Electric- and Magnetic-Charge Renormalization. II

JULIAN SCHWINGER*
Harvard University, Cambridge, Massachusetts
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Further evidence for the universality of charge renormalization is derived by examining the photon radiation and the static interaction of prescribed transverse currents.

IN a previous note bearing this title,[1] evidence was presented that electric and magnetic charges are renormalized by the same factor. The long-range interaction of quasi-static prescribed charges was used as an operational definition of charge. Two aspects of that calculation invite further consideration. The renormalized electric charge, for example, which is derived in this calculation and from general field considerations, differs fundamentally from the expectation value of the total charge operator.[2] There is no explicit reference in the latter to the vacuum polarization effect of magnetic charges. An independent verification of the universality of renormalization would be desirable to confirm the irrelevance of the total charge operator, at least in these considerations of external charges. Then, there is the use of a linear approximation in connecting induced charges and currents with the external charges. The reliability of this treatment might be suspect through the impossibility of weakening quantized total charges. It would be useful to consider another aspect of charge in which the strength of couplings could be varied continuously. For these reasons, we turn our attention to external currents, which can be made arbitrarily weak, and examine the operational definitions of charge associated with photon radiation and Ampèrian interactions.

It is sufficient to consider external electric currents $\mathbf{J}(x)$, which obey

$$\nabla\cdot\mathbf{J}(x)=0.$$

The dependence of the vacuum transformation function upon the external current is specified by

$$\delta\langle 0_+|0_-\rangle^J=i\left\langle 0_+\left|\int(dx)\delta\mathbf{J}(x)\cdot\mathbf{A}(x)\right|0_-\right\rangle^J,$$

or

$$\delta\ln\langle 0_+|0_-\rangle^J=i\int(dx)\delta\mathbf{J}(x)\cdot\langle\mathbf{A}(x)\rangle^J,$$

where

$$\mathbf{A}(x)=\mathbf{A}^T(x)+\int(dx')\mathbf{a}_n(x-x')\,{}^*j^0(x').$$

The analogous equation that refers to external electric charge might have been written

$$\delta\ln\langle 0_+|0_-\rangle^J=-i\int(dx)\delta J^0(x)\langle A^0(x)\rangle^J$$

with

$$A^0(x)=\int(dx')\mathfrak{D}(x-x')(j^0+J^0)(x')+\int(dx')\mathbf{a}_n(x-x')\cdot{}^*\mathbf{j}(x').$$

If charge renormalization is indeed universally given by field strength renormalization, one should be able to make this more evident by constructing the potentials in terms of field strengths. We know that

$$\nabla\times\mathbf{A}(x)=\mathbf{H}(x)+\int(dx')\mathbf{h}_n(x-x')\,{}^*j^0(x')$$

and the corresponding electric field relation is

$$-\partial_0\mathbf{A}(x)-\nabla A^0(x)=\mathbf{E}(x)+\int(dx')\mathbf{h}_n(x-x')\times{}^*\mathbf{j}(x').$$

Now let us write

$$\nabla\times\mathbf{A}(x)=\mathbf{H}(x)+\int(dx')\mathbf{h}_n(x-x')\nabla'\cdot\mathbf{H}(x')$$

$$=\nabla\times\int(dx')\mathbf{h}_n(x-x')\times\mathbf{H}(x'),$$

since

$$\nabla\cdot\mathbf{h}_n(x-x')=-\delta(x-x').$$

This gives the construction

$$\mathbf{A}(x)=\left(\int(dx')\mathbf{h}_n(x-x')\times\mathbf{H}(x')\right)^T,$$

in conformity with the gauge condition[3]

$$\nabla\cdot\mathbf{A}(x)=0.$$

The actual selection of the transverse part can be omitted, however, for this is automatically performed in integrated multiplication with the transverse $\mathbf{J}(x)$. The explicit form of the scalar potential can be pre-

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[1] J. Schwinger, preceding paper, Phys. Rev. **151**, 1048 (1966).

[2] The two approaches differ even in electrodynamics without magnetic charge, but they are reconciled by further discussion of the renormalization of potentials.

[3] Our notation is somewhat unfortunate in that $\mathbf{A}(x)$ and $\mathbf{A}^T(x)$ are both transverse vectors. The latter is specifically defined as the vector potential that represents $\mathbf{H}^T(x)$.

sented as

$$A^0(x)=\int (dx')\mathfrak{D}(x-x')\nabla'\cdot\mathbf{E}(x')+\int (dx')\mathbf{a}_n(x-x')\cdot[-\nabla'\times\mathbf{E}(x')-\partial_0'\mathbf{H}(x')]$$

which is equivalent to[4]

$$A^0(x)=\int (dx')\mathbf{h}_n(x-x')\cdot\mathbf{E}(x')-\partial_0\int (dx')\mathbf{a}_n(x-x')\cdot\mathbf{H}(x').$$

The application of the action principle to transverse external currents has none of the conceptual complications of the charge discussion. We are not impeded by charge conservation in perturbing the initial vacuum state, nor is there any difficulty in justifying a linear regime for induced properties. The result is

$$\langle 0_+|0_-\rangle^J=\exp\left[i\tfrac{1}{2}\int (dx)(dx')\mathbf{J}(x)\cdot i\langle(\mathbf{A}(x)\mathbf{A}(x'))_+\rangle\cdot\mathbf{J}(x')\right]$$

and

$$i\langle(\mathbf{A}(x)\mathbf{A}(x'))_+\rangle=-\int (dx_1)(dx_1')\mathbf{h}_n(x-x_1)\times i\langle(\mathbf{H}(x_1)\mathbf{H}(x_1'))_+\rangle\times\mathbf{h}_n(x'-x_1').$$

The information about field strength vacuum expectation values asserts that

$$i\langle(\mathbf{H}(x)\mathbf{H}(x'))_+\rangle=-(\times\nabla)\cdot(\nabla'\times)\left[A_0D_+(x-x')+\int dm^2(A_e+A_g)(m^2)\Delta_+(x-x',m^2)\right]+\int dm^2m^2A_g(m^2)\Delta_+(x-x',m^2),$$

where

$$D_+(x)=\Delta_+(x,0).$$

The transversality of the current implies the following equivalence with respect to integration,

$$[\mathbf{J}(x)\times\mathbf{h}_n(x-x_1)]\times\nabla_1\to\mathbf{J}(x)\delta(x-x_1)$$

which results in

$$i\langle(\mathbf{A}(x)\mathbf{A}(x'))_+\rangle\to A_0D_+(x-x')+\int dm^2(A_e+A_g)(m^2)\Delta_+(x-x',m^2)+\int (dx_1)(dx_1')(\mathbf{1}-\mathbf{nn})\mathbf{n}\cdot\mathbf{h}_n(x-x_1)\mathbf{n}\cdot\mathbf{h}_n(x'-x_1')\times\int dm^2m^2A_g(m^2)\Delta_+(x_1-x_1',m^2).$$

There are two kinds of uses for this transformation function. In the first one we evaluate the probability of radiation by the current distribution as the complement to the probability that the vacuum state persists despite the effect of the current,

$$|\langle 0_+|0_-\rangle^J|^2=\exp\left[-\int (dx)(dx')\mathbf{J}(x)\cdot\mathrm{Im}\,i\langle(\mathbf{A}(x)\mathbf{A}(x'))_+\rangle\cdot\mathbf{J}(x')\right].$$

Energy-momentum conservation is made explicit by the property

$$2\,\mathrm{Im}\Delta_+(x,m^2)=\int\frac{(dp)}{(2\pi)^3}e^{ipx}\delta(p^2+m^2).$$

Let us assume that the current cannot supply enough energy to create a neutral pair of magnetically charged particles (our whole discussion is predicated on the assumption that this *is* a restriction). Then all terms containing A_g disappear from the imaginary part.[5] The stronger energetic restriction involved in a similar omission of A_e leaves one with the possibility of photon radiation only. It is measured by

$$|\langle 0_+|0_-\rangle^J|^2=\exp\left[-A_0\int (dx)(dx')\mathbf{J}(x)\cdot\mathrm{Im}D_+(x-x')\mathbf{J}(x')\right].$$

This is the conventional radiation formula, with physical currents identified as $A_0^{1/2}\mathbf{J}(x)$ in conformity with the universal charge renormalization constant.

The second application refers to the energy of quasistatic currents. The energy is identified in the phase of the transformation function,

$$\langle 0_+|0_-\rangle^J\cong\exp\left[-i\int dx^0E\right]$$

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[4] The potentials obtained by discarding both the time derivative term of A^0 and the subtractive longitudinal part of $\mathbf{A}$ refer to the axial gauge based on the direction $\mathbf{n}$. This alternative gauge will be discussed in another publication.

[5] This is true of the term containing the functions $\mathbf{h}_n$, despite the additional spatial dependence that they introduce. Being instantaneous time functions, they can only transmit the energy supplied by the currents.

and

$$E=-\frac{1}{2}\int(d\mathbf{x})(d\mathbf{x}')\mathbf{J}(\mathbf{x})\cdot\left[\int_{-\infty}^{\infty}dx^0 i\langle(\mathbf{A}(x)\mathbf{A}(x'))_+\rangle\right]\cdot\mathbf{J}(\mathbf{x}').$$

The time integral of $\Delta_+(x,m^2),m^2>0$, is a short-ranged potential, whereas

$$\int_{-\infty}^{\infty}dx^0 D_+(x)=\mathfrak{D}(\mathbf{x}),$$

the Coulomb potential. We consider the interaction energy of two static current distributions, $\mathbf{J}_1$ and $\mathbf{J}_2$, which are separated by a distance large compared with that characterizing the vacuum polarization mechanism. The term containing $\mathbf{h_n}(\mathbf{x}-\mathbf{x}_1)\mathbf{h_n}(\mathbf{x}'-\mathbf{x}_1')$, $\mathbf{x}_1\sim\mathbf{x}_1'$, can only contribute if $\mathbf{x}-\mathbf{x}'$ and $\mathbf{n}$ have practically the same direction. This possibility has been excluded to simplify the treatment. Then,

$$E_{12}\cong -A_0\int(d\mathbf{x})(d\mathbf{x}')\mathbf{J}_1(\mathbf{x})\cdot\mathfrak{D}(\mathbf{x}-\mathbf{x}')\mathbf{J}_2(\mathbf{x}'),$$

which has the anticipated renormalization constant.

In order to unify the treatment of specified distributions we shall reconsider the interaction energy of static electric charges, for example, using the methods of this paper. One can discard the last term of

$$\delta\langle A^0(x)\rangle^J=\int(dx_1)\mathbf{h_n}(x-x_1)\cdot\delta\langle\mathbf{E}(x_1)\rangle^J-\partial_0\int(dx_1)\mathbf{a_n}(x-x_1)\cdot\delta\langle\mathbf{H}(x_1)\rangle^J,$$

in view of the quasi-static character of the charge distribution. But it should not be assumed that $\partial_0\mathbf{H}$ is completely ineffective in

$$\delta\langle\mathbf{E}(x)\rangle^J=-\int(dx_1')(dx')i\langle(\mathbf{E}(x)\mathbf{E}(x_1'))_+\rangle\cdot\mathbf{h_n}(x'-x_1')\delta J^0(x')-\nabla\int(dx')\mathfrak{D}(x-x')\delta J^0(x')+\int(dx_1')(dx')i\langle(\mathbf{E}(x)\partial_0'\mathbf{H}(x_1'))_+\rangle\cdot\mathbf{a_n}(x'-x_1')\delta J^0(x'),$$

since

$$i\langle(\mathbf{E}(x)\partial_0'\mathbf{H}(x'))_+\rangle=\partial_0'i\langle(\mathbf{E}(x)\mathbf{H}(x'))_+\rangle-(\nabla\times)\delta(x-x'),$$

according to the equal-time commutation relations of $\mathbf{E}$ and $\mathbf{H}$. The latter contribution combines with the gradient term, which expresses the explicit dependence of $\mathbf{E}$ on J^0, and it appears that the interaction energy of two quasi-static charge distributions can be calculated from

$$\int dx^0 E_{12}=-\int(dx)(dx_1)(dx')(dx_1')J_1^0(x)\mathbf{h_n}(x-x_1)\cdot[i\langle(\mathbf{E}(x_1)\mathbf{E}(x_1'))_+\rangle-\delta(x_1-x_1')]\cdot\mathbf{h_n}(x'-x_1')J_2^0(x').$$

Now

$$i\langle(\mathbf{E}(x)\mathbf{E}(x'))_+\rangle-\delta(x-x')=(\partial_0\partial_0'-\nabla\nabla')\times\left[A_0D_+(x-x')+\int dm^2(A_e+A_g)(m^2)\Delta_+(x-x',m^2)\right]-\int dm^2m^2A_g(m^2)\Delta_+(x-x',m^2),$$

and the introduction of the physical context leads immediately to the expected result

$$E_{12}\cong A_0\int(d\mathbf{x})(d\mathbf{x}')J_1^0(\mathbf{x})\mathfrak{D}(\mathbf{x}-\mathbf{x}')J_2^0(\mathbf{x}').$$