Green's Function in Intense-Field Electrodynamics*

HOWARD R. REISS
U. S. Naval Ordnance Laboratory, Silver Spring, Maryland

AND

JOSEPH H. EBERLY†
Department of Physics and Astronomy, University of Rochester, Rochester, New York

(Received 16 May 1966)

The problem of an electron in the presence of an intense monochromatic electromagnetic field is studied from the standpoint of the exact Green's function for the electron in relativistic quantum mechanics. It is shown that the electron acquires a mass increment above the field-free mass. The Green's function is derived and investigated at some length. It is initially derived in a form which is very different from the free-electron Green's function. This form is shown to yield an interesting momentum relation for the electron, and to lead to a seeming level structure of the electron. The Green's function is then transformed to a different form in which the electromagnetic-field effects are manifest as corrections to the free-electron result. This form makes evident the field-strength and electron-momentum conditions which lead to significant quantitative deviations from the free-particle case. Most of the detailed discussion is devoted to the case of the scalar "electron," but results are also given for the spinor electron.

I. INTRODUCTION

THE problem of an electron in an intense plane-wave electromagnetic field has now acquired a considerable literature and also aroused some controversy. The first work on the problem was the exact solution by Volkov[1] in 1935 of the Dirac equation in the presence of a plane-wave field. There was a long pause (in the course of which several other authors independently solved the same problem) before the next step, which was the use of the Volkov wave functions as a set of basis states for the perturbative solution of a physical problem. With this technique, Sengupta[2] calculated the Compton-scattering process, one of the present authors[3] solved the problem of electron pair production by colliding photon beams, and Nikishov and Ritus[4] worked out these processes as well as electron-pair annihilation. Brown and Kibble[5] and Goldman[6] also calculated Compton scattering by this same method of perturbing the Volkov states. All of these calculations were done with a monochromatic background field (i.e., the field encompassed within the Volkov solution), and they are all mutually consistent. Furthermore, they all share the interesting features that an index occurs which serves to count the number of photons from the background field which participate even though the field is not quantized and normally not described in terms of photons; and the mass of the electron in the presence of the field is increased by an amount which depends upon the strength of the field.

Controversy arose primarily because of the appearance of a quite different method of treating the same physical processes. Using the Feynman rules of quantum field theory, Compton scattering was calculated[7] on the assumption that the background field was so strong that its photons could not be depleted, and that interactions with the background field would dominate all self-field radiative effects. It was then possible to find closed-form sums for electron self-energy effects to all orders in the background field. Under these circumstances it would be expected that field theory would give the same results as first-quantized relativistic quantum mechanics. However, in Ref. 7 no momentum or mass shifts were found. This motivated Kibble[8] to ascribe this lack to the use of an infinite plane-wave field in Ref. 7, whereas Brown and Kibble[5] had tried to cast their calculation in the form of an infinite-field limit of a field of finite extent. However, this explanation is not compatible with the other work[2–6] using the (infinite field) Volkov solution, as well as the Brown-and-Kibble paper itself, in which exactly the Volkov solution is obtained by their limiting process. The problem is removed in a recent paper by the present authors[9] (which will be referred to hereafter as I) in which it is shown that a class of diagrams was omitted in Ref. 7 and that this class of diagrams yields exactly the momentum dependence and mass shift found in the first-quantized approach.

One point has become quite clear in these investigations of electrodynamics with a strong plane-wave field. That is that the interesting features which arise in calculating physical processes in terms of first-order perturbation theory with Volkov states, instead of the conventional second-order perturbation theory with free-particle states, are inherent in the Volkov solution

* Partially supported by the U. S. Army Research Office (Durham).

† On leave of absence from U. S. Naval Ordnance Laboratory, Silver Spring, Maryland.

[1] D. M. Volkov, Z. Physik 94, 250 (1935).
[2] N. D. Sengupta, Bull. Math. Soc. (Calcutta) 44, 175 (1952).
[3] H. R. Reiss, J. Math. Phys. 3, 59 (1962).
[4] A. I. Nikishov and V. I. Ritus, Zh. Eksperim. i Teor. Fiz. 46, 776 (1964) [English transl.: Soviet Phys.—JETP 19, 529 (1964)].
[5] L. S. Brown and T. W. B. Kibble, Phys. Rev. 133, A705 (1964).
[6] I. I. Goldman, Phys. Letters 8, 103 (1964).
[7] Z. Fried and J. H. Eberly, Phys. Rev. 136, B871 (1964).
[8] T. W. B. Kibble, Phys. Rev. 138, B740 (1965).
[9] J. H. Eberly and H. R. Reiss, Phys. Rev. 145, 1035 (1966); referred to as I in the text.

itself. Thus the present paper is devoted to a detailed study of the properties and physical implications of the Volkov problem *per se*. Specifically, the present paper gives a direct demonstration of the existence of the mass shift of an electron propagating in a monochromatic plane-wave field, and contains a study of the properties of the exact Green's function for this problem. It has also been found[10] that the Volkov wave functions themselves exhibit some extremely interesting properties that hitherto have not been remarked upon. This investigation of the Volkov wave functions, however, is deferred to a subsequent paper now being prepared.

In the present paper, the Green's function is derived first for the scalar electron. This Green's function has some very interesting properties. It is found to be quite different in form from the usual electron propagator, and it is shown to imply an interesting momentum relation for the electron. The Green's function is given initially in the form of a fourfold momentum integral. When the p^0 integration is done, the form of the result is no longer greatly different from the free-particle case, but is clearly a generalization of the usual free-particle result. There are, however, important differences between the free-particle case and the present result. It is shown that when a particular combination of electromagnetic field and electron parameters approaches unity, that quantitative deviations from free-particle behavior should become manifest. Also, it is found that an index appears which denumerates photons in the unquantized electromagnetic field, and the electron exhibits properties which can be interpreted as representing a level structure. The scalar-particle Green's function derived here is shown to be consistent with the calculation of I in the sense that the leading contribution of the Green's function derived here is identical to the diagonal elements of the Green's operator derived by field-theoretic techniques in I.

Finally, the exact Green's function is presented for a spinor electron in a plane-wave field. This Green's function is found to be related to the scalar case in a manner analogous to (but much more complicated than) the relation between the Green's functions for free scalar and spinor particles.

II. EQUATION OF MOTION

In this paper we consider the problem of an electron in an intense monochromatic electromagnetic field. The calculation is done within the framework of relativistic quantum mechanics. The case of circular polarization of the electromagnetic field will be treated, with only occasional remarks about the differences introduced by plane polarized waves. In addition, since we show that the scalar "electron" exhibits many interesting properties which are preserved (and complicated) in the more difficult spinor case, we shall consider primarily the scalar case. The spinor electron is treated briefly at the end of the paper. The name "electron" will be applied to both the scalar and spinor particles.

We wish to regard the electromagnetic field as being so intense that it can be treated as an external (or background) field. A scalar particle is described, therefore, by the Klein-Gordon equation

$$[(i\partial_\mu - eA_\mu)^2 - m^2]\psi = 0\,, \tag{2.1}$$

where m is the electron mass and A_μ is the four-vector potential

$$A_\mu = a\,\mathrm{Re}(\epsilon_\mu e^{-i\varphi})\,. \tag{2.2}$$

In Eq. (2.2), a is a real scalar amplitude factor, ϵ_μ is the polarization vector of the electromagnetic field, and $\varphi = k\cdot x$, with k_μ the propagation vector of the electromagnetic field. We use units such that $\hbar = c = 1$, and the metric employed is such that the scalar product of two four-vectors k_μ and x_μ is $k\cdot x = k_\mu x^\mu = k^0 x^0 - \mathbf{k}\cdot\mathbf{x}$.

Equation (2.1) may be written in the form

$$[(i\partial_\mu)^2 - m^2] = (2ieA_\mu\partial^\mu - e^2A^2)\psi \tag{2.3}$$

when the Lorentz condition is imposed on A_μ. In this form, Eq. (2.3) has the appearance of the dynamical equation for the field operator ψ, as well as for the wave function ψ, although we adopt the latter point of view here. The terms on the right-hand side of (2.3) are the result of the photon-electron interaction. This interaction is seen to involve a vector coupling $A_\mu\partial^\mu\psi$ and a scalar coupling $A^2\psi$. When the electromagnetic field is circularly polarized, the scalar coupling term exhibits a particularly simple form. If the electromagnetic field is taken to be in the x^3 direction, so that the propagation four-vector is

$$k^\mu = \omega\hat{k}^\mu;\quad \hat{k}^\mu = (1,\hat{e}_3)\,, \tag{2.4}$$

then circular polarization can be described by the polarization vector

$$\epsilon^\mu = 2^{-1/2}(0,\, \hat{e}_1 \pm i\hat{e}_2)\,. \tag{2.5}$$

In Eqs. (2.4) and (2.5), $\hat{e}_i$ represents a unit three-vector along one of the spatial axes x^i. Substitution of the polarization vector (2.5) in (2.2) gives $A^2 = -\frac{1}{2}a^2$, since $\epsilon^2 = \epsilon^{*2} = 0$ and $\epsilon_\mu\epsilon^{\mu *} = -1$. Thus, with the definition $\Delta m^2 = \frac{1}{2}e^2a^2$, the right-hand side of Eq. (2.3) follows from the interaction Lagrangian density

$$\mathcal{L}_I = -2ieA_\mu\psi^*\partial^\mu\psi - \Delta m^2\psi^*\psi\,. \tag{2.6}$$

The second term in (2.6) is exactly of the form of a mass counterterm, and can be viewed as constituting a finite-mass renormalization of the electron due to the presence of the external electromagnetic field. The Klein-Gordon equation (2.3) now takes the form

$$[(i\partial_\mu)^2 - M^2]\psi = 2ieA_\mu\partial^\mu\psi\,, \tag{2.7}$$

where $M^2 = m^2 + \Delta m^2$.

The language used in arriving at the results (2.6) and (2.7) was chosen so as to be applicable to the field-

[10] H. R. Reiss, Bull, Am. Phys. Soc. **10**, 712 (1965); and a paper currently being prepared.

theoretic case as well as to the case of ordinary quantum mechanics which is treated here. When the occupation number of any single mode of the photon field is so large that changes in that number can be neglected, then the result (2.6) can be proved also in the field-theoretic case. This conclusion is demonstrated (although in an indirect way) in I. In a Feynman-diagram approach, the A^2 term with circular polarization leads to diagrams with one photon absorbed and one photon emitted at the same vertex. The replacement of m^2 by M^2 identically removes all such diagrams.

The same Δm^2 mass shift occurs in the linearly polarized case, even though the Δm^2 mass renormalization does not remove the A^2 term entirely. Linear polarization can be described by real space-like ϵ_μ in Eq. (2.2), in which case

$$e^2A^2=-e^2a^2\cos^2\varphi=-\Delta m^2-\tfrac{1}{2}e^2a^2\cos 2\varphi. \quad (2.8)$$

The mass renormalization removes the first term on the right-hand side of (2.8). The effect of the remaining part of e^2A^2 is explicable from the $e^{\pm 2i\varphi}$ terms in $\cos 2\varphi$. In Feynman-diagram language, the mass normalization removes the vertices involving one photon in and one photon out, but does not affect vertices with two photons absorbed or vertices with two photons emitted.

III. GREEN'S FUNCTION—SCALAR ELECTRON

An exact result for the Green's function of Eq. (2.1) will be calculated for circular polarization, i.e., when A_μ is specified by the particular choice (2.5) for the polarization vector in (2.2). The equation to be solved is written most simply in the form which follows from Eq. (2.7),

$$(\partial_\mu\partial^\mu+M^2+2ieA_\mu\partial^\mu)\mathcal{G}(x,x')=-\delta(x-x'). \quad (3.1)$$

The boundary conditions to be satisfied will be discussed when they are imposed upon the general solution. We begin by setting

$$\mathcal{G}(x,x')=(2\pi)^{-4}\int d^4p\, e^{-ip\cdot(x-x')}f(\varphi,\varphi'). \quad (3.2)$$

The reason for this choice is that f is a function of momentum only, in the free-particle case, and departure from free-particle behavior is a consequence of the presence of A^μ, which is a function of $\varphi=k\cdot x$ only. The substitution of the assumed solution (3.2) into (3.1) gives

$$\int d^4p\, e^{-ip\cdot(x-x')}[(p^2-M^2-2ep\cdot A)f+2ip\cdot kf'-1]=0,$$

where $f'=\partial f/\partial\varphi$. This equation can be identically satisfied by setting the square bracket in the integrand equal to zero. In this fashion the first-order differential equation

$$f'+\Phi(\varphi)f=D \quad (3.3)$$

is obtained, where

$$\Phi(\varphi)=i(2p\cdot k)^{-1}(2ep\cdot A-p^2+M^2),$$
$$D=-i(2p\cdot k)^{-1}.$$

The solution of this equation is

$$f=uv,$$
$$u=\exp\left[-\int_{\varphi''}^{\varphi}\Phi(\alpha)d\alpha\right], \quad (3.4)$$
$$v=D\int_{\varphi'}^{\varphi}\exp\left[\int_{\varphi''}^{\beta}\Phi(\alpha)d\alpha\right]d\beta.$$

All the integrations indicated in (3.4) can be carried out explicitly when A^μ represents a monochromatic plane wave. It should be observed that the lower limit φ'' in both integrals over α is quite immaterial as long as it is the same in both integrals, as it must be in order that uv satisfy the differential equation (3.3). The reason φ'' is immaterial is that all dependence on it vanishes when the product of u and v is formed after the evaluation of the α integrals. Hence we shall indicate only an upper limit of integration for the α integrals.

To evaluate $f(\varphi,\varphi')$ explicitly, we introduce the explicit expression for A^μ given by Eqs. (2.2) and (2.5). The result of the integration over α is then

$$\int^{\beta}\Phi(\alpha)d\alpha=i\zeta\sin(\beta-\rho)-i\sigma\beta, \quad (3.5)$$

with the definitions

$$\zeta e^{i\rho}=eap\cdot\epsilon/(p\cdot k),$$
$$\rho=\arctan[(p\cdot\mathrm{Im}\epsilon)/(p\cdot\mathrm{Re}\epsilon)],$$
$$\sigma=(p^2-M^2)/(2p\cdot k).$$

The solution $f(\varphi,\varphi')$ now takes the form

$$f=(2ip\cdot k)^{-1}\exp[-i\zeta\sin(\varphi-\rho)]\times\int_{\varphi'}^{\varphi}d\beta\exp[i\zeta\sin(\beta-\rho)+i\sigma(\varphi-\beta)].$$

If we make the change of integration variable $\theta+2l\pi=\varphi-\beta$, where $0\leq\theta<2\pi$ and l is an integer, then for large values of $\varphi-\varphi'$ we have

$$f=(2ip\cdot k)^{-1}\exp[-i\zeta\sin(\varphi-\rho)]\times\sum_{0}^{L-1}e^{2\pi il\sigma}\int_0^{2\pi}d\theta\exp[-i\zeta\sin(\theta+\rho-\varphi)+i\sigma\theta]. \quad (3.6)$$

The quantity L in the upper limit on the sum over l denotes the smallest integer containing $(\varphi-\varphi')/2\pi$. Now we suppose that the source point φ' goes to $-\infty$ (or, equivalently, the field point-source point separation

$\varphi-\varphi'$ goes to ∞). This corresponds to the large space-time interval that is customarily introduced into scattering problems. Here it signifies explicitly that the electron has passed through many wavelengths of the electromagnetic field in propagating from x' to x. Then the upper limit $L-1$ of the sum in (3.6) becomes ∞. By adding a positive imaginary part to σ, the sum over l is made to converge to

$$\sum_0^\infty e^{2\pi il\sigma}=[1-e^{2\pi i(\sigma+i\epsilon)}]^{-1}. \quad (3.7)$$

A positive imaginary addition to σ is equivalent to the replacement

$$M^2\to M^2-i\epsilon,\quad \epsilon\to 0+. \quad (3.8)$$

As in the free-particle case, (3.8) corresponds to a causal Green's function. The explicit consequences of (3.8) are shown in the next section. We employ (3.7) in the form

$$(1-e^{2\pi i\sigma})^{-1}=\tfrac{1}{2}ie^{-i\pi\sigma}/\sin\pi\sigma.$$

The θ integral in Eq. (3.6) becomes

$$\int_0^{2\pi}d\theta\exp[-i\zeta\sin(\theta+\rho-\varphi)+i\sigma\theta]$$
$$=e^{i\sigma(\varphi-\rho)}\int_{\rho-\varphi}^{2\pi+\rho-\varphi}d\alpha\exp(-i\zeta\sin\alpha+i\sigma\alpha) \quad (3.9)$$

when the change of variables $\alpha=\theta+\rho-\varphi$ is employed. We shall see that σ is required to be an integer, which means that the integrand in (3.9) has period 2π. Since the interval of integration is also 2π, the limits of integration may thus be translated an amount $\rho-\varphi$, i.e.,

$$\int_{\rho-\varphi}^{2\pi+\rho-\varphi}d\alpha\exp(-i\zeta\sin\alpha+i\sigma\alpha)$$
$$=\int_0^{2\pi}d\alpha\exp(-i\zeta\sin\alpha+i\sigma\alpha)$$
$$=2\pi J_{-\sigma}(-\zeta)=2\pi e^{i\pi\sigma}J_{-\sigma}(\zeta) \quad (3.10)$$

when σ is an integer. When Eqs. (3.7) and (3.10) are incorporated into Eq. (3.6), the result is

$$f=(2p\cdot k\sin\pi\sigma)^{-1}\pi J_{-\sigma}(\zeta)$$
$$\times\exp[-i\zeta\sin(\varphi-\rho)+i\sigma(\varphi-\rho)].$$

The exponential function here can be expressed as

$$\exp[-i\zeta\sin(\varphi-\rho)+i\sigma(\varphi-\rho)]=\sum_{-\infty}^{\infty}J_r(\zeta)e^{-i(r-\sigma)(\varphi-\rho)}.$$

When the summation index r is shifted an amount σ, and the final result for the $f(\varphi,\varphi')$ function thereby obtained is substituted into Eq. (3.2), we find that the Green's function for the scalar electron is

$$\mathcal{G}(x,x')=\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot(x-x')}$$
$$\times\sum_{r=-\infty}^{\infty}\frac{\pi J_{-\sigma}(\zeta)J_{\sigma+r}(\zeta)}{2p\cdot k\sin\pi\sigma}e^{-ir(k\cdot x-\rho)}. \quad (3.11)$$

The first thing to be noted about the solution (3.11) is that for $r=0$ the result

$$\mathcal{G}^{(0)}(x-x')=\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot(x-x')}\frac{\pi J_{-\sigma}(\zeta)J_\sigma(\zeta)}{2p\cdot k\sin\pi\sigma} \quad (3.12)$$

is identical to the Green's function derived in Ref. 9 with the use of Feynman rules. Only for the single term $r=0$ does the Green's function depend solely on the difference $x^\mu-x^{\mu\prime}$. This carries the implication that $\mathcal{G}^{(0)}(x-x')$ is the propagator for an electron with momentum unchanged by the electromagnetic field, and thus it constitutes that part of the Green's function which derives from the "external-field self-energy" of the electron propagating in the presence of the electromagnetic field. Even in this unchanged momentum case, the modifications resulting from the external field are manifest in the strikingly different appearance of Eq. (3.12) and the conventional free-electron result, given by Eq. (3.2) with $f=(p^2-m^2)^{-1}$.

A direct way to see the effect of an explicit dependence on x^μ as well as $x^\mu-x^{\mu\prime}$ in the complete Green's function (3.11) is to Fourier transform to the momentum-space Green's function,

$$\mathcal{F}(q,q')=(2\pi)^{-8}\int d^4x\, e^{iq\cdot x}\int d^4x' e^{-iq'\cdot x'}\mathcal{G}(x,x').$$

We look first at the case when there is no dependence other than on $x^\mu-x^{\mu\prime}$, as in $\mathcal{G}^{(0)}(x-x')$. The result then is, trivially,

$$\mathcal{F}^{(0)}(q,q')=\frac{1}{(2\pi)^4}\frac{\pi J_{-\sigma}(\zeta)J_\sigma(\zeta)}{2q\cdot k\sin\pi\sigma}\delta(q-q'), \quad (3.13)$$

where the σ and ζ parameters are now understood to be functions of q^μ rather than p^μ. For the complete Green's function of Eq. (3.11), the corresponding momentum space Green's function is

$$\mathcal{F}(q,q')=\frac{1}{(2\pi)^4}\sum_{r=-\infty}^{\infty}\frac{\pi J_{-\sigma}(\zeta)J_{\sigma+r}(\zeta)}{2q\cdot k\sin\pi\sigma}e^{ir\rho}\delta(q-q'+rk). \quad (3.14)$$

It should be observed that, because $\zeta\ll 1$ in most circumstances, the result is usually dominated by the $\sigma=0$, $r=0$ term. The significant conclusion that follows from Eq. (3.14) is that the Green's function describes not only the propagation of an electron between $x^{\mu\prime}$ and x^μ with unchanged momentum $q=q'$, but it also describes real interactions in which the electron can physically absorb or emit r photons of the external electromagnetic field. Thus the index r has been identified as a photon-number index which counts the net number of photons absorbed or emitted by the electron.

IV. SINGLE INTEGRATION OF GREEN'S FUNCTION

The Green's function we have derived in Eq. (3.11) is greatly different in appearance from the standard result for a free electron. Yet we should expect that the physical consequences of calculating with (3.11) should differ from the usual free-particle case only by the appearance of new terms involving parameters like $\Delta m^2/m^2$. The qualitative similarity of (3.11) to the usual free-particle case can be made manifest by carrying out one of the integrations indicated by $\int d^4p$. Performing this integration also serves to clarify other points of physical interest.

We shall be interested in making comparisons with the well-known result for the free-particle Green's function that

$$\frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon}$$
$$=-\frac{i}{(2\pi)^3}\int d^3p \frac{\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')\mp iE(x^0-x^{0\prime})]}{2E}, \quad (4.1)$$

where $E=(\mathbf{p}^2+m^2)^{1/2}$, and the upper choice of ambiguous sign holds if $x^0-x^{0\prime}>0$ while the lower choice holds if $x^0-x^{0\prime}<0$.

The poles of (3.11) are at $\sigma=l=$integer, which gives, when the definition of σ is introduced,

$$(p-lk)^2=M^2-i\epsilon. \quad (4.2)$$

Equation (4.2), which also arises in the self-energy calculation of I, states than an electron in interaction with a plane-wave electromagnetic field does not possess a uniquely defined momentum vector. Equation (4.2) is interesting also because it does not arise in perturbation theory in any finite order, but arises only in an exact solution as is done here, or in a closed form evaluation of an infinite sum, as in I.

The locations of the poles of the integrand of (3.11) in the complex p^0 plane come from Eq. (4.2). The poles are at

$$p^0=l\omega\pm[\mathcal{E}^2-2l\omega p^3+(l\omega)^2-i\epsilon]^{1/2}, \quad (4.3)$$

where $\mathcal{E}=(\mathbf{p}^2+M^2)^{1/2}$ is defined in analogy with $E=(\mathbf{p}^2+m^2)^{1/2}$. All poles given by the positive square root in (4.3) lie below the real axis, and those given by the negative square root lie above the real axis, as intended. With this understood, we shall no longer write the $i\epsilon$ term. When $\mathcal{E}\gg|l\omega|$, the positive branch of (4.3) gives poles at

$$p^0\approx\mathcal{E}+l\omega[1-(p^3/\mathcal{E})]. \quad (4.4)$$

This means that there are evenly spaced poles about the point $p^0=\mathcal{E}$. On the other hand, if $|l\omega|\gg\mathcal{E}$,

$$p^0\approx 2l\omega-p^3 \quad (4.5)$$

if $l>0$, so that the poles continue to be evenly spaced although with increased interval. When $|l\omega|\gg\mathcal{E}$ and $l<0$, the poles are at

$$p^0\approx p^3+[\mathcal{E}^2-(p^3)^2]/(-2l\omega). \quad (4.6)$$

In this case the poles lie progressively closer to each other as $-l$ increases, and there is an accumulation point at $p^0=p^3$. Analogous results hold for the negative root in (4.3). In this case,

$$p^0\approx-\mathcal{E}+l\omega[1+(p^3/\mathcal{E})] \quad (4.7)$$

when $\mathcal{E}\gg|l\omega|$; Eq. (4.5) holds true when $l<0$ and $-l\omega\gg\mathcal{E}$; and Eq. (4.6) pertains when $l>0$ and $lw\gg\mathcal{E}$.

It should be noted that for optical photons, $|l\omega|=\mathcal{E}$ implies a value of $l\approx3\times10^5$, with the implication that a single electron has absorbed or emitted this many photons. Clearly then, the presumption that the electromagnetic field is not depletable can no longer be sustained, and the entire basis for the calculation becomes invalid. The same is true for the calculation in I. The effects we are describing would cut themselves off through local depletion of the electromagnetic field. Therefore, we shall not be concerned with the possibility $|l\omega|\geq\mathcal{E}$.

We now explicitly carry out the integral

$$I_r=\frac{1}{2\pi}\int dp^0 e^{-ip^0(x^0-x^{0\prime})}\frac{\pi J_{-\sigma}(\zeta)J_{\sigma+r}(\zeta)}{2p\cdot k\sin\pi\sigma} \quad (4.8)$$

over the contour indicated in Fig. 1. In the neighborhood of one of the poles,

$$\sigma\approx l+(p^0-p_l^0)(\partial\sigma/\partial p^0)_l,$$

where p_l^0 is the value of p^0 at the lth pole as found from Eq. (4.3). Since the derivative is

$$(\partial\sigma/\partial p^0)_l=(p\cdot k)^{-1}(p_l^0-\omega l),$$

then the denominator of (4.8) contributes

$$2p\cdot k\sin\pi\sigma\approx\pi(-1)^l(p^0-p_l^0)2(p_l^0-\omega l)$$

in the vicinity of each pole. The result of carrying out the contour integral is then

$$I_r=\mp i\textstyle\sum_l\exp[-ip_l^0(x^0-x^{0\prime})]$$
$$\times\tfrac{1}{2}(p_l^0-\omega l)^{-1}J_l(\zeta_l)J_{l+r}(\zeta_l), \quad (4.9)$$

where the sum over l should go between negative and positive limits whose magnitude is of the order of m/ω. However, because the magnitude of the summand decays very rapidly as $|l|$ becomes large, the limits on the l summation will be taken to be simply $-\infty$ to $+\infty$. The upper choice of ambiguous sign in (4.9) refers to $(x^0-x^{0\prime})>0$, and the lower choice to $((x^0-x^{0\prime})<0$. Of

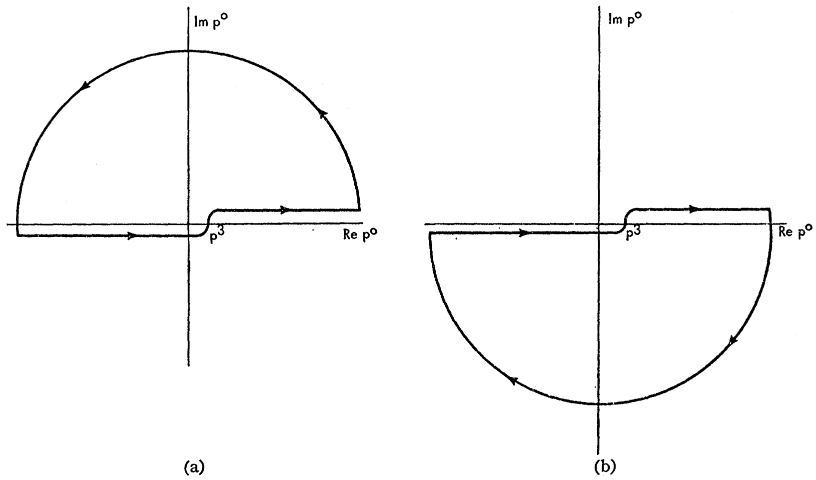

FIG. 1. Contour of integration in the complex p^0 plane for: (a) $(x^0-x^{0\prime})<0$, and for (b) $(x^0-x^{0\prime})>0$.

course the selection of the proper solution for p_l^0 must be made correspondingly. From Eqs. (4.4) and (4.7), we may set

$$p_l^0=\pm\mathcal{E}+l\omega[1\mp(p^3/\mathcal{E})],$$

where here and in the following work the ambiguous signs are ordered as indicated above. Then the exponent in (4.9) becomes

$$\exp[-ip_l^0(x^0-x^{0\prime})]=\exp[\mp i\mathcal{E}(x^0-x^{0\prime})]\times\exp\{-il\omega(x^0-x^{0\prime})[1\mp(p^3/\mathcal{E})]\}.$$

Since, generally, $|p^3|/\mathcal{E}\ll 1$ and $|l\omega|/\mathcal{E}\ll 1$, then we may set

$$(p_l^0-\omega l)^{-1}\approx\pm\mathcal{E}^{-1}$$

in Eq. (4.9). The term $l\omega(p^3/\mathcal{E})$ has been retained in the exponential (4.9) because it makes a qualitative difference in the result. It can be shown that retention of this term in $(p_l^0-\omega l)^{-1}$ produces a similar, but much smaller effect. In the same fashion, we find that

$$\zeta_l\approx ea|p\cdot\epsilon|/(\pm\mathcal{E}-p^3)\omega=\zeta_0$$

is an adequate approximation to insert as the argument of the Bessel coefficients in (4.9), and has the signal advantage of being independent of l. Now I_r in (4.9) can be written as

$$I_r\approx-\tfrac{1}{2}i\mathcal{E}^{-1}\exp[\mp i\mathcal{E}(x^0-x^{0\prime})]\times\textstyle\sum_l e^{il\phi}J_l(\zeta_0)J_{l+r}(\zeta_0),\quad(4.10)$$

where

$$\phi=-\omega(x^0-x^{0\prime})[1\mp(p^3/\mathcal{E})],$$

and ϕ is independent of l. The sum in Eq. (4.10) can be evaluated explicitly by Graf's addition formula (cf., e.g., Ref. 11) to yield the final result for I_r that

$$I_r\approx-\tfrac{1}{2}i\mathcal{E}^{-1}\exp[\mp i\mathcal{E}(x^0-x^{0\prime})]e^{\mp\frac{1}{2}ir(\phi-\pi)}J_r(w),\quad(4.11)$$

with

$$w=2\zeta_0\sin\tfrac{1}{2}\phi.$$

For the special case of the self-energy Green's function, i.e., when $r=0$, the result arising from the insertion of Eq. (4.11) into (3.11) is

$$\mathcal{G}^{(0)}(x,x')\approx-\frac{i}{(2\pi)^3}\times\int d^3p\frac{\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')\mp i\mathcal{E}(x^0-x^{0\prime})]}{2\mathcal{E}}J_0(w).\quad(4.12)$$

When $|\zeta_0|\ll 1$, then $|w|\ll 1$ and $J_0(w)\approx 1$. Thus Eq. (4.12) differs from Eq. (4.1) for the usual free-particle Green's function essentially only by the substitution of $\mathcal{E}$ for E (i.e., M for m). However, even in this case, with $J_0(w)\approx 1$, the presence of $J_0(w)$ in (4.12) is nonetheless interesting since w differs for the particle and antiparticle poles. Explicitly, w is given by

$$w_\pm=\frac{2ea|p\cdot\epsilon|}{(\pm\mathcal{E}-p^3)\omega}\sin[-\tfrac{1}{2}\omega(x_0-x_0')(1\mp p^3/\mathcal{E})].$$

The fact that $J_0(w_+)$ must be used in (4.12) for the case $(x^0-x^{0\prime})>0$ and $J_0(w_-)$ must be used for $(x^0-x^{0\prime})<0$ means that we find a fundamental asymmetry between electrons and positrons which does not occur in the free-particle case. However, this asymmetry is reversed when the sign of p^3 reverses, so that we again achieve a symmetry in this broader sense. When ζ_0 is small, another consequence of the $J_0(w)$ factor in (4.12) can be shown by expanding $J_0(w)$ as

$$J_0(w)\approx 1-(\tfrac{1}{2}w)^2=(1-\tfrac{1}{2}\zeta_0^2)+\tfrac{1}{2}\zeta_0^2\cos\phi.\quad(4.13)$$

The effect of the $\cos\phi$ term is to introduce $\mathcal{E}+\omega$ and $\mathcal{E}-\omega$ into the exponential in (4.12) as well as the original $\mathcal{E}$. Further terms in the expansion (4.13) lead to further small correction terms containing successively higher multiples of ω multiplying $(x^0-x^{0\prime})$ in the exponential. Hence, even though (4.12) closely resembles the free-particle result (4.1), the multiple-

[11] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 45.

photon effects can still be put into evidence. We should point out that a more extensive expansion than (4.13) is not to be trusted, since further terms will be of the same order as quantities which were neglected in arriving at Eq. (4.12). For instance, in going from Eq. (4.9) to Eq. (4.10), a term $l\omega(p^3/\mathcal{E})$ was dropped from $p_l^0-\omega l$. Including this gives a correction proportional to $\cos(\frac{1}{2}\phi)J_1(w)$ to be added to $J_0(w)$ in Eq. (4.12). This term in $J_1(w)$ can be represented approximately as $(p^3/\mathcal{E})(\omega/\mathcal{E})\frac{1}{2}\zeta_0^2\sin\phi$. Its effect is thus much like the last term in (4.13), but with smaller amplitude.

The complete result for the Green's function after the p^0 integration is performed follows from employing I_r in Eq. (3.11). I_r is defined by (4.8) and approximately evaluated in (4.11). When this is done, the summation over the index r is in the form of a generating function for the Bessel coefficients. That is, we have

$$\sum_{-\infty}^{\infty} J_r(w)e^{-\frac{1}{2}ir(\phi-\pi)-ir(k\cdot x-\rho)}=\exp[iw\cos(\tfrac{1}{2}\phi+k\cdot x-\rho)]$$
$$=\exp\{i\zeta_0[\sin(\phi+k\cdot x-\rho)-\sin(k\cdot x-\rho)]\}\,, \quad (4.14)$$

where the last form follows from the definition of w in (4.11). Therefore, the complete Green's function is given by

$$\mathcal{G}(x,x')\approx-\frac{i}{(2\pi)^3}\int d^3p\frac{\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')\mp i\mathcal{E}(x^0-x^{0\prime})]}{2\mathcal{E}}$$
$$\times\exp\{i\zeta_0[\sin(\phi+k\cdot x-\rho)-\sin(k\cdot x-\rho)]\}\,. \quad (4.15)$$

If the exponential function from Eq. (4.14) is expanded when ζ_0 is small, the first term will dominate. Equation (4.15) then becomes [as did Eq. (4.12)] the same as the free-particle result (4.1) with M in place of m. Corrections to this lowest-order result are, however, more important for the complete Green's function than for the self-energy part. The first correction here is of first order in ζ_0 rather than second order. This last fact was evident at a much earlier stage. In Eq. (3.12) for the self-energy part, the dominant contribution for small ζ comes from the pole at $\sigma=0$, since $J_0^2(\zeta)\approx1$. The first correction comes from the poles at $\sigma=\pm1$, which lead to $J_1^2(\zeta)\approx(\frac{1}{2}\zeta)^2$. By contrast, Eq. (3.11) for the complete Green's function shows that the dominant contribution at $\sigma=0$ and $r=0$ can be corrected by the pole at $\sigma=0$ with $r=\pm1$, or by the poles at $\sigma=\pm1$ with $r=\mp1$. Each of these four combinations gives a correction proportional to ζ.

Equation (4.15) exhibits the same sort of asymmetry as (4.12) between particle and antiparticle parts. However, since the exponential factor exhibited in Eq. (4.14) depends on both $x^0-x^{0\prime}$ (through ϕ) and on x^0-x^3, a mixing of coordinates is present in the complete Green's function that does not occur for the self-energy part. The significance of this has already been discussed.

In the above discussion, we have repeatedly mentioned the case when ζ is small. This is not always applicable, however, as we shall show. For an order of magnitude evaluation, we can set $p\cdot k\approx m\omega$. Then since $|p\cdot\epsilon|=2^{-\frac{1}{2}}|p_\perp|$, we have

$$|\zeta|\approx(\Delta m^2/m^2)^{1/2}(|p_\perp|/\omega)\,,$$

where $p_\perp$ is the component of the electron's three-momentum perpendicular to the electromagnetic field direction. At present, the maximum available value of $\Delta m^2/m^2$ from a pulsed-ruby laser is about 10^{-8}. Hence, an electron beam of about 20 keV is adequate to yield $|\zeta|\approx1$. The results presented in Eqs. (4.12) and (4.15) are valid for the parameter magnitudes we are quoting here. Thus, values of $|\zeta|$ of the order of unity may be employed in these equations, and clearly lead to major deviations from the free-particle Green's function.

A further fact should be emphasized. We have generally had in mind photon fields of optical frequency or higher, so that one would expect an electron to propagate essentially as a free particle even in the presence of the field. This is true when ζ is small. On the other hand, for microwave or rf fields, ζ is generally very large and the electron does not behave at all like a free particle. This is to be expected for a charged particle in the presence of classical electric and magnetic fields.

V. GREEN'S FUNCTION—SPINOR ELECTRON

The second-order Dirac equation for the Green's function of a spinor particle is

$$[(i\partial_\mu-eA_\mu)(i\partial^\mu-eA^\mu)-m^2-\tfrac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu}]G(x,x')=\delta(x-x')\,, \quad (5.1)$$

where $\sigma_{\mu\nu}=\frac{1}{2}[\gamma_\mu,\gamma_\nu]$, and $F^{\mu\nu}$ is the electromagnetic field tensor, $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$. Since Eq. (5.1) can be factored into

$$(i\gamma\cdot\partial-e\gamma\cdot A-m)[(i\gamma\cdot\partial-e\gamma\cdot A+m)G(x,x')]=\delta(x-x')\,,$$

we find that the Green's function for the first-order Dirac equation $\tilde{G}(x,x')$, which satisfies

$$(i\gamma\cdot\partial-e\gamma\cdot A-m)\tilde{G}(x,x')=\delta(x-x')\,,$$

is given directly by

$$\tilde{G}(x,x')=(i\gamma\cdot\partial-e\gamma\cdot A+m)G(x,x')\,. \quad (5.2)$$

Equation (5.1) can be solved by reference to the solution of the scalar electron case in (3.1). The tensor product term in (5.1) can be reduced to

$$\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}=i\gamma\cdot k(d/d\varphi)\gamma\cdot A\,.$$

By following our earlier procedure, we are led again to Eq. (3.3), where $\Phi(\varphi)$ is replaced now by

$$\Psi(\varphi)=i(2p\cdot k)^{-1}[2ep\cdot A-p^2+M^2+ie\gamma\cdot k(d/d\varphi)\gamma\cdot A]\,,$$

and we shall denote the solution by $g(\varphi)$ instead of $f(\varphi)$. It can be verified readily that the solution is still given by Eq. (3.4), even though $\Psi(\varphi)$ contains matrix quantities. The result for the α integral in the solution for $g(\varphi)$ becomes

$$\int^{\beta}\Psi(\alpha)d\alpha=i\zeta\sin(\beta-\rho)-i\sigma\beta-(2p\cdot k)^{-1}e\gamma\cdot k\gamma\cdot A(\beta)$$

instead of (3.5). The explicit form taken by $u(\varphi)$ is

$$u=[1+(2p\cdot k)^{-1}e\gamma\cdot k\gamma\cdot A(\varphi)]\times\exp[-i\zeta\sin(\varphi-\rho)+i\sigma\varphi]. \quad (5.3)$$

The reason this simple form is possible is that the transversality condition $k\cdot A=0$ implies that $\gamma\cdot k\gamma\cdot A=-\gamma\cdot A\gamma\cdot k$, and the null nature of k^μ means that $(\gamma\cdot k)^2=0$. Therefore, if an exponential in $\gamma\cdot k\gamma\cdot A$ is expanded in a power series, all terms beyond the first order are identically zero.

The function v takes the form

$$v=(2ip\cdot k)^{-1}\int_{\varphi'}^{\varphi}d\beta[1-(2p\cdot k)^{-1}e\gamma\cdot k\gamma\cdot A(\beta)]\times\exp[i\zeta\sin(\beta-\rho)-i\sigma\beta], \quad (5.4)$$

where in both u and v we have

$$\gamma\cdot A(\varphi)=\tfrac{1}{2}a(\gamma\cdot\epsilon e^{-i\varphi}+\gamma\cdot\epsilon^* e^{i\varphi}),$$

with the ϵ^μ appropriate to circular polarization.

Exactly as in the scalar case, we form the uv product with Eqs. (5.3) and (5.4), change variable of integration, obtain a sum over an index l, and sum it to get the result

$$g=\frac{\pi\exp[-i\zeta\sin(\varphi-\rho)+i\sigma(\varphi-\rho)]}{2p\cdot k\sin\pi\sigma}\times\left[1+\frac{ea\gamma\cdot k}{4p\cdot k}(\gamma\cdot\epsilon e^{-i\varphi}+\gamma\cdot\epsilon^* e^{i\varphi})\right]\times\left\{J_{-\sigma}+\frac{ea\gamma\cdot k}{4p\cdot k}[\gamma\cdot\epsilon e^{-i\rho}J_{-\sigma-1}+\gamma\cdot\epsilon^* e^{i\rho}J_{-\sigma+1}]\right\}. \quad (5.5)$$

We then introduce the representation

$$\exp[-i\zeta\sin(\varphi-\rho)+i\sigma(\varphi-\rho)]=\sum_{-\infty}^{\infty}J_{\sigma+r}(\zeta)e^{-ir(\varphi-\rho)},$$

and multiply it into the first square bracket in (5.5) to obtain

$$\exp[-i\zeta\sin(\varphi-\rho)+i\sigma(\varphi-\rho)][1+ea\gamma\cdot k(4p\cdot k)^{-1}(\gamma\cdot\epsilon e^{-i\varphi}+\gamma\cdot\epsilon^* e^{i\varphi})]=\sum_{-\infty}^{\infty}e^{-ir(\varphi-\rho)}\{J_{\sigma+r}+ea\gamma\cdot k(4p\cdot k)^{-1}[\gamma\cdot\epsilon e^{-i\rho}J_{\sigma+r-1}+\gamma\cdot\epsilon^* e^{i\rho}J_{\sigma+r+1}]\}. \quad (5.6)$$

Finally, inserting the result (5.6) into Eq. (5.5) yields

$$g=\frac{\pi}{2p\cdot k\sin\pi\sigma}\sum_{r=-\infty}^{\infty}e^{-ir(\varphi-\rho)}\left\{J_{-\sigma}J_{\sigma+r}+\frac{ea\gamma\cdot k}{4p\cdot k}[\gamma\cdot\epsilon e^{-i\rho}(J_{-\sigma-1}J_{\sigma+r}+J_{-\sigma}J_{\sigma+r-1})+\gamma\cdot\epsilon^* e^{i\rho}(J_{-\sigma+1}J_{\sigma+r}+J_{-\sigma}J_{\sigma+r+1})]\right\}, \quad (5.7)$$

where all the Bessel functions have argument ζ. The sum over the single product of Bessel functions, $J_{-\sigma}J_{\sigma+r}$, has now expanded to a sum over five such products. The Green's function for the second-order Dirac equation (5.1) follows directly from (5.7).

Equation (5.2) specifies that the spinor electron Green's function for the first-order Dirac equation is found from

$$\tilde{G}(x,x')=(i\gamma\cdot\partial-e\gamma\cdot A+m)(2\pi)^{-4}\times\int d^4p\, e^{-ip\cdot(x-x')}g(\varphi,\varphi').$$

Carrying out the indicated operations leads to a factor in the integrand given by

$$(\gamma\cdot p-e\gamma\cdot A+m)g+(\gamma\cdot k)g',$$

where the prime on g indicates differentiation with respect to φ. The simple properties $\gamma\cdot k\Psi=\gamma\cdot k\Phi$, where Φ is given in Eq. (3.3), and $\gamma\cdot kg=\gamma\cdot kf$, where f is the solution (3.4), lead to the chain of equalities

$$\gamma\cdot kg'=\gamma\cdot k(D-\Psi g)=\gamma\cdot k(D-\Phi g)=\gamma\cdot k(D-\Phi f)=\gamma\cdot kf'.$$

Hence, the spinor Green's function is

$$\tilde{G}(x,x')=(2\pi)^{-4}\int d^4p\, e^{-ip\cdot(x-x')}\times[(\gamma\cdot p-e\gamma\cdot A+m)g+\gamma\cdot kf'], \quad (5.8)$$

where g is given by Eq. (5.7) and

$$f'=\pi(2p\cdot k\sin\pi\sigma)^{-1}\textstyle\sum_r(-ir)e^{-ir(\varphi-\rho)}J_{-\sigma}(\zeta)J_{\sigma+r}(\zeta).$$

The denominator of the integrand in (5.8) contains

$\sin\pi\sigma$, exactly as in the scalar electron case, so that the singularities on the real axis for a p^0 integration remain unchanged here. The appearance of $(\gamma\cdot p - e\gamma\cdot A + m)$ in the integrand of the Green's function is no surprise, but the additive term $\gamma\cdot kf'$ is a novel feature. Considerable further structure is introduced into the Green's function by the somewhat complicated character of the function g.

VI. DISCUSSION

In this paper the dynamics of an electron in the presence of an intense monochromatic plane-wave electromagnetic field have been explored. The approach employed was to examine the consequences of the exact quantum-mechanical equation of motion in which the electromagnetic field was included as a potential. A considerable amount of physical information could be deduced even though no cross sections were calculated, and no perturbations were introduced.

It was shown that the somewhat controversial mass shift of the electron due to the intense electromagnetic field can be exhibited directly in the equation of motion of the electron or in the Lagrangian density which leads to this equation. In the Lagrangian density which defines the problem, the mass shift appears explicitly as a finite-mass renormalization term.

The properties of the exact Green's function for the electron were examined in order to achieve a better understanding of the explicit intense-field effects experienced by the electron. A salient feature of this Green's function is that it exhibits a multiplicity of poles in a complex p^0 space, which in turn implies the relatively complicated kinematical relationship $(p-lk)^2 = m^2+\Delta m^2$. There is a separate contribution to the Green's function by each integer l, which decreases rapidly as $|l|$ becomes large.

It was shown that the momentum-space Green's function contains a delta function connecting different values of the p parameter separated by integer multiples of the electromagnetic field quantum. This property may be regarded as ascribing to the electron a uniformly spaced level structure whose levels are tagged with the index r, and which are symmetrical about $r=0$. The electron can absorb radiation from the electromagnetic field, or experience induced emission, to an $r\neq 0$ level.

Because the original expression obtained for the Green's function of an electron in an intense field differed so radically from the free-particle case, it was recast in a form which emphasized the nature of the intense-field effects as corrections to the free electron problem. Modifications of several types to the free-particle case were exhibited among these corrections, including self-energy effects, exchanges of energy between the electron and the electromagnetic field, and a certain asymmetry between particle and antiparticle states. It was shown that these modifications to the free-particle Green's function could become important even for nonrelativistic electrons, given a sufficiently intense (but realizable) electromagnetic field. Explicit spin effects of the electron were examined also, although briefly.

The Green's function derived here is applicable also in the very low-frequency or classical limit of electromagnetic fields, in which case the electron's behavior is not at all like the free-particle behavior. This Green's function, which includes the complete coupling between the electron and an external plane-wave field, provides a novel approach to the classical problem. It complements the usual classical treatment in which one uses the Green's function for the electromagnetic field in the presence of a source due to the electron, i.e., the Lienard-Wiechert potentials. This will be explored in a separate investigation.

The work presented in this paper refers to the case where the electromagnetic field consists of a monochromatic plane wave, filling all of space. The question can be raised whether the results achieved here will persist (albeit in modified form) when the electromagnetic field consists of a wave packet, or whether they are peculiar to the purely monochromatic case. This question has been answered in the sense that a wave-packet electromagnetic field of narrow width in frequency and of Gaussian form has been explored[12,13] in terms of the mass-shift effect and in terms of the pole behavior of the Green's function in a complex p^0. It was found[12] that the mass shift of an electron at the center of a Gaussian wave packet has the same value as in the monochromatic case, and this value decreases very slowly even in the small amplitude "tails" of the packet. It was also found[13] that the poles in the p^0 plane in the monochromatic case appear as damped resonances of small width in the wave-packet calculation. These results are being prepared for publication by one of the present authors. The significance of these conclusions is that the monochromatic results given in the present paper are essentially valid even for a packet of electromagnetic waves, provided that this packet is of narrow width in frequency.

ACKNOWLEDGMENTS

One of us (H. R. R.) wishes to thank Professor G. Wataghin for his hospitality at the Istituto di Fisica dell'Università di Torino during the academic year 1963–64, where this work was begun. He also wishes to thank Professor S. Fubini for his interest and constructive comments. We are grateful to Professor Z. Fried and Professor H. M. Nussenzveig, and Dr. A. L. Licht for useful discussions.

[12] H. R. Reiss, Bull. Am. Phys. Soc. **11**, 96 (1966).
[13] H. R. Reiss, Bull. Am. Phys. Soc. **11**, 323 (1966).