.039	−0.086	0.128	−0.261	−0.285	0.400
$\delta R^*(10,8_{\theta^*})$	0.005	0.543	0.340	0.000	0.000	0.000	−0.129	0.095	0.290	−0.589	0.243	−0.267
$\delta R^*(10,10)$	0.300	−0.103	0.182	−0.000	−0.000	−0.000	0.523	0.696	−0.092	0.056	0.300	0.073
$\delta R^*(10,27)$	0.639	0.042	−0.083	0.000	0.000	0.000	−0.232	−0.319	0.024	−0.003	0.467	0.454

TABLE X. The eigenvalues and eigenvectors of A^{RR} for the same conditions as in Table IX, except with $\theta=-40°$ instead of $\theta=-28°$.

	Eigenvalues											
	0.944	0.598	0.524	0.059	0.000	−0.000	−0.000	−0.082	−0.318	−0.476	−0.738	−0.746
	Eigenvectors											
$\delta R(8_\theta,1)$	0.100	−0.008	0.142	0.537	−0.587	0.331	0.003	−0.252	0.348	0.111	0.073	0.151
$\delta R(8_\theta,8_\theta(s))$	−0.235	−0.393	−0.081	0.218	0.313	0.217	−0.452	−0.280	0.099	0.114	−0.242	−0.473
$\delta R(8_\theta,8_{\theta^*}(s))$	−0.037	0.459	−0.205	0.179	0.475	0.442	0.138	−0.110	0.205	−0.438	−0.028	0.157
$\delta R(8_\theta,8_\theta(a))$	0.555	−0.039	−0.232	−0.209	−0.047	0.149	−0.380	0.020	0.088	−0.170	0.576	−0.229
$\delta R(8_\theta,8_{\theta^*}(a))$	0.076	0.253	−0.205	0.434	−0.042	−0.471	−0.164	−0.427	−0.481	−0.170	0.059	−0.020
$\delta R(8_\theta,10)$	0.053	0.144	0.343	0.110	0.287	−0.128	−0.619	0.166	0.108	0.211	0.034	0.526
$\delta R(8_\theta,\overline{10})$	0.325	−0.244	0.206	0.238	0.445	0.154	0.417	−0.139	−0.265	0.397	0.296	0.053
$\delta R(8_\theta,27)$	0.092	−0.110	−0.019	0.151	0.213	−0.601	0.214	−0.052	0.687	−0.071	0.096	−0.111
$\delta R^*(10,8_\theta)$	−0.062	−0.403	−0.712	−0.034	−0.000	0.000	−0.000	−0.058	0.030	0.130	−0.012	0.550
$\delta R^*(10,8_{\theta^*})$	0.015	0.556	−0.342	−0.082	−0.000	−0.000	−0.000	−0.019	0.139	0.705	−0.043	−0.216
$\delta R^*(10,10)$	0.299	−0.047	−0.208	0.487	−0.000	0.000	0.000	0.714	−0.096	−0.039	−0.283	−0.155
$\delta R^*(10,27)$	0.638	−0.014	0.067	−0.246	0.000	−0.000	−0.000	−0.313	0.046	−0.027	−0.646	0.089

TABLE XI. The eigenvalues and eigenvectors of A^{RR} for the same conditions as in Table IX, except with linear D replaced by curved D functions, as described in the text.

	Eigenvalues											
	0.748	0.529	0.370	0.000	0.000	−0.000	−0.015	−0.050	−0.242	−0.416	−0.571	−0.633
	Eigenvectors											
$\delta R(8_\theta,1)$	0.070	0.043	−0.212	−0.596	−0.201	−0.245	0.367	−0.226	0.502	0.056	−0.130	0.180
$\delta R(8_\theta,8_\theta(s))$	−0.252	−0.254	0.072	0.358	0.244	−0.475	0.234	−0.324	0.258	0.037	0.372	−0.290
$\delta R(8_\theta,8_{\theta^*}(s))$	−0.027	0.553	0.193	−0.020	0.603	−0.018	0.113	−0.092	−0.032	0.452	−0.066	0.241
$\delta R(8_\theta,8_\theta(a))$	0.572	0.069	0.260	0.116	−0.164	−0.349	0.006	−0.152	−0.134	0.160	−0.447	−0.412
$\delta R(8_\theta,8_{\theta^*}(a))$	−0.090	0.287	0.061	0.236	−0.350	0.291	0.491	−0.513	−0.297	−0.189	0.043	0.096
$\delta R(8_\theta,10)$	0.047	0.163	−0.315	0.574	−0.153	−0.362	−0.000	0.223	0.160	−0.089	−0.227	0.500
$\delta R(8_\theta,\overline{10})$	0.302	−0.231	−0.271	0.062	0.587	0.236	0.211	−0.139	0.051	−0.465	−0.303	−0.020
$\delta R(8_\theta,27)$	0.121	−0.080	0.014	0.335	−0.126	0.564	0.022	−0.002	0.587	0.405	−0.063	−0.133
$\delta R^*(10,8_\theta)$	−0.012	−0.329	0.756	0.000	0.000	−0.000	−0.055	−0.092	0.161	−0.188	−0.149	0.472
$\delta R^*(10,8_{\theta^*})$	0.003	0.582	0.235	0.000	−0.000	−0.000	−0.140	0.143	0.409	−0.551	0.118	−0.281
$\delta R^*(10,10)$	0.296	−0.060	0.180	−0.000	0.000	0.000	0.637	0.608	−0.075	0.018	0.304	0.001
$\delta R^*(10,27)$	0.631	0.029	−0.085	0.000	−0.000	−0.000	−0.284	−0.281	0.012	−0.002	0.599	0.270

TABLE XII. The eigenvalues and eigenvectors obtained by diagonalizing the *unprojected* matrix A^{RR} for $P=+$, $C=+$, $T=+$, $S=27$ couplings. Linear D functions and $\theta=-28°$ were used in the calculation.

	Eigenvalues										
	0.963	0.796	0.583	0.180	0.086	0.014	−0.212	−0.342	−0.551	−0.762	−0.887
	Eigenvectors										
$\delta R(8_\theta,8_\theta)$	0.010	0.093	−0.280	0.127	0.025	0.840	0.054	0.099	0.349	0.223	−0.056
$\delta R(8_\theta,8_{\theta^*})$	0.028	0.195	−0.435	−0.342	0.086	−0.303	−0.289	−0.451	0.444	0.260	−0.042
$\delta R(8_\theta,10)$	−0.328	0.208	0.281	−0.607	0.039	0.255	−0.092	0.001	−0.009	−0.158	0.546
$\delta R(8_\theta,\overline{10})$	0.457	0.175	0.277	−0.033	0.324	0.250	−0.058	−0.494	−0.056	−0.445	−0.252
$\delta R(8_\theta,27)$	−0.026	−0.588	−0.001	−0.255	0.212	0.209	−0.239	−0.207	−0.458	0.418	−0.115
$\delta R(8_\theta,27')$	0.391	−0.036	0.065	−0.438	−0.259	0.012	−0.376	0.545	0.106	−0.093	−0.349
$\delta R^*(10,8_\theta)$	−0.259	0.169	0.227	0.413	0.345	−0.026	−0.725	0.165	0.063	0.051	0.001
$\delta R^*(10,8_{\theta^*})$	−0.056	−0.124	−0.357	−0.164	0.745	−0.107	0.213	0.375	0.035	−0.266	−0.030
$\delta R^*(10,10)$	0.067	0.544	0.295	−0.137	0.244	−0.077	0.293	0.143	−0.183	0.573	−0.240
$\delta R^*(10,\overline{10})$	−0.027	−0.433	0.551	−0.026	0.142	−0.087	0.199	−0.020	0.645	0.142	−0.046
$\delta R^*(10,27)$	0.672	−0.045	−0.015	0.141	0.131	−0.059	−0.067	0.109	0.001	0.227	0.660

TABLE XIII. The eigenvalues and eigenvectors obtained by diagonalizing the matrix A^{RR} for $P=+$, $C=+$, $T=+$, $S=27$ couplings *after* the projection of Sec. V. Linear D functions and $\theta=-28°$ were used in the calculation. The zero eigenvalues refer to the two $C=-$ couplings.

	Eigenvalues										
	0.878	0.627	0.465	0.085	0.000	−0.000	−0.086	−0.326	−0.392	−0.519	−0.862
	Eigenvectors										
$\delta R(8_\theta,8_\theta)$	−0.061	0.491	−0.247	−0.052	0.544	−0.029	−0.113	−0.113	−0.433	0.392	−0.157
$\delta R(8_\theta,8_{\theta^*})$	0.125	0.244	0.272	0.101	−0.143	0.771	−0.102	−0.300	0.219	0.274	−0.019
$\delta R(8_\theta,10)$	0.392	−0.205	−0.401	−0.025	0.220	0.356	0.252	0.009	−0.220	−0.154	0.574
$\delta R(8_\theta,\overline{10})$	−0.214	0.072	−0.441	0.296	−0.464	−0.006	−0.071	−0.537	−0.292	−0.250	−0.081
$\delta R(8_\theta,27)$	−0.384	−0.285	0.004	0.176	0.573	0.303	−0.066	−0.100	0.135	−0.457	−0.277
$\delta R(8_\theta,27')$	−0.285	−0.031	−0.182	−0.285	−0.299	0.429	0.012	0.587	−0.341	−0.016	−0.258
$\delta R^*(10,8_\theta)$	0.319	−0.131	−0.097	0.384	0.000	0.000	−0.797	0.288	−0.060	−0.007	0.004
$\delta R^*(10,8_{\theta^*})$	−0.047	0.118	0.328	0.736	−0.000	−0.000	0.399	0.273	−0.311	0.017	0.043
$\delta R^*(10,10)$	0.291	0.170	−0.528	0.228	−0.000	−0.000	0.293	0.196	0.498	0.051	−0.423
$\delta R^*(10,\overline{10})$	−0.204	−0.655	−0.157	0.151	−0.000	0.000	0.071	−0.112	−0.000	0.678	−0.045
$\delta R^*(10,27)$	−0.566	0.280	−0.231	0.139	−0.000	0.000	−0.128	0.208	0.382	0.100	0.557

TABLE XIV. The matrix A^{MR} for $P=+$, $C=+$, $T=+$, $S=8$ couplings calculated with $\theta=-28°$ and linear D functions, using Table VII and Eqs. (7.9)–(7.12).

	$\delta R(8_\theta,1)$	$\delta R(8_\theta,8_\theta(s))$	$\delta R(8_\theta,8_\theta{}^*(s))$	$\delta R(8_\theta,8_\theta(a))$	$\delta R(8_\theta,8_\theta{}^*(a))$	$\delta R(8_\theta,10)$	$\delta R(8_\theta,\overline{10})$	$\delta R(8_\theta,27)$	$\delta R^*(10,8_\theta)$	$\delta R^*(10,8_\theta{}^*)$	$\delta R^*(10,10)$	$\delta R^*(1,27)$
$\delta M^B(s)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	−0.04	−0.08	0.00
$\delta M^B(a)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.02	0.00	0.12
δM^Δ	0.00	−0.08	0.02	0.00	0.00	−0.06	0.06	0.05	0.05	−0.01	0.00	0.02

shift changes the position of the direct-channel pole, but not the residue which controls its coupling to the final state. (ii) In deriving the operator for projecting out the reaction $8_\theta \to X'$ in Sec. IV, we started with the direct channel amplitude

$$-\Big(\sum_\beta G_{\beta\nu_2}{}^{\nu_1}\delta G_{\nu_4\beta}{}^{\bar\nu_3}\Big)(W-M^B)^{-1}.$$

The δG factor was expressed as

$$\delta G_{\nu_4\beta}{}^{\bar\nu_3}=Z_{X'}\delta G_S(X')\sum_\nu \begin{pmatrix} 8 & S & X' \\ \nu_4 & \sigma & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & X' \\ \bar\nu_3 & \beta & \nu \end{pmatrix}, \quad (3.9)$$

which, by (4.5), can be re-expressed as

$$\delta G_{\nu_4\beta}{}^{\bar\nu_3}=Z_{X'}(-1)^{Q_{\nu_3}}\delta G_S(X')\times\sum_\nu \begin{pmatrix} 8 & S & X' \\ \beta & \sigma & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & X' \\ \nu_3 & \nu_4 & \nu \end{pmatrix} \quad (7.4)$$

(we are only considering couplings with $C=+$ here). For mass shifts, the direct-channel amplitude changes to

$$-\Big(\sum_{\beta\nu} G_{\beta\nu_2}{}^{\nu_1}\delta M_{\beta\nu}{}^B G_{\nu_4\nu}{}^{\bar\nu_3}\Big)(W-M^B)^{-2}, \quad (7.5)$$

where

$$\delta M_{\beta\nu}{}^B=Z'\delta M_S\begin{pmatrix} 8 & S & 8 \\ \beta & \sigma & \nu \end{pmatrix} \quad (7.6)$$

is the baryon mass-shift matrix for a given S, and Z' is an arbitrary normalization factor. For mass shifts, then, factor (7.4) is replaced by

$$\sum_\nu \delta M_{\beta\nu}{}^B G_{\nu_4\nu}{}^{\bar\nu_3}=\sum_\nu \left[Z'\delta M_S\begin{pmatrix} 8 & S & 8 \\ \beta & \sigma & \nu \end{pmatrix}\right]\times\left[G\begin{pmatrix} 8 & 8 & 8_\theta \\ \bar\nu_3 & \nu & \nu_4 \end{pmatrix}\right] \quad (7.7)$$

which, by (4.5), can be replaced by

$$\sum_\nu \delta M_{\beta\nu}{}^B G_{\nu_4\nu}{}^{\bar\nu_3}=Z'(-1)^{Q_{\nu_3}}\delta M_S G\times\sum_\nu \begin{pmatrix} 8 & S & 8 \\ \beta & \sigma & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_3 & \nu_4 & \nu \end{pmatrix}. \quad (7.8)$$

The projection operator resulting from (7.8) is, of course, the same as that resulting from (7.4), except that Z' replaces $Z_{X'}$ in (4.17).

In view of the foregoing, one can obtain the elements of A^{MR} from the corresponding elements of A^{RR} [Eqs. (4.26)–(4.29)] by the following prescription (we confine our remarks to linear D for simplicity):

(i) $A^{R(8_\theta,X'=8_\theta)R}$ and $A^{R(8_\theta,X'=8_\theta)R^*}$ give A^{M^BR} and $A^{M^BR^*}$ respectively, while $A^{R^*(10,X'=10)R}$ and $A^{R^*(10,X'=10)R^*}$ give $A^{M^\Delta R}$ and $A^{M^\Delta R^*}$, subject to the modifications below.

(ii) Z' and Z^* replace Z_{8_θ} and $Z_{10}{}^*$. In order to make contact with the results of Ref. 2 for mass shifts, we must use $Z'=(8)^{-1/2}$ and $Z^{*\prime}=(10)^{-1/2}$, which correspond to the convention of Sec. V, Ref. 2. These values are to be compared with $Z_{8_\theta}=(4)^{-1/2}$ and $Z_{10}{}^*=(10)^{-1/2}$ (Table IV). As a result of the change from Z to Z', A^{M^BR} and $A^{M^BR^*}$ are multiplied by $\sqrt{2}$ relative to A^{RR} and A^{RR^*}.

(iii) Expression (7.1) for δT_i is fed into Eq. (3.17) for δM_i rather than Eqs. (3.15) and (3.16) for δR. For linear D, one finds (7.3) $A^{M_iR_j}=C_{ij}(M^x-M)M^{-1}G^{-2}$ as compared with $A^{R_iR_j}=-C_{ij}$. Therefore, A^{M^BR} is multiplied by zero relative to A^{RR}, $A^{M^BR^*}$ is multiplied by $(M^\Delta-M^B)(M^B)^{-1}G^{-2}$, $A^{M^\Delta R}$ by $(M^\Delta-M^B)(M^\Delta)^{-1}\times(G^*)^{-2}$, and $A^{M^\Delta R^*}$ by $2(M^\Delta-M^B)(M^\Delta)^{-1}(G^*)^{-2}$. Putting all the factors together, we have

$$A^{M^BR(8_\theta,X)}=0\,, \quad (7.9)$$

$$A^{M^BR^*(10,X)}=[2(M^\Delta-M^B)/G^2M^B]A^{R(8_\theta,8_\theta)R^*(10,X)}\,, \quad (7.10)$$

$$A^{M^\Delta R(8_\theta,X)}=[(M^\Delta-M^B)/G^{*2}M^\Delta]A^{R^*(10,10)R(8_\theta,X)}\,, \quad (7.11)$$

$$A^{M^\Delta R^*(10,X)}=[2(M^\Delta-M^B)/G^{*2}M^\Delta]A^{R^*(10,10)R^*(10,X)}\,. \quad (7.12)$$

The numerical values of the coefficients in (7.9)–(7.12) are, using (2.4) and (2.5), about 0, $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{6}$, respectively. The matrix A^{MR} obtained with these coefficients is tabulated in Table XIV. One sees that the largest element has magnitude 0.12, and for nonlinear D the value would be even smaller. Thus the approximation of neglecting A^{MR} was quite well justified.

It is instructive to see how the dominant coupling shifts drive the mass shifts by forming $f_i=\sum_j A_{ij}{}^{MR}X_j{}^R$, where $X_j{}^R$ is the leading eigenvector for coupling shifts (the first column of Table IX). We find

$$\begin{aligned} f(M^B(s))&=-0.023\,,\\ f(M^B(a))&=0.078\,,\\ f(M^\Delta)&=0.049\,. \end{aligned} \quad (7.13)$$

TABLE XV. The matrix $A^{RM(\text{exch})}$ for $P=+$, $C=+$, $T=+$, $S=1$ couplings estimated with $\theta=-28°$ and *curved* D functions of the form Eq. (8.6) with $W_0\approx(7/3)M^\Delta$ for $J=\frac{3}{2}^+$, $W_0\approx(7/3)M^B$ for $J=\frac{1}{2}^+$.

	δM^B	δM^Δ
$\delta R(8_\theta,8_\theta)$	0	0.29
$\delta R(8_\theta,8_{\theta*})$	0.02	-0.19
$\delta R^*(10,10)$	0.32	0.04

By comparison, the leading mass eigenvector [Eq. (5.58) of Ref. 2] is X_i^M, with

$$X(M^B(s))=-0.15\,,\quad X(M^B(a))=0.60\,,\quad X(M^\Delta)=0.78\,. \tag{7.14}$$

Since the ratios in (7.13) and (7.14) are similar, the main effect of the coupling shifts is to drive the leading mass shift. The magnitude of the effect is small, $\sum_{ij} X_i^M \times A_{ij}^{MR} X_j^R \approx 0.09$, but even if we have underestimated the magnitude of A^{MR}, the fact that it drives mainly the leading mass shift ensures that including it would not change the determination of the mass-shift ratios by much.

VIII. CALCULATION OF A^{RM}

Our remaining task, before turning to the experimental consequences of the model, is the study of the influence of changes in the mass of exchanged and external particles on the coupling shifts.

Like the other elements of A, A^{RM} is composed of a product of Clebsch-Gordan coefficients and dynamical factors. Proceeding in close parallel to Sec. VII, we consider first changes in the mass of an exchanged particle and find that the Clebsch-Gordan product for $A^{R(8_\theta\to X)M(B\,\text{exch})}$ is the same as for $A^{R(8_\theta\to X)R(8_\theta\to 8_\theta)}$, while $A^{R^*(10\to X)M(\Delta\,\text{exch})}$ is the same as for $A^{R^*(10\to X)R^*(10\to 10)}$. Consequently, $A^{RM(\text{exch})}$ differs from A^{RR} by (a) normalization factors, and (b) dynamical factors. One finds, in the present case, that one must replace the normalization factor Z_X by Z', as in Sec. VII. This replacement has no effect on the elements of A^{RM^Δ}, while decreasing the elements of A^{RM^B} uniformly by a factor $\sqrt{2}$.

To estimate the dynamical factors in $A^{RM(\text{exch})}$, one may consider as an example the equation

$$\delta R^*(10\to 27)=\frac{1}{D_{10}'(M^\Delta)D_{27}(M^\Delta)} \times\frac{1}{2\pi i}\int\frac{D_{10}(W')\delta T(10\to 27)D_{27}(W')dW'}{W'-M^\Delta}. \tag{8.1}$$

In the study of the effect of a mass shift on δR, δT has the form

$$\delta T(10\to 27)\sim C\delta M/(W-M)^2. \tag{8.2}$$

TABLE XVI. The matrix $A^{RM(\text{exch})}$ for $P=+$, $C=+$, $T=+$, $S=8$ couplings estimated with $\theta=-28°$ and *curved* D functions of the form Eq. (8.6) with $W_0\approx(7/3)M^\Delta$ for $J=\frac{3}{2}^+$, $W_0\approx(7/3)M^B$ for $J=\frac{1}{2}^+$.

	$\delta M^B(s)$	$\delta M^B(a)$	δM^Δ
$\delta R(8_\theta,1)$	0.08	0.03	0.03
$\delta R(8_\theta,8_\theta(s))$	0	0	-0.21
$\delta R(8_\theta,8_{\theta*}(s))$	0.03	0.12	0.13
$\delta R(8_\theta,8_\theta(a))$	0	0	0
$\delta R(8_\theta,8_{\theta*}(a))$	-0.10	0	0
$\delta R(8_\theta,10)$	0.07	0.11	-0.44
$\delta R(8_\theta,\overline{10})$	0.06	0.13	0.44
$\delta R(8_\theta,27)$	-0.21	0.01	0.31
$\delta R^*(10,8_\theta)$	0.19	0.30	0.16
$\delta R^*(10,8_{\theta*})$	-0.36	0.16	-0.02
$\delta R^*(10,10)$	-0.18	0	0
$\delta R^*(10,27)$	0	1.12	0.07

In the linear-D approximation, we have

$$D_{10}(W')=(W'-M)D_{10}'(M^\Delta)\,, \tag{8.3}$$

$$D_{27}(W')=D_{27}(M^\Delta)\,, \tag{8.4}$$

and, as a result, Eq. (8.1) gives

$$\delta R^*(10\to 27)=\frac{C\delta M}{2\pi i}\int\frac{dW'}{(W'-M)^2}=0\,. \tag{8.5}$$

As usual, we must check the sensitivity of this result to the form of the D function. Using the better expression,

$$D_{10}(W')=(W'-M^\Delta)(M^\Delta-W_0)/(W'-W_0)\,, \tag{8.6}$$

with $W_0\approx 2M^\Delta$, one finds a relatively *large* value for δR^*. We conclude that, in contrast to A^{RR} and A^{MR}, the elements of A^{RM} are strongly model-dependent and, in particular, that they are sensitive to the details of the denominator function. Consequently, in this paper, we shall not place any reliance on results that depend on the absolute magnitude of the elements of A^{RM}. We shall, however, draw some conclusions from *ratios* of elements of A^{RM}, which are less model-dependent.

Bearing this proviso in mind, one may proceed to calculate the dynamical factors which appear in $A^{RM(\text{exch})}$. Since the calculations in the present case are so involved, the results so model-dependent, and the method similar to that used in previous sections, we

TABLE XVII. The matrix $A^{RM(\text{ext})}$ for $P=+$, $C=+$, $T=+$, $S=1$ couplings, expressed in terms of the dynamical parameters C_8 and C_{10}.

	δM^B
$\delta R(8_\theta,8_\theta)$	$C_8/\sqrt{2}$
$\delta R(8_\theta,8_{\theta*})$	0
$\delta R^*(10,10)$	$(\sqrt{\frac{5}{8}})C_{10}$

TABLE XVIII. The matrix $A^{RM(\mathrm{ext})}$ for $P=+$, $C=+$, $T=+$, $S=8$ couplings, expressed in terms of the mixing angle θ and the dynamical parameters C_8, C_8', C_{10}, and C_{10}'.

	$\delta M^B(s)$	$\delta M^B(a)$
$\delta R(8_\theta,1)$	$(\cos\theta/2\sqrt{2})C_8'$	$(\sin\theta/2\sqrt{2})C_8'$
$\delta R(8_\theta,8_\theta(s))$	$(\sin^2\theta/2\sqrt{2}-3\cos^2\theta/10\sqrt{2})C_8$	$(\sin\theta\cos\theta/\sqrt{2})C_8$
$\delta R(8_\theta,8_{\theta^*}(s))$	$-\frac{4}{5}\sin\theta\cos\theta C_8'$	$\frac{1}{2}(\sin^2\theta-\cos^2\theta)C_8'$
$\delta R(8_\theta,8_\theta(a))$	$(\sin\theta\cos\theta/\sqrt{2})C_8$	$C_8/2\sqrt{2}$
$\delta R(8_\theta,8_{\theta^*}(a))$	$\frac{1}{2}(\sin^2\theta-\cos^2\theta)C_8'$	0
$\delta R(8_\theta,10)$	$(-\sin\theta/2+\cos\theta/\sqrt{5})C_8'$	$\frac{1}{2}\cos\theta C_8'$
$\delta R(8_\theta,\overline{10})$	$(\sin\theta/2+\cos\theta/\sqrt{5})C_8'$	$-\frac{1}{2}\cos\theta C_8'$
$\delta R(8_\theta,27)$	$-\frac{3}{10}(\sqrt{\frac{3}{2}})\cos\theta C_8'$	$-(\sqrt{\frac{3}{8}})\sin\theta C_8'$
$\delta R^*(10,8_\theta)$	$(\sin\theta/2-\cos\theta/\sqrt{5})C_{10}'$	$-\frac{1}{2}\cos\theta C_{10}'$
$\delta R^*(10,8_{\theta^*})$	$(-\cos\theta/2-\sin\theta/\sqrt{5})C_{10}'$	$-\frac{1}{2}\sin\theta C_{10}'$
$\delta R^*(10,10)$	$C_{10}/4$	$(\sqrt{5}/4)C_{10}$
$\delta R^*(10,27)$	$(3\sqrt{30}/20)C_{10}'$	$-(\sqrt{\frac{3}{8}})C_{10}'$

TABLE XX. The matrix $A^{RM(\mathrm{ext})}$ for $P=+$, $C=+$, $T=+$, $S=8$ couplings estimated with $\theta=-28°$ and *curved* D functions of the form Eq. (8.6) with $W_0\approx(7/3)M^\Delta$ for $J=\frac{3}{2}^+$, $W_0\approx(7/3)M^B$ for $J=\frac{1}{2}^+$.

	$\delta M^B(s)$	$\delta M^B(a)$
$\delta R(8_\theta,1)$	−0.53	0.28
$\delta R(8_\theta,8_\theta(s))$	0	0.12
$\delta R(8_\theta,8_{\theta^*}(s))$	−0.57	0.48
$\delta R(8_\theta,8_\theta(a))$	0.12	−0.14
$\delta R(8_\theta,8_{\theta^*}(a))$	0.48	0
$\delta R(8_\theta,10)$	−1.08	−0.75
$\delta R(8_\theta,\overline{10})$	−0.27	0.75
$\delta R(8_\theta,27)$	0.55	−0.49
$\delta R^*(10,8_\theta)$	1.08	0.40
$\delta R^*(10,8_{\theta^*})$	0.75	−0.40
$\delta R^*(10,10)$	−0.11	−0.25
$\delta R^*(10,27)$	−1.40	1.05

shall content ourselves here with merely presenting the results.

The results for the exchange mass shifts $A_S{}^{RM(\mathrm{exch})}$ are presented in Tables XV and XVI.

We now turn to the effect of the external mass shifts. There are four independent elements of $A^{RM^B(\mathrm{ext})}$ whose determination requires dynamical calculations; these can be chosen to be C_8, C_{10}, C_8' (describing $8_\theta\to X\neq 8_\theta$) and C_{10}' (describing $10\to X\neq 10$). The remaining elements of $A^{RM^B(\mathrm{ext})}$ may be expressed in terms of these four through the group-theory ratios given in Tables XVII to XIX.

To illustrate how these group-theory ratios were calculated, consider $A^{R(8_\theta,X)M^B(\mathrm{ext})}$. The group-theory ratios are the same as for the bubble diagram of Fig. 2, and can be expressed as

$$A^{R(8_\theta,X)M^B(\mathrm{exch})}=\frac{CN_S}{(N_8N_X)^{1/2}}\sum_{ijk}\sum_{\alpha\beta}\begin{pmatrix}8 & 8 & 8_\theta\\ j & i & \alpha\end{pmatrix}\times\begin{pmatrix}8 & 8 & X\\ j & k & \beta\end{pmatrix}\begin{pmatrix}8 & S & 8\\ i & \sigma & k\end{pmatrix}\begin{pmatrix}8 & S & X\\ \alpha & \sigma & \beta\end{pmatrix}, \tag{8.7}$$

where C is a dynamical factor. Here, the first two factors project ΠB onto the appropriate initial and final states,[31] the third factor represents the external mass splitting, and the fourth factor projects onto the desired type of coupling shift.

The dynamical parameters C_8, C_{10}, C_8', and C_{10}' were estimated by two different methods:

(1) Scale invariance gives conditions on the "diagonal elements" C_8 and C_{10} which enter into $S=1$ shifts, although it does not determine C_8' or C_{10}'.

(2) The reciprocal bootstrap used in this paper gives an explicit model for the amplitude, with two poles on the left. Using this model, we can estimate all four of the dynamical parameters.[32] The results of the two methods differ, but do agree as to sign and order of magnitude. The dynamical factors estimated in this way, plus the group theory factors, lead to the results for $A^{RM(\mathrm{ext})}$ presented in Table XX.

TABLE XIX. The matrix $A^{RM_B(\mathrm{ext})}$ for $P=+$, $C=+$, $T=+$, $S=27$ couplings, expressed in terms of the mixing angle θ and the dynamical parameters C_8, C_8', C_{10}, and C_{10}'.

$\delta R(8_\theta,8_\theta)$	$(\cos^2\theta/5-\sin^2\theta/3)C_8/\sqrt{2}$
$\delta R(8_\theta,8_{\theta^*})$	$(8/15)\sin\theta\cos\theta C_8'$
$\delta R(8_\theta,10)$	$(\sin\theta/3+\cos\theta/\sqrt{5})C_8'$
$\delta R(8_\theta,\overline{10})$	$(\sin\theta/3-\cos\theta/\sqrt{5})C_8'$
$\delta R(8_\theta,27)$	$-[(14)^{1/2}/5]\cos\theta C_8'$
$\delta R(8_\theta,27')$	$-(\sqrt{\frac{2}{3}})\sin\theta C_8'$
$\delta R^*(10,8_\theta)$	$(\sin\theta/3+\cos\theta/\sqrt{5})C_{10}'$
$\delta T^*(10,8_{\theta^*})$	$(-\cos\theta/3+\sin\theta/\sqrt{5})C_{10}'$
$\delta R^*(10,10)$	$\frac{1}{6}\sqrt{\frac{7}{2}}C_{10}$
$\delta R^*(10,\overline{10})$	$-[(10)^{1/2}/6]C_{10}'$
$\delta R^*(10,27)$	$-[(\sqrt{7})/15]C_{10}'$

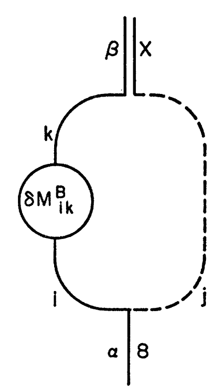

FIG. 2. Bubble diagram which contains the group-theory factors needed for $A^{R(8_\theta,X)M^B(\mathrm{ext})}$.

[31] The phase of the projected state depends on whether the meson index j is placed first or second in the Clebsch-Gordan coefficients. We place it first for consistency with the convention employed in calculating A^{RR}.

[32] In addition to the left-hand poles, one must integrate over the right-hand cut in equations such as (8.1). This is most important for C_{10} where the factor $(W-M^\Delta)^{-2}$ is large near threshold (the average Δ mass lies only slightly below the average ΠB threshold).

TABLE XXI. Strongly perturbed ${}_kB_j\Pi_i\bar{B}$ couplings. The first column gives the unperturbed couplings $G_{kj}{}^i/G$. The second column gives the coupling shifts $\delta G_{kj}{}^i/G$ corresponding to the leading eigenvector of $A_8{}^{gg}$ in Table IX, in an arbitrary normalization defined in the text. The final column gives the total unperturbed coupling $(G_{kj}{}^i+x\delta G_{kj}{}^i)/G$. The strength parameter $x=-13.4$ is obtained from the ratio $\Gamma(Y_1{}^*\to\Lambda\pi)/\Gamma(\Xi^*\to\Xi\pi)$, as explained in the text.

B_k	B_j	Π_i	$G_{kj}{}^i/G$	$10\delta G_{kj}{}^i/G$	$(G_{kj}{}^i-13.4\delta G_{kj}{}^i)/G$
p	p	π^0	0.477	−0.195	0.739
	p	η	0.037	0.009	0.025
	n	π^+	−0.675	0.276	−1.046
	Σ^+	K^0	−0.292	0.000	−0.292
	Σ^0	K^+	0.206	0.000	0.206
	Λ	K^+	−0.432	0.025	−0.465
n	p	π^-	0.675	−0.276	1.046
	n	π^0	−0.477	0.195	−0.739
	n	η	0.037	0.009	0.025
	Σ^0	K^0	−0.206	0.000	−0.206
	Σ^-	K^+	0.292	0.000	0.292
	Λ	K^0	−0.432	0.024	−0.465
Σ^+	p	$\bar{K}^0$	−0.292	0.000	−0.292
	Σ^+	π^0	0.271	−0.020	0.297
	Σ^+	η	0.395	0.111	0.246
	Σ^0	π^+	−0.271	0.020	−0.297
	Ξ^0	K^+	−0.675	−0.138	−0.490
	Λ	π^+	0.395	−0.077	0.498
Σ^0	p	K^-	−0.206	0.000	−0.206
	n	$\bar{K}^0$	−0.206	0.000	−0.206
	Σ^+	π^-	0.271	−0.020	0.297
	Σ^0	η	0.395	0.111	0.246
	Σ^-	π^+	−0.271	0.020	−0.297
	Ξ^0	K^0	−0.477	−0.098	−0.347
	Ξ^-	K^+	−0.477	−0.098	−0.347
	Λ	π^0	0.395	−0.077	0.498
Σ^-	n	K^-	−0.292	0.000	−0.292
	Σ^0	π^-	0.271	−0.020	0.297
	Σ^-	π^0	−0.271	0.020	−0.297
	Σ^-	η	0.395	0.111	0.246
	Ξ^-	K^0	−0.675	−0.138	−0.490
	Λ	π^-	0.395	−0.077	0.498
Ξ^0	Σ^+	K^-	0.675	0.138	0.490
	Σ^0	$\bar{K}^0$	−0.477	−0.098	−0.347
	Ξ^0	π^0	−0.206	0.000	−0.207
	Ξ^0	η	−0.432	−0.126	−0.264
	Ξ^-	π^+	0.292	0.000	0.293
	Λ	$\bar{K}^0$	0.037	−0.009	0.050
Ξ^-	Σ^0	K^-	0.477	0.098	0.347
	Σ^-	$\bar{K}^0$	−0.675	−0.138	−0.490
	Ξ^0	π^-	−0.292	0.000	−0.292
	Ξ^-	π^0	0.206	0.000	0.206
	Ξ^-	η	−0.432	−0.126	−0.264
	Λ	K^-	0.037	−0.009	0.050
Λ	p	K^-	0.432	−0.025	0.465
	n	$\bar{K}^0$	−0.432	0.025	−0.465
	Σ^+	π^-	−0.395	0.077	−0.498
	Σ^0	π^0	0.395	−0.077	0.498
	Σ^-	π^+	−0.395	0.077	−0.498
	Ξ^0	K^0	0.037	−0.009	0.050
	Ξ^-	K^+	−0.037	0.009	−0.050
	Λ	η	−0.395	−0.043	−0.338

Now let us see what conclusions can be drawn from these results.

(i) To get a clear idea of the way in which A^{RM} affects our calculations, it is useful to evaluate the quantity

$$|\sum_{ij} X_i{}^R(A_{ij}{}^{RM\,(\mathrm{exch})}+A_{ij}{}^{RM\,(\mathrm{ext})})X_j{}^{M\,(\mathrm{leading})}| \quad (8.8)$$

[$X_j{}^{M\,(\mathrm{leading})}$ is given by Eq. (7.14)]. The numerical results show that the term in A^{RM} connecting the enhanced eigenvector of A^{MM} to the leading eigenvector A^{RR} is ≈ 1.1. Although the over-all strength of A^{RM} is admittedly not reliable, this result of order unity indicates that the leading eigenvector of A^{RR} will acquire a double enhancement according to Eq. (1.4);

$$\delta R=(1-A^{RR})^{-1}A^{RM}(\delta M/M)|_{M\text{ enhancement}}. \quad (8.9)$$

(ii) The numerical results show that the elements of A^{RM} connecting the enhanced mass shifts to the second and third eigenvectors of A^{RR} are ≈ 0.2 and ≈ 0.0, respectively. Thus the leading eigenvector is much more strongly enhanced than the others. In view of the uncertainties in the calculation of A^{RM}, we remark that the dominance of the leading eigenvector does not depend on any delicate cancellations. It comes about because the largest term in A^{RM} turns out (in agreement with the calculations of Wali and Warnock[8]) to involve $\delta R^*(10\to 27)$, which happens to be strongly present in the leading eigenvector of $A_8{}^{RR}$, but not in the next two.

(iii) Since δM_{27} is very small, the eigenvalue of $A_{27}{}^{RR}$ lying near one receives only a single enhancement.

These conclusions depend on the *relative values* of the various elements of A^{RM}, and are not too sensitive to the model employed or to the details of the D function. When in the next section we start to extract experimental consequences from the results of the foregoing sections, we will find that we need to have the *absolute value* of the elements of A^{RM}. This over-all scale parameter will then be regarded as a physical parameter, and will be determined by fitting to the data.

IX. APPLICATION TO STRONG INTERACTIONS

In Sec. VII, we found that A^{Mg} is small. Approximating it by zero, Eqs. (1.1) and (1.2) for δM and δg have the solution (matrix equations are understood here)

$$\delta M=(1-A^{MM})^{-1}D^M, \quad (1.3)$$

$$\delta g=(1-A^{gg})^{-1}(A^{gM}\delta M+D^g). \quad (1.4)$$

In Sec. VI we found that A^{gg} has several eigenvectors with eigenvalues near one, which should dominate (1.4). Finally, in Sec. VIII we found that $A^{gM}\delta M$, with δM taken from experiment and A^{gM} from theory, strongly favors the leading eigenvector of $A_{S=8}{}^{gg}$ over the other eigenvectors with eigenvalues near one. Assuming that the already-enhanced term $A^{gM}\delta M$ dominates D^g, we can conclude that the strong-coupling shifts follow the leading eigenvector of $A_{S=8}{}^{gg}(A^{gg}\approx 0.93)$. This eigenvector was printed out in the first column of Table IX. The individual particle couplings are obtained from it by applying Eqs. (4.30) and (4.31) with $S=8$ and σ equal to the $I=0$, $Y=0$ member (which we call "8") of the octet:

$$\delta G_{kj}{}^{i}=-G^{-1}\sum_{X}(n_X)^{-1/2}\delta R_8(8_\theta,X)\times\sum_{\nu}\begin{pmatrix}8 & 8 & X\\ k & 8 & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & X\\ i & j & \nu\end{pmatrix}, \quad (9.1)$$

$$\delta G_{kj}{}^{*i}=-\frac{G^{*-1}}{\sqrt{2}}\sum_{X}(n_X{}^*)^{-1/2}\delta R_8{}^*(10,X)\times\sum_{\nu}\begin{pmatrix}10 & 8 & X\\ k & 8 & \nu\end{pmatrix}\begin{pmatrix}8 & 8 & X\\ i & j & \nu\end{pmatrix}. \quad (9.2)$$

Our predictions for the strongly perturbed couplings will have the form $(G+x\delta G)_{kj}{}^{i}$, where $G_{kj}{}^{i}$ is the $SU(3)$-symmetric coupling, $\delta G_{kj}{}^{i}$ is given by (9.1) or (9.2) with the eigenvector δR_8 (arbitrarily normalized to 1) taken from Table IX, and x is an over-all strength parameter for the perturbation. $G_{kj}{}^{i}/G$ and $G_{kj}{}^{*i}/G^*$ are obtained from (2.9) and (2.10), and the nonzero elements are listed in the first columns of Tables XXI and XXII, respectively. The perturbations $\delta G_{kj}{}^{i}/G$ and $\delta G_{kj}{}^{*i}/G^*$ are listed in the second columns of these tables.

The strength parameter x could, in principle, be estimated from (1.4) by using the eigenvalue $A_8{}^{gg}\approx 0.93$, the physical δM, the calculated A^{gM}, and by dropping D^g. This estimate is highly uncertain, both because small changes in the eigenvalue of A^{gg} produce large changes in $(1-A^{gg})^{-1}$ and because, as explained in Sec. VIII, the magnitude of A^{gM} is highly sensitive to details of the D function. For these reasons, we preferred to estimate x from the experimental ratio of $\frac{3}{2}^+$ resonance decay widths. Of these, the decay $Y_1{}^*\to\Sigma\pi$ is too poorly known experimentally. The N^* decay width is well known experimentally, but the static model we are using does not reproduce its shape well and thus does not give an accurate estimate for its width. (The static model greatly overestimates the width of the high-energy tail of this resonance; such energy-dependent effects may be less for the other, considerably narrower, members of the decimet.) Thus we used the ratio $\Gamma(Y_1{}^*\to\Lambda\pi)/\Gamma(\Xi^*\to\Xi\pi)$ to determine x. The result obtained in Ref. 1 was $x=-13.4$.[33] The total couplings $(G+x\delta G)_{kj}{}^{i}/G$ and $(G^*+x\delta G^*)_{kj}{}^{i}/G^*$, obtained using this value of x, are listed in the third columns of Tables XXI and XXII, respectively. In view of the uncertain situation in the $\frac{3}{2}^+$ decay widths, the precise value of x used here should not be taken seriously, but the sign and order of magnitude appear to be reasonable.

Table II in our previous writeup of these results[1] was constructed by squaring $(G+x\delta G)_{kj}{}^{i}/G$ and, for a given

[33] The value of x obtained from the theoretical $(1-A^{gg})^{-1}$ and A^{gM} has the same sign and order of magnitude. This can be taken as evidence that the leading eigenvalue of A^{gg} is a little less than, rather than a little greater than, one.

TABLE XXII. Strongly perturbed $\overline{\Delta}_kB_j\Pi_i$ couplings. The first column gives the unperturbed couplings $G_{kj}{}^{*i}/G^*$. The second column gives the coupling shifts $\delta G_{kj}{}^{*i}/G^*$ corresponding to the leading eigenvector of $A_8{}^{gg}$ in Table IX, in an arbitrary normalization defined in the text. The final column gives the total perturbed coupling $(G_{kj}{}^{*i}+x\delta G_{kj}{}^{*i})/G^*$. The strength parameter $x=-13.4$ is obtained from the ratio $\Gamma(Y_1{}^*\to\Lambda\pi)/\Gamma(\Xi^*\to\Xi\pi)$, as explained in the text.

Δ_k	B_j	Π_i	$G_{kj}{}^{*i}/G^*$	$10\delta G_{kj}{}^{*i}/G^*$	$(G_{kj}{}^{*i}-13.4\delta G_{kj}{}^{*i})/G^*$
N^{*++}	p	π^+	0.707	−0.398	1.241
	Σ^+	K^+	−0.707	−0.235	−0.392
N^{*+}	p	π^0	0.577	−0.325	1.013
	n	π^+	0.408	−0.230	0.716
	Σ^+	K^0	−0.408	−0.136	−0.226
	Σ^0	K^+	−0.577	−0.192	−0.320
N^{*0}	p	π^-	0.408	−0.230	0.716
	n	π^0	0.577	−0.325	1.013
	Σ^0	K^0	−0.577	−0.192	−0.320
	Σ^-	K^+	−0.408	−0.136	−0.226
N^{*-}	n	π^-	0.707	−0.398	1.241
	Σ^-	K^0	−0.707	−0.235	−0.392
Y^{*+}	p	$\bar{K}^0$	0.408	−0.144	0.602
	Σ^+	π^0	−0.289	−0.007	−0.280
	Σ^+	η	−0.500	−0.185	−0.252
	Σ^0	π^+	0.289	0.007	0.280
	Ξ^0	K^+	−0.408	−0.135	−0.228
	Λ	π^+	0.500	−0.185	0.748
Y^{*0}	p	K^-	0.289	−0.102	0.425
	n	$\bar{K}^0$	0.289	−0.102	0.425
	Σ^+	π^-	−0.289	−0.007	−0.280
	Σ^0	η	−0.500	−0.185	−0.252
	Σ^-	π^+	0.289	0.007	0.280
	Ξ^0	K^0	−0.289	−0.095	−0.161
	Ξ^-	K^+	−0.289	−0.095	−0.161
	Λ	π^0	0.500	−0.185	0.748
Y^{*-}	n	K^-	0.408	−0.144	0.602
	Σ^0	π^-	−0.289	−0.007	−0.280
	Σ^-	π^0	0.289	0.007	0.280
	Σ^-	η	−0.500	−0.185	−0.252
	Ξ^-	K^0	−0.408	−0.135	−0.228
	Λ	π^-	0.500	−0.185	0.748
Ξ^{*0}	Σ^+	K^-	−0.408	−0.095	−0.281
	Σ^0	$\bar{K}^0$	0.289	0.067	0.198
	Ξ^0	π^0	−0.289	−0.006	−0.281
	Ξ^0	η	−0.500	−0.183	−0.254
	Ξ^-	π^+	0.408	0.008	0.397
	Λ	$\bar{K}^0$	0.500	−0.080	0.608
Ξ^{*-}	Σ^0	K^-	−0.289	−0.067	−0.198
	Σ^-	$\bar{K}^0$	0.408	0.095	0.281
	Ξ^0	π^-	−0.408	−0.008	−0.397
	Ξ^-	π^0	0.289	0.006	0.281
	Ξ^-	η	−0.500	−0.183	−0.254
	Λ	K^-	0.500	−0.080	0.607
Ω^-	Ξ^0	K^-	−0.707	−0.163	−0.489
	Ξ^-	$\bar{K}^0$	0.707	0.163	0.489

k, summing over the i and j within a given isospin multiplet. For example, the $NN\Pi$ coupling strength is

$$(G_{pn}{}^{\pi^+}+x\delta G_{pn}{}^{\pi^+})^2/G^2+(G_{pp}{}^{\pi^0}+x\delta G_{pp}{}^{\pi^0})^2/G^2.$$

The physical implications of the results were discussed in Ref. 1: The "medium strong" coupling shifts are very large, and have the right sign to suppress K couplings

TABLE XXIII. Electromagnetic perturbations on $\bar{B}_k B_j \Pi_i$ couplings. The first column gives the unperturbed couplings $G_{kj}{}^i/G$. The second column gives the coupling shifts $\delta G_{kj}{}^i/G$ defined in Eq. (10.1), calculated using the leading eigenvector of $A_8{}^{gg}$ in Table IX. The final column gives the total perturbed coupling $(G_{kj}{}^i+x^{\mathrm{em}}\delta G_{kj}{}^i)/G$. The strength parameter $x=-0.25$ is obtained from universality and the experimental electromagnetic mass shifts among baryons, as explained in the text. This table does not include the effects of strong symmetry breaking on $G_{kj}{}^i$.

B_k	B_j	Π_i	$G_{kj}{}^i/G$	$10\delta G_{kj}{}^i/G$	$(G_{kj}{}^i-0.25\delta G_{kj}{}^i)/G$
p	p	π^0	0.4775	−0.013	0.4778
	p	η	0.0373	−0.078	0.0393
	n	π^+	−0.6753	0.000	−0.6752
	Σ^+	K^0	−0.2920	0.000	−0.2919
	Σ^0	K^+	0.2064	−0.066	0.2081
	Λ	K^+	−0.4322	0.078	−0.4341
n	p	π^-	0.6753	0.000	0.6753
	n	π^0	−0.4775	−0.013	−0.4772
	n	η	0.0373	0.078	0.0353
	Σ^0	K^0	−0.2064	−0.065	−0.2048
	Σ^-	K^+	0.2920	0.000	0.2920
	Λ	K^0	−0.4322	−0.078	−0.4302
Σ^+	p	$\bar{K}^0$	−0.2920	0.000	−0.2919
	Σ^+	π^0	0.2711	−0.115	0.2739
	Σ^+	η	0.3949	−0.137	0.3983
	Σ^0	π^+	−0.2711	0.036	−0.2720
	Ξ^0	K^+	−0.6753	0.239	−0.6813
	Λ	π^+	0.3949	−0.058	0.3963
Σ^0	p	K^-	−0.2064	0.066	−0.2081
	n	$\bar{K}^0$	−0.2064	−0.066	−0.2048
	Σ^+	π^-	0.2711	−0.036	0.2720
	Σ^0	π^0	0.0000	−0.043	0.0011
	Σ^0	η	0.3949	0.000	0.3949
	Σ^-	π^+	−0.2711	−0.036	−0.2702
	Ξ^0	K^0	−0.4775	−0.032	−0.4767
	Ξ^-	K^+	−0.4775	0.032	−0.4783
	Λ	π^0	0.3949	0.000	0.3949
	Λ	η	0.0000	−0.077	0.0019
Σ^-	n	K^-	−0.2920	0.000	−0.2919
	Σ^0	π^+	0.2711	0.036	0.2702
	Σ^-	π^0	−0.2711	−0.114	−0.2682
	Σ^-	η	0.3949	0.137	0.3915
	Ξ^-	K^0	−0.6753	−0.239	−0.6693
	Λ	π^-	0.3949	0.058	0.3934
Ξ^0	Σ^+	K^-	0.6753	−0.239	0.6813
	Σ^0	$\bar{K}^0$	−0.4775	−0.032	−0.4767
	Ξ^0	π^0	−0.2064	0.122	−0.2095
	Ξ^0	η	−0.4322	0.117	−0.4351
	Ξ^-	π^+	0.2920	0.000	0.2920
	Λ	$\bar{K}^0$	0.0373	−0.020	0.0378
Ξ^-	Σ^0	K^-	0.4775	−0.032	0.4783
	Σ^-	$\bar{K}^0$	−0.6753	−0.239	−0.6693
	Ξ^0	π^-	−0.2920	0.000	−0.2920
	Ξ^-	π^0	0.2064	0.122	0.2034
	Ξ^-	η	−0.4322	−0.117	−0.4292
	Λ	K^-	0.0373	0.020	0.0368
Λ	p	K^-	0.4322	−0.078	0.4341
	n	$\bar{K}^0$	−0.4322	−0.078	−0.4302
	Σ^+	π^-	−0.3949	0.058	−0.3963
	Σ^0	π^0	0.3949	0.000	0.3949
	Σ^0	η	0.0000	−0.077	0.0019
	Σ^-	π^+	−0.3949	−0.058	−0.3934
	Ξ^0	K^0	0.0373	−0.020	0.0378
	Ξ^-	K^+	−0.0373	−0.020	0.0368
	Λ	π^0	0.0000	0.111	−0.0028
	Λ	η	−0.3949	0.000	−0.3949

relative to π couplings, in agreement with experiment, and generally decrease the coupling strengths to high-mass channels. The latter feature makes it reasonable to neglect certain high-mass channels in approximate dynamical calculations[34] even though they might appear to enter in an important way from $SU(3)$-symmetry considerations.

It is interesting to see how the near self-consistency of the dominant symmetry breaking works out in terms of specific attractions and repulsions in the broken-$SU(3)$ bootstrap. We cite two examples:

(i) According to Tables XXI and XXII, N and N^* couple almost exclusively to πN, rather than ΣK, etc., in the broken-$SU(3)$ bootstrap. Thus, one is led back to the original self-consistent $SU(2)$ model for N and N^*.[35]

(ii) In $SU(3)$, the potential for $\pi\Sigma$ scattering in the $J^P=\frac{3}{2}^+$ state receives a repulsion from Λ exchange, and an attraction from Σ and $Y_1{}^*$ exchange.[36] In broken $SU(3)$, the strength of Λ exchange is enhanced relative to Σ and $Y_1{}^*$ exchange, leading to a more repulsive $\pi\Sigma$ potential. This repulsion provides the detailed mechanism by which the $Y_1{}^*$ decay into the $\Sigma\pi$ channel is reduced.[37]

X. ELECTROMAGNETIC APPLICATIONS

Since the A matrix is independent of the "direction" taken by the symmetry violation in $SU(3)$ space,[1] we can estimate that the first eigenvector of $A_{S=8}{}^{gg}$ in Table IX dominates electromagnetic perturbations of order e^2 as well as strong perturbations on the B and Δ couplings. The individual-particle couplings are again obtained from it by applying Eqs. (4.30) and (4.31) with $S=8$. This time, the interesting couplings involve σ equal to the $I=1$, $I_3=0$, $Y=0$ rather than the $I=0$, $Y=0$ member of the octet ($\sigma=$"3" rather than "8"):

$$\delta G_{kj}{}^i=-G^{-1}\sum_X (n_X)^{-1/2}\delta R_8(8_\theta,X) \times\sum_\nu \begin{pmatrix} 8 & 8 & X \\ k & 3 & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & X \\ i & j & \nu \end{pmatrix}, \quad (10.1)$$

$$\delta G_{kj}{}^{*i}=-\frac{G^{*-1}}{\sqrt{2}}\sum_X (n_X{}^*)^{-1/2}\delta R_8{}^*(10,X) \times\sum_\nu \begin{pmatrix} 10 & 8 & X \\ k & 3 & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & X \\ i & j & \nu \end{pmatrix}. \quad (10.2)$$

Our predictions for the perturbed couplings now take the form $(G^{\mathrm{strong}}+x^{\mathrm{em}}\delta G^{\mathrm{em}})_{kj}{}^i$ where $(G^{\mathrm{strong}})_{kj}{}^i$ is the outcome of Sec. IX, $(\delta G^{\mathrm{em}})_{kj}{}^i$ is given by (10.1) or (10.2)

[34] See, for example, B. Kayser, Phys. Rev. **138**, B1244 (1964); F. Gilman, *ibid.* **147**, 1094 (1966).
[35] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[36] E. Golowich, Phys. Rev. **139**, B1297 (1965).
[37] B. Kayser and E. Bloom, Phys. Rev. **144**, 1176 (1966). The authors are indebted to Dr. P. Carruthers for an informative discussion on this and related points.

with δR taken from Table IX, and x^{em} is an over-all strength parameter for the electromagnetic perturbation. As a further consequence of the fact that the A matrix is the same for electromagnetic and strong perturbations, we can estimate that the same ratio $\delta G/\delta M$ holds for both. If this is true, we can take $x^{\text{em}}/x^{\text{strong}}$ equal to $\delta M^{B\,\text{em}}/\delta M^{B\,\text{strong}}$, which is known experimentally. (By δM^B we mean the coefficient of the normalized octet mass matrix as defined in Ref. 2; this coefficient is the "strength parameter" for mass shifts, just as x is the strength parameter for couplings.) By such means we obtain $x^{\text{em}}=-0.25$. The results obtained for $BB\Pi$ coupling shifts, using this value of x^{em}, are presented in Table XXIII. $\Delta B\Pi$ coupling shifts can similarly be calculated from Eq. (10.2).

There are no firm data on any electromagnetic shifts in $BB\Pi$ or $\Delta B\Pi$ couplings, but at least a few cases have some experimental interest. One sees from Table XXIII that the corrections $\delta G_{NN\pi}$ due to the leading octet eigenvector are extremely small, a point which is relevant to possible violations of charge independence in nuclear physics, where one-pion exchange is an important part of the two-nucleon potential. Similarly, a calculation of electromagnetic shifts in $N^*N\pi$ coupling, using (10.2), yields results relevant to the recent experimental search for differences between $N^{*++}\to p+\pi^+$ and $N^{*-}\to n+\pi^-$.[38] For the $\Delta I=1$ couplings which participate in octet $SU(3)$ breaking, we estimate crudely $\Gamma(N^{*-}\to n+\pi^-)-\Gamma(N^{*++}\to p+\pi^+)\approx 1$ MeV.

XI. APPLICATIONS TO WEAK NONLEPTONIC INTERACTIONS

In this section we discuss the coupling shifts induced by the weak interactions. We will work under the assumption that CP is conserved; some discussion of CP-violating couplings will be given in Sec. XIII.

As usual, we specify the character of a weak violation of $SU(3)$ by σ, S, $\mathcal{C}$, and P, which stands for the σ component of a representation S whose $I=0$, $Y=0$ member has charge conjugation $\mathcal{C}$ and parity P. To avoid possible confusion, we would like to stress that for the strangeness-violating weak interaction, $\mathcal{C}$ and C may be different, a situation which did not arise in our previous studies of strong and electromagnetic corrections to $SU(3)$. That is, $\mathcal{C}$ remains the same for all components σ of the representation S, while C equals $\mathcal{C}$ for the $I=0$, $Y=0$ component, but is negative for some of the other components. For example, the $K_1{}^0$ meson has $\mathcal{C}=+1$ and $C=+1$, while the $K_2{}^0$ has $\mathcal{C}=+1$ (since it belongs to the same octet), but $C=-1$. Similarly, a strangeness-changing weak Hamiltonian with $C=+1$ can contain a piece which acts like the $K_1{}^0$ from an octet with $\mathcal{C}=+1$, or a piece which acts like the "$K_2{}^0$" from an octet with $\mathcal{C}=-1$, or both. Concerning this point, the current-current interaction in the Cabibbo form predicts $\mathcal{C}=+1$ for the parity-conserving part of the nonleptonic weak interaction and $\mathcal{C}=-1$ for the parity-violating part.[39] Apart from this attractive hypothesis, however, there is little evidence either for or against these $\mathcal{C}$ assignments. Furthermore, whatever $\mathcal{C}$ properties the weak interaction has in the $SU(3)$ limit are likely to be modified by the large strong violations of $SU(3)$. For these reasons, we have studied weak couplings with $\mathcal{C}=\pm1$ for each of the cases, parity conservation and parity nonconservation.

It is important to note that the A matrix refers to a definite $\mathcal{C}$ and does not connect violations with different $\mathcal{C}$.[40] We can therefore treat $\mathcal{C}=+1$ and $\mathcal{C}=-1$ separately.

We now proceed to outline our calculation and results for the four cases $P=\pm1$ and $\mathcal{C}=\pm1$, remaining always with $CP=1$. We begin with $P=1$, $\mathcal{C}=1$, then proceed to $P=1$, $\mathcal{C}=-1$ and take up $P=-1$ in the latter part of the section. In our discussion of the parity-conserving weak interaction, we restrict ourselves to the strangeness-changing $\Delta Y\neq0$ part; the tiny strangeness-conserving, parity-conserving couplings induced by the weak interactions are, at most, of academic interest.

Our treatment of $P=1$, $\mathcal{C}=1$, $\Delta Y\neq0$ coupling shifts follows along the same lines as the treatment of the strong and electromagnetic δG's, but differs in one important way: There are no strangeness changing mass shifts δM.

That the $\Delta Y\neq0$ weak interaction produces no first-order mass shifts is quite obvious from a physical point of view, but it is instructive to see formally how this comes about. To this end, let us consider a calculation of δg and δM correct to first order in strong-$SU(3)$ violations, electromagnetism and weak interactions. For simplicity, we suppose that $A^{Mg}=0$ and write

$$\delta M=[1/(1-A^{MM})]D^M, \qquad (1.3)$$

$$\delta g=[1/(1-A^{gg})](D^g+A^{gM}\delta M), \qquad (1.4)$$

where D^M and D^g each contain three terms, one from each of the strong, electromagnetic, and weak interactions. We now wish to isolate the strangeness-changing ($\Delta Y\neq0$) perturbations, which is equivalent to picking out the perturbations which point in a direction perpendicular to the 3 and 8 axes in $SU(3)$ space. One must, however, be rather careful here. In a totally $SU(3)$-symmetric world, the orientation of the 3 and 8 axes would be arbitrary. It is only because $SU(3)$ is violated that we can give a unique meaning to the 3 and 8 axes. The direction of these axes is, in fact, defined solely by the requirement that the physical particles have definite values of I_3 and Y, which is equivalent to

[38] G. Gidal, A. Kernan, and S. Kim, Phys. Rev. **141**, 1261 (1966).

[39] M. Gell-Mann, Phys. Rev. Letters **12**, 155 (1964).

[40] For the $I=0$, $Y=0$ component of a given representation S, $\mathcal{C}$ is the same as C, and A connects only couplings of the same $C=\mathcal{C}$. Under $SU(3)$ rotations to other components σ of S, A, and $\mathcal{C}$ remain unchanged, so A continues to connect only couplings of the same $\mathcal{C}$. It also connects couplings of a given C only to themselves, but *which* C is involved varies with the component σ.

TABLE XXIV. Eigenvalues of A^{gg} for parity-conserving couplings of various C and S, and the contributions of the associated eigenvector to the observable $B \to B+\pi$ decay amplitudes. The evaluation was made at $\theta=-28°$ and only eigenvalues $\gtrsim 0.5$ are included. The normalization and over-all phase of each column are arbitrary; it is the ratios which are significant.

Decay \ S / C / Eigenvalue	8 / + / 0.93	8 / + / 0.66	8 / + / 0.49	8 / − / 0.89	8 / − / 0.82	27 / + / 0.88	27 / + / 0.63	27 / − / 0.83	Experiment[a]
$\Lambda \to \pi^0+n$	0.40	−0.20	0.00	0.10	0.13	−0.14	0.18	0.03	−1.4±0.6
$\Lambda \to \pi^-+p$	−0.56	0.28	0.00	−0.14	−0.18	0.19	−0.25	−0.05	+2.0±0.3
$\Sigma^+ \to \pi^++n$	−0.40	−0.33	−0.51	0.20	0.14	0.07	−0.23	−0.41	4.1±0.1
$\Sigma^+ \to \pi^0+p$	0.42	0.43	0.05	0.36	−0.23	−0.09	0.00	0.06	{−3.6±0.4, −1.7±0.2
$\Sigma^- \to \pi^-+n$	0.19	0.28	−0.44	0.71	−0.19	−0.03	−0.23	−0.32	−0.4±0.6
$\Xi^0 \to \pi^0+\Lambda$	−0.11	0.06	−0.18	0.11	−0.32	−0.03	0.24	0.19	−1.0±0.2
$\Xi^- \to \pi^-+\Lambda$	−0.16	0.08	−0.25	0.15	−0.46	−0.04	0.34	0.27	−1.4±0.1

[a] See Ref. 50.

saying that the mass matrix δM has no components perpendicular to the 3 and 8 axes. (This is somewhat easier to see if one imagines a world in which the π meson is sufficiently massive so that decays like $\Lambda \to \pi N$ are energetically forbidden even though strangeness is not conserved.) Thus, by definition, there are no $\Delta Y \neq 0$ mass shifts and for the strangeness-changing corrections to $SU(3)$, we have[41]

$$\delta G(\Delta Y\neq 0)=[1/(1-A^{gg})]D_{\Delta Y\neq 0}{}^{g}. \qquad (11.1)$$

The matrix A^{gg} in (11.1) is (for the $\mathcal{C}=+1$ δg's under consideration) the same as the A^{gg} which appears in the strong violations of $SU(3)$ (since A is independent of σ). We know that A^{gg} has several eigenvalues near one, so that there is no lack of enhancement for these couplings. Specifically, there are five eigenvalues or order $\frac{1}{2}$ or greater, three for $S=8$, and two for $S=27$; the contributions of each of these five eigenvectors to the seven observed hyperon decay amplitudes is given in Table XXIV.

One will recall that for the strong and electromagnetic violations of $SU(3)$, the mass shifts drove mostly the one leading octet eigenvector, thus singling it out as doubly enhanced. The absence of $\Delta Y\neq 0$ mass shifts, however, prevents us from singling out a unique eigenvector for the weak interactions. In this sense, the parity-conserving weak interactions do not share the single-enhanced-eigenvector "universality" which seems to be present in the strong and electromagnetic corrections to $SU(3)$.

We now turn to $P=1$, $\mathcal{C}=-1$ perturbations. Again, there will be no strange mass shifts and we deal with an equation like (11.1). The matrix A^{gg} is, however, different in this case.

To calculate A^{gg} for $\mathcal{C}=-1$, we note that since A is independent of any direction in $SU(3)$ space. We may as well[40] construct it by considering a violation in the $I=0$, $Y=0$ direction which has $C=\mathcal{C}=-1$, even though we will ultimately be interested in a direction where $C=-\mathcal{C}=+1$. We may proceed, then, exactly as in the construction of A^{gg} for $\mathcal{C}=+1$, except that:

(i) C is now equal to -1, where it appears explicitly in the equations of Sec. IV.

(ii) As a result of taking $C=-1$, the "diagonal" coupling shifts $\delta R(8_\theta,8_\theta(s \text{ or } a))$ and $\delta R^*(10,10)$ do not contribute to Eqs. (4.26)–(4.29). Thus the projection procedure of Sec. V operates on the reduced basis of "off-diagonal" coupling shifts. The diagonal components $[\delta R(8_\theta,8_\theta(s \text{ or } a))]$ must be removed from the $C=-$ vectors of Table V before they are employed in the projection procedure.

(iii) In Sec. V, we are now instructed to project out the couplings with $C=+1$ rather than $C=-1$, as we did to obtain A^{gg} for $\mathcal{C}=+1$. If P_{ij} is the projection matrix which removes the $C=-1$ $BB\Pi$ couplings [modified in accordance with (ii) above], then the complementary projection $P_{ij}'=\delta_{ij}-P_{ij}$ (for $BB\Pi$ couplings) and δ_{ij} (for $\Delta B\Pi$ couplings) will remove the $C=+1$ $BB\Pi$ couplings. Again, we have to check the sensitivity of the leading eigenvectors to this projection procedure.

Numerically, we found that for $\mathcal{C}=-1$ there are three eigenvalues near one, two for $S=8$, and one for $S=27$. These eigenvalues and the contributions of their associated eigenvectors to the observed hyperon decays are shown in Table XXIV.

[41] Although we have shown that $\delta M_{\Delta Y\neq 0}$ is rotated away, one might wonder whether the effect of $\delta M_{\Delta Y\neq 0}$ on δg is not simply replaced by the effect of the rotation. After all, the rotation does change the wave functions at a vertex such as $\sum_{\alpha\beta k} G_{\alpha\beta}{}^k \bar{\psi}_\alpha \gamma_5 \psi_\beta \phi_k$ by an amount

$$\delta\psi_1=[\langle 1|H_{\text{weak}}|2\rangle/(E_2-E_1)]\psi_2,$$

where E_2-E_1 is the energy splitting introduced by the strong symmetry breaking. (The way in which mixing of this type appears in our formalism was described in footnote 14 of Ref. 45.) We have not estimated this effect for the following reasons: (i) To the extent that the strong and weak mass shifts (before rotation) are dominated by a single eigenvector of A, a *single* $SU(3)$ rotation removes $\delta M_{\Delta Y\neq 0}$ for *all* supermultiplets. A uniform $SU(3)$ rotation of all supermultiplets leaves couplings which were initially $SU(3)$ scalars *unchanged*, as stressed by Coleman and Glashow (Ref. 9). (ii) Actually, the mass shifts contain small admixtures of other eigenvectors as well. Therefore, somewhat different $SU(3)$ rotations are needed to remove $\delta M_{\Delta Y\neq 0}$ from different supermultiplets, and this leads to coupling shifts. But the leading effects of the rotation do cancel as indicated above, and the small residual shifts, not being controlled by the leading eigenvector, are hard to predict. (iii) Since the effect depends on *both* the weak and strong mass shifts, it is "nonlinear" and technically is part of the driving term rather than the A matrix.

The sensitivity of these eigenvalues and eigenvectors to the parameters, θ and the curvature of D, in our model was roughly the same as for the $\mathcal{C}=+1$ case. The sensitivity to the projection which enforces vertex symmetry was also comparable to the $\mathcal{C}=+1$ case.

Since we have a total of eight enhanced eigenvectors for $P=+$, five for $\mathcal{C}=+1$, and three for $\mathcal{C}=-1$, comparison with experiment is difficult. Some phenomenology was discussed in Ref. 1; here, we simply note that the numbers quoted in Ref. 1 were derived from Table XXIV.

We turn now to the parity-violating ($P=-1$) weak interaction.

The static Chew-Low approximation which we employ has the special feature that orbital angular momentum is preserved under the operation of crossing from the s to u channel. It will turn out that this fact greatly simplifies the treatment of parity-violating couplings.

The crossing properties of orbital angular momentum are determined by the relation between the scattering angles $\cos\theta_s=1+t/2q_s^2$ and $\cos\theta_u=1+t/2q_u^2$ in the two channels, where t is the momentum transfer and q_u and q_s are the c.m. momenta in the two channels. Clearly, if the two angles are equal, orbital angular momentum is the same in both channels. Now, in the static region around $W=M^B$, one readily verifies that to order $(W-M^B)(2M^B)^{-1}$, q_s^2 and q_u^2 are equal so that $\cos\theta_s\approx\cos\theta_u$ and, within our approximation, orbital angular momentum is preserved under crossing.

The importance of this result is seen as follows: To study the parity-violating $BB\Pi$ couplings, we look at the scattering amplitude for $\Pi+B$ ($J=\frac{1}{2}$ P wave) $\rightarrow\Pi+B$ ($J=\frac{1}{2}$ S wave). It follows from the discussion of the above paragraph, that the cross reaction which determines the nearby part of the left cut must be S waves $\rightarrow$ P waves. Such a reaction must proceed through a $J=\frac{1}{2}$ state, which tells us that $J=\frac{1}{2}$ B-exchange contributes to the left cut, but not $J=\frac{3}{2}$ Δ-exchange. Similarly, if we want to study the $P=-1$ $\Delta B\Pi$ coupling, we look at ($J=\frac{3}{2}$ P wave) $\rightarrow$ ($J=\frac{3}{2}$ D wave) which has only $J=\frac{3}{2}$ in the cross channel, and B-exchange does not contribute. Thus, in the notation of Sec. IV, we have $\eta^{\Delta B}=\eta^{B\Delta}=0$ for parity-violating couplings.

The previous paragraph may be summarized by the statement that for parity-violating processes, total angular momentum J as well as orbital angular momentum is preserved under crossing. This is not, of course, the case for parity-conserving processes where we have, for example, ($J=\frac{1}{2}$ P wave) $\rightarrow$ ($J=\frac{1}{2}$ P wave) which crosses to P waves with both $J=\frac{1}{2}$ and $J=\frac{3}{2}$, thereby complicating the treatment of parity-conserving processes.

There is still a further simplification in the $P=-1$ case. Returning to the reaction $\Pi^{\nu_1}+B^{\nu_2}(J=\frac{1}{2}\,P\text{ wave})\rightarrow \Pi^{\nu_3}+B^{\nu_4}(J=\frac{1}{2}\,S\text{ wave})$ which crosses to $\Pi^{\bar\nu_3}+B^{\nu_2}(J=\frac{1}{2}\,S\text{ wave})\rightarrow\Pi^{\bar\nu_1}+B^{\nu_4}(J=\frac{1}{2}\,P\text{ wave})$, we note that Π^{ν_3} in the direct channel and its crossed partner $\Pi^{\bar\nu_3}$ both are in S waves and couple to the baryon pole with the parity-violating coupling δG while Π^{ν_1} and $\Pi^{\bar\nu_1}$ both are in P waves and have the symmetric coupling G. Referring to Fig. 1, we see that, for $P=-1$, diagram (1b) does not appear, which means $\eta_b{}^{BB}=\eta_b{}^{\Delta\Delta}=0$.

Thus, we have found that for $P=-1$ all the "dynamical" factors vanish except $\eta_a{}^{BB}$ and $\eta_a{}^{\Delta\Delta}$. The calculation of these factors is straightforward and we obtain

$$\eta_a{}^{BB}=D_{8_\theta}'(M^B)^{-1}[D_{8_\theta}(W)/(W-M^B)]_{W=M^B}=1 \quad (11.2)$$

and

$$\eta_a{}^{\Delta\Delta}=[D_{10}'(M^\Delta)]^{-1}\times[D_{10}(W)/(W-M^\Delta)]_{W=2M^B-M^\Delta}. \quad (11.3)$$

Given the knowledge of the dynamical factors η, the remaining task is to evaluate the Clebsch-Gordan factors. We shall discuss two ways of doing this. The first way is to proceed exactly as in Sec. IV, obtaining Eqs. (4.26) and (4.29) with the following modifications:

(i) The η factors are now to be taken from above. Note that, unlike the η's for parity-conserving couplings, there is no distinction between $\eta_a{}^{\Delta\Delta}(X=10)$ and $\eta_a{}^{\Delta\Delta}(X\neq 10)$ because all parity-violating couplings are "off-diagonal." This "off-diagonal" nature is also responsible for the remaining modifications, which involve factors of 2.

(ii) In Table IV, relating to the Z_X and n_X factors, the 8_θ row now takes on the same "off-diagonal" value as the $8_{\theta*}$ row, and the 10 row takes on the same values as the $\overline{10}$ row.

(iii) In Table V, relating to the $C=-$ projection, the components $\delta R(8_\theta,8_\theta)$, $\delta R(8_\theta,8_\theta(s))$, and $\delta R(8_\theta,8_\theta(a))$ are to be multiplied by $\sqrt{2}$ (note that these components are now present in A both for $C=+$ and $C=-$). Taking the appropriate values for all these factors, we evaluated (4.26) and (4.29), obtaining the eigenvalues and eigenvectors of $A^{R^*R^*}$ and verifying the eigenvalues and eigenvectors previously reported by A^{RR} in Ref. 3.

The second way of evaluating the Clebsch-Gordan factors involves a different set of basis states than Sec. IV, but is ultimately easier and yields more insight. By using this second method in Ref. 3, we were able to obtain A^{RR} without resorting to machine calculation (unfortunately, these advantages of the second method apply only to parity-violating decays).

To see why a different choice of basis state yields simpler equations, recall that only Fig. 1(a) contributes to $\delta R_{\nu_1\nu_2,\nu_3\nu_4}$ for the $P_{1/2}\rightarrow S_{1/2}$ reaction. Projecting $\nu_1\nu_2$ onto the incoming state 8_θ, k, we can express the contribution of Fig. 1(a) to δG diagrammatically by Figs. 3(a) and 3(b). For comparison, diagrams expressing the Clebsch-Gordan content of $A_{MM(\mathrm{ext})}$ are presented in Figs. 3(c) and 3(d). One sees that if in the coupling calculation, Π_k and S_σ (expressing the transformation property of the symmetry breaking) are combined into

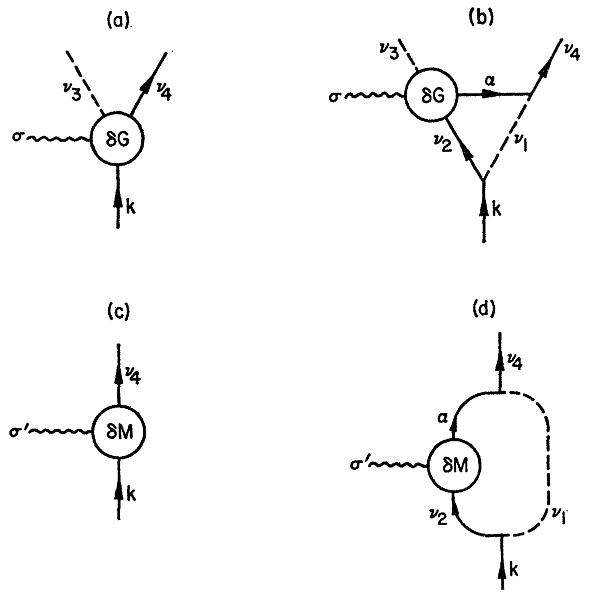

FIG. 3. Figure 3(b) represents the effect of baryon exchange [Fig. 1(a)] on the parity-violating reaction $B_k \to \Pi_{\nu_3} + B_{\nu_4}$ in Fig. 3(a). In each case, the wavy line represents the $SU(3)$ violation, transforming like the σ component of representation S. Figure 3(d) represents the group-theory factors in the contribution of the external baryon mass shift to the baryon mass shift [Fig. 3(c)] transforming like the σ' component of representation S'.

the nth member of representation N, then N, n plays the same role in $A^{GG}{}_{(\text{parity-violating})}$ as S', σ' plays in $A^{MM(\text{ext})}$. Thus we can read off the group-theoretical factor for $A^{GG}{}_{(\text{parity-violating})}$ from the relatively simple factor for $A^{MM(\text{ext})}$, provided we express the coupling shifts in terms of the basis where $B_{\nu_4}\bar{B}_k$ are combined into representation N instead of the basis of Sec. IV, where $B_{\nu_4}\Pi_{\nu_3}$ are combined into representation X.

We now proceed to work out the equations for the new basis in detail. We wish to calculate

$$\delta G_{kj}{}^{i} = \sum_{i'j'k'} A_{kj,k'j'}{}^{ii'} \delta G_{k'j'}{}^{i'} + D_{kj}{}^{i}. \qquad (11.4)$$

In the parity-conserving case, we changed from the $\delta G_{kj}{}^{i}$ basis to the $\delta R_S(8_\theta, X)$ basis by means of the transformation (4.30). In the present case, it is more convenient to remain in the $\delta G_{kj}{}^{i}$ basis for a time, before transforming to the new basis.

The basic equation from which A can be deduced is (3.22). Since the form of Eq. (3.22) is independent of X, the conversion of (3.22) from the X to the individual-particle basis is trivial; we obtain

$$\delta G_{\nu_4 k}{}^{\bar{\nu}_3} = -\frac{1}{G D_{8_\theta}'(M^B) 2\pi i} \int_C \frac{D_{8_\theta} \delta T_{8_\theta k,\nu_3\nu_4} dW'}{W' - M^B}, \qquad (11.5)$$

where we have used $\delta R = -G\delta G$, $\delta T_{8_\theta k,ij}$ is the amplitude for ΠB in the k component of the 8_θ state to go to $B^{\nu_4} + \Pi^{\nu_3}$, and the labeling on δG then follows from Eq. (4.1). Specifically, $\delta T_{8_\theta k,\nu_3\nu_4}$ is given by

$$\delta T_{8_\theta k,\nu_3\nu_4} = \sum_{\nu_1\nu_2} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \nu_2 & k \end{pmatrix} \delta T_{\nu_1\nu_2,\nu_3\nu_4}. \qquad (11.6)$$

Evaluating the exchange diagrams according to (4.10) and taking account of the fact that $\eta_b = 0$, we obtain

$$\delta G_{\nu_4 k}{}^{\bar{\nu}_3} = \frac{-1}{G D_{8_\theta}'(M^B) 2\pi i} \sum_{\nu_1\nu_2\alpha} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \nu_2 & k \end{pmatrix} \times \int_C \frac{D_{8_\theta} T_a \delta G_{\alpha\nu_2}{}^{\bar{\nu}_3} G_{\nu_4\alpha}{}^{\nu_1} dW'}{W' - M^B}. \qquad (11.7)$$

Writing

$$G_{\nu_4\alpha}{}^{\nu_1} = G \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \alpha & \nu_4 \end{pmatrix},$$

and using (4.19) and our previous result that $\eta_a{}^{BB} = 1$ to eliminate the dispersion integral, we reduce (11.7) to

$$\delta G_{\nu_4 k}{}^{\bar{\nu}_3} = \sum_{\nu_1\nu_2\alpha} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \nu_2 & k \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \alpha & \nu_4 \end{pmatrix} \delta G_{\alpha\nu_2}{}^{\bar{\nu}_3}. \qquad (11.8)$$

Comparing (11.8) with the A term in (11.4), we finally obtain

$$A_{\nu_4 k,\alpha\nu_2}{}^{\bar{\nu}_3 j} = \delta_{\bar{\nu}_3 j} \sum_{\nu_1} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \nu_2 & k \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_\theta \\ \nu_1 & \alpha & \nu_4 \end{pmatrix}, \qquad (11.9)$$

which will be recognized as Eq. (6) of our previous paper.[3]

We now turn to the new basis, where

$$\delta G_{kj}{}^{i} = \sum_N Z_N \delta G_S(N) \times \sum_n \begin{pmatrix} 8 & S & N \\ i & \sigma & n \end{pmatrix} \begin{pmatrix} 8 & 8 & N \\ j & \bar{k} & n \end{pmatrix}. \qquad (11.10)$$

Equation (11.10) is analogous to (3.9), with the second Clebsch-Gordan coefficient representing the projection of $B_j\bar{B}_{\bar{k}}$ onto representation N, the first Clebsch-Gordan coefficient representing the combination of N with Π_i to form a coupling transforming like S, $\delta G_S(N)$ representing the strength of this coupling, and Z_N representing a normalization factor. The projection necessary to invert (11.10) can be derived from (4.14) and (4.15) and turns out to be

$$\delta G_S(N) \equiv P\delta G_{kj}{}^{i} = \frac{N_S}{N_N Z_N} \sum_{ijkn} \begin{pmatrix} 8 & S & N \\ i & \sigma & n \end{pmatrix} \begin{pmatrix} 8 & 8 & N \\ j & \bar{k} & n \end{pmatrix} \delta G_{kj}{}^{i}. \qquad (11.11)$$

Replacing $\delta G_{\alpha\nu_2}{}^{\bar{\nu}_3}$ by (11.10) on the right side of (11.8), and multiplying both sides by the projection operator of

(11.11), we obtain

$$\delta G_S(N)=\frac{N_S}{N_N Z_N}\sum_{\nu_1\nu_2\alpha n'}\sum_{\nu_3\nu_4 k n}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\times\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\begin{pmatrix}8 & S & N\\ \bar{\nu}_3 & \sigma & n\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ k & \bar{\nu}_4 & n\end{pmatrix}\times\sum_{N'}\begin{pmatrix}8 & S & N'\\ \bar{\nu}_3 & \sigma & n'\end{pmatrix}\begin{pmatrix}8 & 8 & N'\\ \nu_2 & \bar{\alpha} & n'\end{pmatrix}Z_{N'}\delta G_S(N'). \quad (11.12)$$

Equation (11.12) is analogous to, and essentially as complicated as, (4.18). The simplification comes when we recognize that the indices ν_3 and σ appear only in the Clebsch-Gordan coefficients involving S. Now we know, in general, that A is independent of σ. Therefore, we may write the part of (11.12) that depends on S, σ, and ν_3 as

$$\sum_{\bar{\nu}_3}\begin{pmatrix}8 & S & N\\ \bar{\nu}_3 & \sigma & n\end{pmatrix}\begin{pmatrix}8 & S & N'\\ \bar{\nu}_3 & \sigma & n'\end{pmatrix}=\frac{1}{N_S}\sum_{\sigma\bar{\nu}_3}\begin{pmatrix}8 & S & N\\ \bar{\nu}_3 & \sigma & n\end{pmatrix}\begin{pmatrix}8 & S & N'\\ \bar{\nu}_3 & \sigma & n'\end{pmatrix}=\frac{1}{N_S}\delta_{nn'}\delta_{NN'}, \quad (11.13)$$

so that

$$\delta G_S(N)=\frac{1}{N_N}\sum_{\nu_1\nu_2\alpha}\sum_{\nu_4 k n}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\times\begin{pmatrix}8 & 8 & N\\ k & \bar{\nu}_4 & n\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ \nu_2 & \bar{\alpha} & n\end{pmatrix}\sum_{N'}\delta_{NN'}\delta G_S(N'). \quad (11.14)$$

Thus the A matrix for parity-violating couplings is diagonal in N. For a given N, it is the same for all S contained in $\mathbf{8}\times\mathbf{N}$.

The analogy between (11.14) and the equation for $A^{MM(\mathrm{ext})}$ becomes even clearer if we recognize that each n contributes the same amount to (11.14). This allows us to deduce that

$$A_S^{G(N)G(N')}=\delta_{NN'}\sum_{\nu_1\nu_2\alpha}\sum_{\nu_4 k}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\times\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ k & \bar{\nu}_4 & n\end{pmatrix}\begin{pmatrix}8 & 8 & N\\ \nu_2 & \bar{\alpha} & n\end{pmatrix}. \quad (11.15)$$

This is just the group theory factor appearing in $A^{MM(\mathrm{ext})}$ [Fig. 3(d)], provided one replaced N, n by S, σ.

The comparison between the leading eigenvector of $A^{GG}{}_{\text{parity-violating}}$ and experiment is excellent, as described in Refs. 1 and 3. All six ratios among the observed parity-violating $B\to B+\pi$ amplitudes are well accounted for, which makes this our "best case." It is therefore interesting to note that A for this case is less parameter-dependent than usual.

Note that:

(i) The part of A which affects parity-violating $BB\Pi$ couplings is independent of the form of the denominator functions, since the $BB\Pi$ decay decouples from the $\Delta B\Pi$ decay and the B pole lies at the same energy in the direct and crossed channels.

(ii) The form of $A^{GG}{}_{\text{parity-violating}}$ ensures that vertex symmetry holds in this case. To see this, it is easiest to start with Eq. (11.8):

$$\delta G_{\nu_4 k}{}^{\bar{\nu}_3}=\sum_{\nu_1\nu_2\alpha}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\delta G_{\alpha\nu_2}{}^{\bar{\nu}_3}. \quad (11.8)$$

We recall that vertex symmetry holds if the coupling on the left side of (11.8) has the same Hermiticity property,

$$\delta G_{\alpha\beta}{}^k=C(-1)^{Q_k}\delta\bar{G}_{\beta\alpha}{}^{\bar{k}}, \quad (4.3)$$

as the coupling on the right side of (11.8). This motivates, us to compare (11.8) with the corresponding relation

$$\delta\bar{G}_{k\nu_4}{}^{\nu_3}=\sum_{\nu_1\nu_2\alpha}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & \nu_4\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & k\end{pmatrix}\delta\bar{G}_{\alpha\nu_2}{}^{\nu_3}, \quad (11.16)$$

where the reality of the Clebsch-Gordan coefficients has been used. Since d and ν_2 are summed over, we can interchange them in (11.6) to obtain

$$\delta\bar{G}_{k\nu_4}{}^{\nu_3}=\sum_{\nu_1\nu_2\alpha}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\delta\bar{G}_{\nu_2\alpha}{}^{\nu_3}. \quad (11.17)$$

Next we employ the Hermiticity property (4.3) of the *input* δG on the right side of (11.7) to obtain

$$\delta\bar{G}_{k\nu_4}{}^{\nu_3}=C_{\text{input}}(-1)^{Q_{\bar{\nu}_3}}\sum_{\nu_1\nu_2\alpha}\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \alpha & \nu_4\end{pmatrix}\times\begin{pmatrix}8 & 8 & 8_\theta\\ \nu_1 & \nu_2 & k\end{pmatrix}\delta G_{\alpha\nu_2}{}^{\bar{\nu}_3}. \quad (11.18)$$

Comparing (11.18) with (11.8), we see that

$$\delta G^{\bar{\nu}_3}{}_{\nu_4 k\,\text{output}}=C_{\text{input}}(-1)^{Q_{\bar{\nu}_3}}\delta\bar{G}^{\nu_3}{}_{k\nu_4\,\text{output}}, \quad (11.19)$$

so that δG_{output} retains the same Hermiticity property as δG_{input}.

It is also interesting to compare the parity-violating $BB\Pi$ couplings, associated with the leading eigenvector of A^{GG}, with the predictions obtained from $SU(6)$ under the assumption of **35** dominance.

Leading eigenvector. As described in Ref. 3, the leading eigenvector of $A^{GG}{}_{\text{parity-violating}}$ gave $B\bar{B}$ in the dominantly antisymmetric octet state $\mathbf{8}_a+\frac{1}{4}\mathbf{8}_s$ (the analogy with $A^{MM(\mathrm{ext})}$ and the fact that $A^{MM(\mathrm{ext})}$ dominates A^{MM^2} explains why this is the same combination that occurs in the leading eigenvector for the baryon mass shift). This placement of $B\bar{B}$ in the octet state determined five ratios among the observed $B\rightarrow B+\pi$ decays, and the assumption of $\mathcal{C}=-$ for the parity-violating decay determined the sixth ratio.

SU(6). In the $SU(6)$ treatment, it is found[42] that the **35** representation dominates the parity-violating $\bar{B}B\Pi$ coupling which can thus be thought of as a unitary singlet ($\bar{B}_{\overline{56}}B_{56}\Pi_{35}$ spurion$_{35}$) coupling. Since

$$\overline{\mathbf{56}}\times\mathbf{56}=\mathbf{1}+\mathbf{35}+\mathbf{405}+\mathbf{2695} \qquad (11.20)$$

and

$$\mathbf{35}\times\mathbf{35}=\mathbf{1}_s+\mathbf{35}_s+\mathbf{35}_a+\mathbf{189}_s+\mathbf{280}_a+\overline{\mathbf{280}}_a+\mathbf{405}_s, \qquad (11.21)$$

there are four independent ways to construct an over-all unitary singlet. However, the singlet in $\overline{\mathbf{56}}\times\mathbf{56}$ cannot produce observable strangeness-changing decays. Moreover, if we assume $\mathcal{C}=-$ for the spurion, then since $\mathcal{C}=+$ for $\overline{\mathbf{56}}\times\mathbf{56}$ and for Π, an over-all $\mathcal{C}=+$ can be obtained only from the antisymmetric products of $\mathbf{35}\times\mathbf{35}$. This last condition singles out $\mathbf{35}_a$ from $\mathbf{35}\times\mathbf{35}$. Then $\overline{\mathbf{56}}\times\mathbf{56}$ must be in **35**, and since this is the adjoint, $\bar{B}B$ is in $\mathbf{8}_a$, which is close to our $\mathbf{8}_a+\frac{1}{4}\mathbf{8}_s$ above.

In addition to $A^{G(BB\Pi)G(BB\Pi)}$ we have also evaluated $A^{G^*(\Delta B\Pi)G^*(\Delta B\Pi)}$, which is relevant to Ω^- decays. For the parity-conserving amplitude, we recall there were several eigenvalues near one, so Ω^- decay was enhanced but the ratios could not be predicted. For the parity-violating amplitude the largest eigenvalue of $A^{G^*G^*}$ was 0.5 if linear D was used, and less if the curved Balazs D was used. Thus we are again unable to predict ratios, but we do expect the parity-conserving amplitude to predominate somewhat.

XII. COMPARISON WITH OTHER STUDIES

The present work is descended from the initial work on octet enhancement by Cutkosky and Tarjanne,[43] and the study of strong $B\Delta\Pi$ coupling shifts by Wali and Warnock.[8,7] Roughly speaking, Wali and Warnock estimated A^{gM} but not A^{gg}.[44] Since the largest term in $A^{gM}\delta M$ feeds the same eigenvector that is favored by A^{gg}, their results obtained using only A^{gM} are in qualitative agreement with ours. By including A^{gg}, we obtain a somewhat fuller picture of strong coupling shifts, as well as the new results we have enumerated for the weak interactions.

Technically, the method of Wali and Warnock[8] is somewhat different from ours. They use the N/D method, keeping the numerator $SU(3)$ symmetric but putting the physical masses into ρ in

$$D=1-\int N\rho(W'-W)^{-1}dW'.$$

This procedure varies the position and residue of the direct-channel singularities, but not the position of the exchange singularities. The resulting equations are considerably simpler than ours. In terms of parts of the A matrix, external mass effects on the direct channel are well taken into account, the (numerically less important) exchange-mass shifts are neglected, external mass-shift effects on the left cut are neglected, and exchange-coupling shifts are not systematically accounted for by their method.

Ernst, Wali, and Warnock[7] have stressed two difficulties common to all these studies: (i) The approximations do not guarantee "vertex symmetry" (Sec. V); and (ii) The shifts are so large that higher order effects represent an important and interesting correction to the linear perturbation theory we have been using.

Difficulty (i) does not happen to be serious for our leading eigenvectors—it was shown in Secs. V and XI that they possess the required symmetry to within a few percent. Difficulty (ii) would become really important if higher order effects drove eigenvectors of A^{gg} with eigenvalues far from one much more strongly than the eigenvector with eigenvalue near one, or if they drove the leading 27 eigenvectors as strongly as the leading octet eigenvector. It is not known whether this happens for strong coupling shifts. Empirically, we have seen that the linear theory gives good results for mass shifts and parity-violating decays, and that higher order effects on parity-conserving nonleptonic decays (producing abnormal $\mathcal{C}$ through the combined action of strong symmetry breaking and weak interactions) are comparable to but not dominant over the linear effects.

In another recent study, Diu, Rubinstein, and Van Royen[45] have calculated A^{gg} for $BB\Pi$ and $\Delta B\Pi$ coupling shifts by the same approach as ours, and obtained eigenvalues in complete agreement with ours.

Another approach to symmetry breaking is the "tadpole" theory,[9] involving octets of 0^+ mesons. In a previous paper[46] it was shown that if a low-mass 0^+ octet exists, A can easily have an eigenvalue near unity, so that tadpole theory and the methods of the present paper may actually be related. The connection does not *necessarily* hold, however; 0^+ particles do not *require* an eigenvalue of A near one or vice versa.[47] Now in the

[42] G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters **14**, 70 (1965); K. Kawarabayashi, Phys. Rev. Letters **14**, 86 (1965); **14**, 169 (1965); P. Babu, *ibid.* **14**, 166 (1965); S. P. Rosen and S. Pakvasa, *ibid.* **13**, 733 (1964); M. Suzuki, Phys. Letters **14**, 64 (1965).

[43] R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963).

[44] Another paper in which mass shifts are used to derive $B\Delta\Pi$ coupling shifts is that of E. Johnson and E. R. McCliment, Phys. Rev. **139**, B951 (1965).

[45] B. Diu, H. Rubinstein, and R. Van Royen, Nuovo Cimento **43A**, 961 (1966).

[46] R. Dashen and S. Frautschi, Phys. Rev. **140**, B698 (1965).

[47] In Ref. 45 we showed that low-mass 0^+ hadrons would imply an eigenvalue of X, the matrix expressing the self-consistent effects in 0^+ emission, near unity at $q^2=0$. They do *not*, however, imply that the *submatrix* of $X(q^2=0)$ connecting monopole couplings (couplings which persist in the limit $q_\mu\rightarrow 0$) has an eigenvalue near unity, and it is this submatrix which has the same eigenvalues as A. Thus the statement made in that reference that low-mass 0^+ hadrons *ensure* an eigenvalue of A near one is not correct.

present paper we calculated, and made physical use of, several different eigenvalues of A^{gg} near one. They included positive-parity octets of either charge conjugation, and a negative-parity octet of charge conjugation opposite to the $\pi K\eta$ octet. It is possible that an octet of 0^+ particles corresponding to one of these eigenvalues of A exists. But it seems unlikely that separate octets of tadpoles exist corresponding to *each* eigenvalue of A^{gg} near one as well as the eigenvalue of A^{MM} near one (which requires a separate set of tadpoles according to Coleman and Glashow[9]).

XIII. *CP*-VIOLATING COUPLINGS

The recent discovery of CP violation in K decays[48] has opened the possibility that CP is violated in weak $BB\Pi$ and $\Delta B\Pi$ couplings, or perhaps even in semistrong couplings.[49] The method of the present paper cannot tell us whether or not CP violations occur, or their over-all strength, but does give information on the ratios of couplings if such violations do occur.

The procedure for calculating the A matrix for $P=+$, $C=-$, and $P=-$, $C=+$ couplings has already been given in Sec. XI, where we were interested in terms with abnormal $\mathfrak{C}$. The only change comes in Eqs. (4.30) and (4.31) for obtaining the couplings corresponding to an eigenvector of A: the $P=+$, $C=-$, $S=8$ coupling comes from the eighth component ($\sigma=8$) for strong interactions, $\sigma=3$ or 8 for electromagnetic interactions, and $\sigma=6$ for strangeness-changing weak interactions—instead of $\sigma=7$ for "abnormal" C- and P-conserving weak interactions.

As discussed in Sec. XI and Ref. 3, there is no lack of eigenvalues of A^{gg} near one for CP-violating couplings. For $P=+$, $C=-$, $S=8$, eigenvalues 1.0 and 0.7 are found, for $P=-$, $C=+$, $S=1$ or 27, the eigenvalue 0.7. Thus if CP violation exists, it can readily become enhanced and competitive with CP-conserving couplings.

The only possible consequence of CP violation we shall discuss here is the question: What happens to our predictions for the weak interactions if CP is violated? We can make the following comments:

(i) The phase relations between amplitudes for reactions like $\Lambda\to p+\pi^-$ and $p\to\Lambda+\pi^+$ depend on C [Eq. (4.5)]. If both reactions could be observed, these relations would give information on C. In practice, however, due to the mass spectrum of the baryons, only decays with $\Delta Y=+1$ are observed ($\Xi\to\Lambda\pi$, $\Lambda\to N\pi$, $\Sigma\to N\pi$), so C cannot be determined in this way. ($\sigma=6$ cannot be directly distinguished from $\sigma=7$ in the observed decays.)

(ii) According to Ref. 3 and Sec. XI, the leading eigenvector for $P=-$ decays predicts $\Sigma_+{}^+=0$, the ratios of $\Xi_0{}^0$ to $\Sigma_0{}^0$ to Λ_0, and the ratios of $\Xi_-{}^-$ to $\Sigma_-{}^-$ to Λ_-, independently of charge-conjugation properties. CP conservation implies only one further relation[3] (which can be taken as the $\Delta I=\frac{1}{2}$ rule for Λ decay). This last relation is well-satisfied experimentally but might have some other explanation; thus the success of our theory for $P=-$ decays does not tell us much about CP properties.

(iii) The usual phenomenological analysis of nonleptonic baryon decays[50] is made with the simplifying assumption $T=+$. If CP-violating terms are present in the amplitude, they have $T=-$ by the TCP theorem and would be 90° out of phase with CP-conserving terms according to Eq. (4.7). What the experimental "asymmetry parameter" in baryon decay gives us, then, is the interference between the S-wave amplitude and that part of the P wave which has the same time-reversal properties as the S wave (assuming final-state interactions are small). Redoing the phenomenological analysis with T violation in mind, one finds that the magnitudes of the S wave are essentially unchanged and the "in-phase" part of the P waves not much changed, although there is room for "out-of-phase" P waves comparable to the "in-phase" P waves. The considerations of Sec. XI still apply to the "in-phase" P waves: Both "normal" and "abnormal" C are required for us to fit them, independently of CP conservation.

To summarize, then, the possibility of CP violation in baryon decays affects our conclusions very little.

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APPENDIX A. CONVERGENCE OF THE DISPERSION INTEGRALS

As discussed in Ref. 2, the integrals (3.15) and (3.16) for δR and (3.17) for δM converge well when the D functions (3.19) and (3.20), which approach a constant as $W\to\infty$, are used. In fact, the convergence of the integrals for δR and δM is better than the convergence of appropriate dispersion integrals representing the unperturbed strong interactions in many practical situations. To see this, it is best to change temporarily from the static amplitude [Eqs. (2.1) and (2.2)] to kinematic, singularity-free amplitudes. For the $J=\frac{1}{2}^+$ amplitude, for example, we have

$$T(W)=\frac{W^2}{[(W-M^B)^2-(M^\pi)^2]}\frac{e^{2i\eta}-1}{2iq}. \quad (A1)$$

[48] J. Christenson, J. Cronin, V. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964).

[49] T. D. Lee and L. Wolfenstein, Phys. Rev. **138**, B1490 (1965); J. Prentki and M. Veltman, Phys. Letters **15**, 88 (1965).

[50] See, for example, M. Stevenson, J. Berge, J. Hubbard, G. Kalbfleisch, J. Shafer, F. Solmitz, S. Wojcicki, and P. Wohlmut, Phys. Letters **9**, 349 (1964); R. Dalitz, 1964 Varenna lectures (to be published).

Now consider some specific term such as the contribution of Δ exchange to the $\frac{1}{2}^+$ channels. It is well known that spin-$\frac{3}{2}^+$ exchange is divergent, contributing a term

$$T(W) \xrightarrow[W\to\infty]{} (\text{constant})$$

(we ignore $\ln W$'s) which exceeds the unitarity bound by one power of W at large W. A perturbation on this term, δT, will generally also approach a constant at large W, but this behavior combined with the behavior

$$D(W) \xrightarrow[W\to\infty]{} (\text{constant})$$

produces only logarithmic divergences in Eqs. (3.15)–(3.17) for δR and δM. Thus the convergence at large W is better by one power of W for first-order perturbations than it is for iterations which one usually makes in treating the strong interactions.

This advantage applies only to first-order perturbations; in second order, the unitarity relation $\mathrm{Im}T = \bar{T}\rho T$, with input $T\sim$ constant at large W and the phase factor $\rho\sim W$ at large W, introduces a worse asymptotic behavior.

Another difficulty with N/D calculations of strong interactions is that when correct threshold behavior is imposed, divergences develop at high energy for all but the lowest partial waves. Again, no difficulty of this type occurs in our treatment of first-order perturbations: e.g., in the above example,

$$\delta T^l(W) \xrightarrow[W\to\infty]{} \text{constant and } D^l(W) \xrightarrow[W\to\infty]{} \text{constant}$$

for all l so (3.15)–(3.17) converge for all l.

We believe that the good convergence of the perturbation integral is responsible for the relatively successful results of calculations on perturbed bootstraps. Contributions from W which are far from M are usually approximated or left out of bootstrap calculations, both perturbed and unperturbed. For normal unperturbed bootstraps, the resulting errors are serious; the method is quite successful in showing which channels have strong attractions and therefore resonances or bound states, but quantitative success in predicting such things as the positions of resonances is generally not achieved. Our studies of perturbations on the B-Δ reciprocal bootstrap, on the other hand, keeping just the usual singularities near $W=M$, but with the advantage of improved convergence, have yielded results within 30% of the data for: (i) the neutron-proton mass difference,[19] (ii) the ratios of mass differences within the B and Δ multiplets,[2] (iii) the ratios of parity-violating nonleptonic decay amplitudes of baryons,[1,3] (iv) the ratios of various electromagnetic couplings of B and Δ, such as the D/F ratio for baryon magnetic moments,[51] and (v) the ratios of various weak couplings of baryons to leptons.[51]

[51] R. Dashen and S. Frautschi, Phys. Rev. **143**, 1171 (1966).

APPENDIX B. CHOICE OF D FUNCTIONS

In the present study of perturbations, as well as in the earlier treatment of B and D mass shifts,[2] only terms appearing in the static limit have been considered. Also the form used for the D function, Eq. (3.19), did not contain physical effects (such as the Roper resonance) which take one beyond the static model. Thus our results are to be interpreted as results of the static model.

While the connection of (3.19) (or its linearized version) to the static model is the most straightforward reason for using this form of D, it is interesting to consider what D function would be appropriate if one went beyond the static model and attempted a more exact calculation. In the present Appendix we give some arguments on this difficult question.

For a single-channel amplitude, there is a unique denominator function which has the phase of the amplitude along the right cut and no Castillejo-Dalitz-Dyson (CDD) singularities. This unique D function was prescribed[52] for use in relations such as

$$\delta M = \frac{1}{R[D'(M)]^2}\frac{1}{2\pi i}\int_C \frac{D^2(W')\delta T(W')}{W'-M}dW' \quad \text{(B1)}$$

which occur in the study of perturbations on the amplitude.

In practice, however, strong interactions always couple many channels together. Any one channel can be described in terms of various phases, such as the phase $\mathrm{Re}\eta$ occurring in the S matrix $e^{2i\eta}$, or the phase of the single-channel amplitude

$$T = \rho[e^{-2\,\mathrm{Im}\eta}e^{2i\,\mathrm{Re}\eta} - 1]/2i \quad \text{(B2)}$$

which differs from $\mathrm{Re}\eta$ in the presence of absorption. Corresponding to each choice of phase, a different D function can be defined.[53]

Thus we are unavoidably faced with a decision; *which* D function, among various possibilities, will we use in equations such as (B1)? This problem was not noticed in the original single-channel derivation of Eq. (B1), but we wish to bring it out into the open now. The denominator function (3.19) used in this and previous papers[2] will emerge from this discussion as an especially convenient choice, although it is certainly not "*the* physical D function."

Lest the reader become too nervous about this apparent arbitrariness, we hasten to add that the main results of this paper are not so sensitive to the details of the D function. Among the various parts of the A matrix, A^{RR} is not very sensitive to details of D, as discussed in Sec. VI. The overall magnitude of A^{RM} is

[52] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).

[53] For an excellent discussion of the two choices mentioned in this paragraph, see J. Hartle and C. Jones, Ann. Phys. (N. Y.) **38**, 348 (1966).

highly sensitive, so we used the theoretical estimates of A^{RM} only to estimate *ratios*. A^{MR} and A^{MM} are again less sensitive in our model.[54]

Let us now review the properties required of D.

(i) Singularities of D on the physical sheet are confined to the right-hand cut and possible CDD poles [note that if D has a CDD pole, the pole gives rise to an additional term in the contour integrals of Eqs. (B1) and (3.15), and (3.16)].

(ii) $D(W)$ should be suitably bounded at large W to allow the integrals (3.15) to (3.16) and (B1) to converge. This implies that the representation for D must include any CDD poles that are present, instead of multiplying both N and D by a (divergent) factor $(W-W_{\text{CDD}})$.

(iii) Along the right-hand cut, D has the phase factor $e^{-i\delta}$, where δ is the physical phase shift in the case of elastic scattering but has various possible definitions (such as the phase of the S matrix or the phase of the single-channel amplitude) when inelasticity is present.

(iv) $D=0$ at the bound states or resonances under study.

These properties are incorporated in the Omnes representation for D in the presence of one bound state and N CDD poles,

$$D(W)=\prod_{i=1}^{N}\frac{(W-M)}{(W-W_{\text{CDD}})_i}\times\exp\left\{-\frac{(W-M)}{\pi}\int_{\text{thres}}^{\infty}\frac{\delta(W')dW'}{(W'-M)(W'-W)}\right\}. \quad \text{(B3)}$$

The open questions here are the choice of δ, and the CDD poles.

One choice which has been studied in detail recently by Shaw and Wong[55] is the D function one gets by taking δ to be the phase $\text{Re}\eta$ occurring in the S-matrix $e^{2i\eta}$, and assuming no CDD poles in the low-energy region. The πN phase shift $\text{Re}\eta$ for $I=\frac{1}{2}$, $J^P=\frac{1}{2}^+$ scattering is known to be small and negative at low energies. Recently, it has been found to turn positive above 150 MeV, becoming large[56] or very likely passing through a resonance[57] by 600 MeV. At higher energies, its behavior is unknown, so Shaw and Wong let it come back down again in a smooth fashion to give $D(W)$ a bounded behavior as W approaches ∞.

Inserting this phase into the Omnes formula, Shaw and Wong obtain a D function whose curvature differs considerably from our Balázs form (3.19). In particular, the positive sign of η above 150 MeV causes their D to increase *faster* than $|W-M^B|$ in a sizeable region around M^B before settling down and approaching a constant limit. This is in contrast to the Balázs D function (3.19), which increases slower than $|W-M^B|$ at all points along the left cut. As a result, use of the Shaw-Wong D would require a careful evaluation of $D\delta T$ and $D^2\delta T$ in the integrands of (B1) and (3.15), and (3.16) out to considerably larger values of $|W-M^B|$ than when the Balázs form, which damps the integrand at large $|W-M^B|$, is used.[58] This would be a serious disadvantage because only the singularities of δT near $W=M^B$ are somewhat well understood.

We prefer to use the Balázs D function rather than the Shaw-Wong form for two reasons. The first reason has to do with the fact that the $I=\frac{1}{2}$, $J^P=\frac{1}{2}^+$ scattering becomes highly inelastic in the region of the Roper resonance[56]—if there is a resonance here, it is not primarily associated with the πN channel. Thus it is very likely that the Roper phenomenon behaves like an "effective CDD pole" in the πN channel.[59] The statement of Levinson's theorem *for this channel* would then become

$$\text{Re}\eta(\infty)-\eta(0)=\pi(N_{\text{CDD}}-N_{\text{bound}})=0. \quad \text{(B4)}$$

[The *sum* over eigenphases of all the coupled channels would, of course, still go to $-\pi$ if there is no elementary particle involved,[60] but this condition does not prevent the single-channel phase from behaving as in (B4).] In this case, the large W behavior of (B3),

$$D(W)\sim W^{N_B-N_{\text{CDD}}+[\delta(\infty)-\delta(0)]/\pi}, \quad \text{(B5)}$$

must be brought down to a constant limit by including the CDD pole. Since the πN amplitude has a zero about 150 MeV above threshold, it is natural to place the CDD pole of D at this point.[61] Replacing the Shaw-Wong D function by one with this pole, one finds that it grows considerably less rapidly along the left cut and behaves more like the Balázs D. [Essentially the convergent factor $(W-W_{\text{CDD}})^{-1}$ is almost cancelling the divergent factor $(W-W_{\text{Roper}})$ along the left cut.] Roughly speaking, the Balázs D can be obtained by the approximation $(W-W_{\text{Roper}})/(W-W_{\text{CDD}})=1$, which is not so inaccurate on the left and avoids the new term that would have

[54] It happens that A^{MM} becomes much more sensitive in the $N-N^*$ than in the $B-\Delta$ reciprocal bootstrap, as stressed recently by G. L. Shaw and D. Y. Wong (Ref. 5).

[55] G. L. Shaw and D. Y. Wong (Ref. 5).

[56] P. Auvil, C. Lovelace, A. Donnachie, and A. Lea, Phys. Letters 12, 76 (1964).

[57] L. D. Roper, Phys. Rev. Letters **12**, 340 (1964).

[58] For example, Shaw and Wong point out that Δ exchange gets multiplied by a considerably larger factor when their D is used. Exchange of the higher ΠB resonances would also gain in importance.

[59] This type of situation has recently been discussed by a number of authors; for example, J. Hartle and C. Jones, Phys. Rev. **140**, B90 (1965); M. Bander, P. Coulter, and G. L. Shaw, Phys. Rev. Letters 14, 270 (1965); E. Squires, Nuovo Cimento **34**, 1751 (1964); D. Atkinson, K. Deitz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966).

[60] L. Cook and B. Lee, Phys. Rev. **127**, 283 (1962).

[61] In their paper, Shaw and Wong present two different models, one of which involves a CDD pole. Their CDD pole, however, is placed $8m_\pi$ above threshold and therefore affects the D function differently.

to be evaluated on the right-hand cut in (B1) and (3.15) and (3.16) at an (uncancelled) CDD pole.[62]

Next we turn to the second reason for preferring the Balázs D which applies even if the CDD pole was incorrectly identified in the first argument. The second argument runs as follows: The dispersion relations (B1) and (3.11) hold *exactly* for any D that satisfies conditions (i), (ii), and (iv) above, independently of how the phase of D is defined and its CDD poles are located, as one can verify by reviewing the derivation of the equations. Thus if we knew δT exactly, it would not matter what phase we gave D or how we located its CDD poles. In practice, however, only the nearby singularities of δT in the dispersion relations (B1) and (3.11) can be evaluated. The problem, then, is to choose from among various exact equations (corresponding to various choices of D) one which weights the known nearby singularities heavily compared to intermediate and distant singularities of $D^T\delta TD$. Now as we have already pointed out, the Balázs D damps intermediate and distant parts of the left cut much better than the Shaw-Wong D, and this makes it far preferable. The philosophy here is somewhat analogous to the recent evaluation of matrix elements of current commutators by Fubini and Furlan,[63] where the kinematic conditions are chosen partly with an eye to improving the convergence of the sum over intermediate states.

As was stated above, Eqs. (B1) and (3.11) are still *exact*, even if D does not have the Shaw-Wong choice of phase along its right cut. The price that is paid for using a different phase is an additional right-hand singularity of $D^T\delta TD$. To see what happens, it is sufficient to consider

$$T=\rho[e^{2i\eta}-1]/2i \tag{B2}$$

and

$$D=|D|e^{-i\delta}. \tag{B6}$$

In terms of these parameters, the perturbation on δT is

$$\delta T=\delta\rho[e^{2i\eta}-1]/2i+\rho\delta\eta e^{2i\eta} \tag{B7}$$

and one finds

$$\begin{aligned}\mathrm{Im}[D^2\delta T]=&\tfrac{1}{2}\delta\rho|D|^2[\cos 2\delta-e^{-2\,\mathrm{Im}\eta}\cos 2(\mathrm{Re}\eta-\delta)]\\&+(\delta\,\mathrm{Im}\eta)|D|^2\rho e^{-2\,\mathrm{Im}\eta}\cos 2(\mathrm{Re}\eta-\delta)\\&+(\delta\,\mathrm{Re}\eta)|D|^2\rho e^{-2\,\mathrm{Im}\eta}\sin 2(\mathrm{Re}\eta-\delta).\end{aligned} \tag{B8}$$

The first term, involving the variation of the kinematic factor ρ, occurs all along the right cut for any D function. The second term, involving perturbations on the absorption cut of the πN channel, would also be present above inelastic threshold for any D function (unless we considered the matrix problem with all channels included, which of course has its own complications). It is the third term which occurs only if a D function with phase $\delta\neq\mathrm{Re}\eta$ is used.

The status of the three terms along the right cut in our treatment using the Balázs D function is as follows. The $\delta\rho$ term is a mass-shift term, and is crudely incorporated into our treatment either through direct evaluation of $\delta M^B(\partial\rho/\partial M^B)$ along the right cut, or implicitly through the condition of mass-scale invariance (both methods are used to estimate A^{RM} in Sec. VIII). The third term is small until $\mathrm{Re}\eta$ turns positive above 150 MeV, allowing the coefficient $\sin 2(\mathrm{Re}\eta-\delta)$ to grow large. (The Balázs function corresponds roughly to a phase which is small and negative at low energies, passes through $-90°$ at W_0, and approaches $-\pi$ as W approaches ∞.) The second, inelastic, term begins at about the same place. Since no good model exists for the Roper phenomenon and the strong inelasticity above a couple of hundred MeV in this channel, we have no way to estimate either the second or third term in this region. Thus we find that, using either the Balázs or the Shaw-Wong D function, the dispersion relation receives a contribution above inelastic threshold which is poorly known because of our lack of understanding of δT there.

[62] We are thinking of the Roper phenomenon as a resonance or large phase shift mainly associated with inelastic channels, rather than as an elementary particle. It is worth commenting, however, on what the situation would be for real elementary particles. Elementary particles introduce arbitrary parameters into the calculation of mass and coupling perturbations. If D is defined to include any CDD poles, the arbitrary parameters arise from the contour integrals around the CDD poles in (3.15) and (3.16). On the other hand, if the CDD terms are inserted as zeros in N rather than poles in D, the arbitrary parameters arise from the subtractions required to make (3.15) and (3.16) converge. In the previous discussion of this subject in Sec. III, Ref. 2, we omitted the possibility of including the CDD poles in D.

[63] S. Fubini and G. Furlan, Physics 4, 229 (1965).