Integrating by parts we obtain

$$I(\alpha,\xi,\beta)=(-i/(\alpha-i\epsilon))\exp\left[i(\alpha-i\epsilon)z+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]\Bigg|_{-\infty}^{0}+(\xi/[\alpha-i\epsilon])\int_{-\infty}^{0}dz$$

$$\times\exp\left[i\alpha z-\beta z^2+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]-(i/[\alpha+i\epsilon])\exp\left[i(\alpha+i\epsilon)z+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]\Bigg|_{0}^{\infty}$$

$$+(\xi/[\alpha+i\epsilon])\int_{0}^{\infty}dz\exp\left[i\alpha z-\beta z^2+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]. \quad \text{(B3)}$$

Inserting the upper and lower limits we get

$$I(\alpha,\xi,\beta)=(2\pi)\delta(\alpha)\exp[i(\xi/2)(\pi/\beta)^{1/2}]+\xi[P(1/\alpha)]\int_{-\infty}^{+\infty}\exp\left[i\alpha z-\beta z^2+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]$$

$$-\pi\delta(\alpha)\int_{-\infty}^{0}dz\frac{d}{dz}\exp\left[i\xi\int_{-\infty}^{-z}\exp(-\beta z'^2)dz'\right]+\pi\delta(\alpha)\int_{0}^{\infty}dz\frac{d}{dz}\exp\left[i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right] \quad \text{(B4)}$$

which in turn can be written as

$$I(\alpha,\xi,\beta)=\pi\delta(\alpha)[1+\exp\{i\xi(\pi/\beta)^{1/2}\}]+\xi[P(1/\alpha)]\int_{-\infty}^{+\infty}dz\exp\left[i\alpha z-\beta z^2+i\xi\int_{-\infty}^{-z}dz'\exp(-\beta z'^2)\right]. \quad \text{(B5)}$$

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Electric- and Magnetic-Charge Renormalization. I

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An important question in the field theory of electric and magnetic charge is the relative renormalization of the two kinds of charges. A general view of renormalization, as a scale change introduced in proceeding from the field to the particle level of description, indicates the universality of charge renormalization. This is confirmed by an explicit calculation of the long-range interaction of static charges.

THE quantization of electrical charge produced by the existence of magnetic charge can be considered at two different dynamical levels. There is the fundamental, or field level, and the phenomenological, or particle level. Similar considerations operate to produce the analogous but distinct quantization conditions[1]

$$e_0g_0/\hbar c=n_0=0,1,2,\cdots,$$
$$eg/\hbar c=n=0,1,2,\cdots.$$

The distinction between particle charges e,g and the charges e_0,g_0 carried by fields or combinations of fields is described as charge renormalization. It is a consequence of the physical process of vacuum polarization. Can this mechanism produce a renormalization of magnetic charge different from that of electrical charge? I have already argued to the contrary,[2] by asserting that charge renormalization is a property of the electromagnetic field, not of any specific entity that interacts with it. Currents induced in the vacuum tend to counteract the inducing field. When observed at some distance from their sources, fields are thereby effectively reduced in scale and the single scale factor determines the renormalization of all charges.

This note is devoted to an explicit verification of that general field viewpoint. We evaluate the long-range interaction of static charges, as modified by vacuum polarization phenomena, and confirm that

$$e/e_0=g/g_0=C<1.$$

The first section is concerned with the renormalization of the electromagnetic field, as produced by interaction with both electric and magnetic currents.

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[1] The field theory is presented in J. Schwinger, Phys. Rev. **144**, 1087 (1966). A nonrelativistic particle formulation is described in another paper (to be published).

[2] J. Schwinger, *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, January, 1966* (W. H. Freeman and Company, San Francisco, California, 1966).

FIELD RENORMALIZATION

We consider the vacuum expectation value of the product of two field strengths,

$$\langle F_{\mu\nu}(x)F_{\lambda\kappa}(x')\rangle=\langle F_{\mu\nu}e^{iP(x-x')}F_{\lambda\kappa}\rangle.$$

In the second version, all field operators are referred to a common point, and P^μ is the energy-momentum operator of the system. The explicit introduction of the physical energy-momentum spectrum gives the four-dimensional momentum integral form

$$\langle F_{\mu\nu}(x)F_{\lambda\kappa}(x')\rangle=\int\frac{(dp)}{(2\pi)^3}dm^2e^{ip(x-x')}\times\delta(p^2+m^2)\eta_+(p)F_{\mu\nu,\lambda\kappa}(p),$$

where the factor $\delta(p^2+m^2)\eta_+(p)$, $m^2\geq0$, makes explicit the physical restriction to nonspacelike momenta and positive energy. The tensor $F_{\mu\nu,\lambda\kappa}(p)$ is the matrix element, with respect to the states $\langle F_{\mu\nu}$ and $F_{\lambda\kappa}\rangle$, of the projection operator for all states with the specified momentum. As such, it is a Hermitian, non-negative matrix. We also require invariance of the vacuum expectation value under complex conjugation and space-time reflection, the so-called *TCP* operation. Its consequence is the reality of the matrix,

$$F_{\mu\nu,\lambda\kappa}(p)=F_{\mu\nu,\lambda\kappa}(p)^*=F_{\lambda\kappa,\mu\nu}(p),$$

and symmetry appears as the appropriate form of the Hermitian property. The elements of the matrix are also antisymmetrical in μ and ν, λ and κ.

Covariance with respect to the group of proper, orthochronous Lorentz transformations specifies the form of $F_{\mu\nu,\lambda\kappa}$. It is

$$F_{\mu\nu,\lambda\kappa}(p)=(p_\mu p_\lambda g_{\nu\kappa}-p_\nu p_\lambda g_{\mu\kappa}+p_\nu p_\kappa g_{\mu\lambda}-p_\mu p_\kappa g_{\nu\lambda})A(m^2)+(g_{\mu\lambda}g_{\nu\kappa}-g_{\nu\lambda}g_{\mu\kappa})m^2A'(m^2)-\epsilon_{\mu\nu\lambda\kappa}m^2A''(m^2),$$

where the three weight functions A, A', A'' are real. The totally antisymmetrical tensor $\epsilon_{\mu\nu\lambda\kappa}$ is normalized by

$$\epsilon^{0123}=+1.$$

To impose the positiveness requirement on this matrix, we consider some submatrices. Thus

$$F_{03,03}=[(p^0)^2-(p_3)^2]A-m^2A'=[(p_1)^2+(p_2)^2]A+m^2(A-A')$$

must be positive for all p_1 and p_2. Hence

$$A(m^2)\geq0,\quad m^2[A(m^2)-A'(m^2)]\geq0.$$

Similarly,

$$F_{12,12}=[(p_1)^2+(p_2)^2]A+m^2A'$$

implies that

$$m^2A'(m^2)\geq0.$$

On including the nondiagonal element

$$F_{03,12}=m^2A''(m^2),$$

positiveness of the two-dimensional submatrix requires that

$$m^2(A-A')m^2A'-(m^2A'')^2\geq0.$$

Thus, in addition to the generally valid inequality $A(m^2)\geq0$, we have for all $m^2>0$,

$$A\geq A'\geq0,\quad (A-A')A'\geq(A'')^2.$$

There are no other general positiveness restrictions, as we shall recognize from a more specific physical interpretation of these weight functions.

One further general relation exists, in the form of a sum rule. It is a consequence of the equal-time commutation relations among field strengths. In view of the reality of $F_{\mu\nu,\lambda\kappa}$,

$$\langle[F_{\mu\nu}(x),F_{\lambda\kappa}(x')]\rangle=\int\frac{(dp)}{(2\pi)^3}dm^2e^{ip(x-x')}\times\delta(p^2+m^2)\epsilon(p)F_{\mu\nu,\lambda\kappa}(p),$$

where $\epsilon(p)$ states the algebraic sign of p^0. At equal times, only a term in $F_{\mu\nu,\lambda\kappa}$ that is linear in p^0 can contribute. All such terms have the weight factor $A(m^2)$. The nonvanishing structures are

$$-i\langle[F^{0k}(x),F_{lm}(x')]\rangle=(\delta_l^k\partial_m-\delta_m^k\partial_l)\delta(\mathbf{x}-\mathbf{x}')\int dm^2A(m^2)$$

and comparison with the well-known equal-time commutators of field strengths gives

$$\int dm^2A(m^2)=1.$$

One consequence of the sum rule should be noted in relation to the positiveness requirements. The existence of the integral implies that $A(m^2)$ is no more singular than $\delta(m^2)$ in the infinitesimal neighborhood of $m^2=0$. Accordingly, $m^2A(m^2)$ vanishes at $m^2=0$, and the combinations $m^2A'(m^2)$, $m^2A''(m^2)$, which appear explicitly in $F_{\mu\nu,\lambda\kappa}$, are also zero at $m^2=0$.

In the general form that embodies both electric and magnetic currents, the Maxwell equations are

$$\partial_\nu F^{\mu\nu}=j^\mu,\quad \partial_\nu {}^*F^{\mu\nu}={}^*j^\mu,$$

where

$${}^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}.$$

Given the specification of the vacuum expectation value $\langle F_{\mu\nu}(x)F_{\lambda\kappa}(x')\rangle$ by means of the matrix $F_{\mu\nu,\lambda\kappa}$, we can derive the three vacuum expectation values: $\langle j_\mu(x)\times j_\nu(x')\rangle$, $\langle j_\mu(x)\,{}^*j_\nu(x')\rangle$, and $\langle {}^*j_\mu(x)\,{}^*j_\nu(x')\rangle$, which are analogously expressed in terms of matrices $j_{\mu\nu}(p)$, ${}^*j_{\mu\nu}(p)$, and ${}^{**}j_{\mu\nu}(p)$. The latter are obtained as

$$\begin{Bmatrix} j_{\mu\nu}(p)\\ {}^*j_{\mu\nu}(p)\\ {}^{**}j_{\mu\nu}(p)\end{Bmatrix}=(p_\mu p_\nu-p^2g_{\mu\nu})\begin{Bmatrix} m^2A_e(m^2)\\ m^2A_{eg}(m^2)\\ m^2A_g(m^2)\end{Bmatrix},$$

where $(m^2>0)$

$$A_e(m^2)=A(m^2)-A'(m^2)\geq 0,$$
$$A_g(m^2)=A'(m^2)\geq 0,$$
$$A_{eg}(m^2)=A''(m^2), \quad (A_{eg})^2\leq A_eA_g.$$

Just these inequalities are regained on remarking that the matrices which specify the current vacuum expectation values are also subject to positiveness requirements, as illustrated by

$$j_{00}(p)=\mathbf{p}^2m^2A_e(m^2)\geq 0,$$

and

$$^{**}j_{00}(p)=\mathbf{p}^2m^2A_g(m^2)\geq 0.$$

The inference that A_e and A_g are non-negative, for $m^2>0$, is supplemented by $(A_{eg})^2\leq A_eA_g$. This is obtained by considering an arbitrary linear combination of the currents j^μ and $^*j^\mu$. The equality sign applies only if electric and magnetic currents are identical, apart from a numerical constant.

From the weight functions $A_e(m^2)$, $A_g(m^2)$, $m^2>0$, which have a direct physical meaning in terms of current excitations, we construct

$$m^2>0:\quad A(m^2)=A_e(m^2)+A_g(m^2).$$

The complete form recognizes the exceptional nature of $m^2=0$,

$$A(m^2)=A_0\delta(m^2)+A_e(m^2)+A_g(m^2), \quad (A_0\geq 0),$$

and the sum rule determines A_0, a number lying in the interval between zero and unity,

$$1=A_0+\int_0^\infty dm^2[A_e(m^2)+A_g(m^2)].$$

If $m^2=0$ is to be in the physical spectrum, it is necessary that

$$\int_0^\infty dm^2[A_e(m^2)+A_g(m^2)]<1.$$

The vacuum expectation value $\langle F_{\mu\nu}(x)F_{\lambda\kappa}(x')\rangle$, describing the correlation of electromagnetic field fluctuations in the vacuum, can now be exhibited as the additive superposition of four contributions. Three of these, which are computed from A_e, A_g, and A_{eg}, characterize the field fluctuations that accompany current fluctuations. They all have the energy-momentum property $m^2>0$. The fourth contribution, proportional to A_0 and uniquely associated with $m^2=0$, describes the role of the photon in establishing correlations between phenomena in different space-time regions. If only the latter agency is significant, the process of vacuum polarization produces an effective reduction of field strengths by the factor $A_0^{1/2}$. Since the long-range interaction of charged particles operates through this mechanism, we anticipate that all charges are effectively reduced, or renormalized, by the universal constant

$$C=A_0^{1/2}<1,$$

where A_0 must also be a ratio of integers.

To aid the work of the next section we shall present vacuum expectation values, illustrated by

$$\langle j_\mu(x)j_\nu(x')\rangle=(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\int dm^2A_e(m^2)\Delta^{(+)}(x-x',m^2),$$

$$\Delta^{(\pm)}(x,m^2)=\int\frac{(dp)}{(2\pi)^3}e^{ipx}\delta(p^2+m^2)\eta_\pm(p),$$

in the form of vacuum expectation values of time-ordered products, a Green's function structure. Now

$$(j_\mu(x)j_\nu(x'))_+=\eta_+(x-x')j_\mu(x)j_\nu(x')+\eta_-(x-x')j_\nu(x')j_\mu(x),$$

and

$$i\eta_+(x-x')(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\Delta^{(+)}+i\eta_-(x-x')(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\Delta^{(-)}=(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\Delta_++(g_{\mu\nu}+n_\mu n_\nu)\delta(x-x'),$$

where $\Delta_+(x-x',m^2)$ is the outgoing wave Green's function

$$\Delta_+(x,m^2)=\int\frac{(dp)}{(2\pi)^4}e^{ipx}\frac{1}{p^2+m^2-i\epsilon}\bigg|_{\epsilon\to+0}$$

and $g_{\mu\nu}+n_\mu n_\nu$ represents the spatial projection of the metric tensor. Accordingly,

$$i\langle(j_\mu(x)j_\nu(x'))_+\rangle-(g_{\mu\nu}+n_\mu n_\nu)\delta(x-x')\int dm^2m^2A_e(m^2)=(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\int dm^2m^2A_e(m^2)\Delta_+(x-x',m^2)$$

in which the four-dimensional delta-function term appears to maintain the divergenceless nature of the left-hand side of this equation. Thus,

$$\partial_\mu(j^\mu(x)j_k(x'))_+=\delta(x^0-x^{0\prime})[j^0(x),j_k(x')]$$

and it is necessary that $(x^0=x^{0\prime})$

$$\langle i[j^0(x),j_k(x')]\rangle=\partial_k\delta(\mathbf{x}-\mathbf{x}')\int dm^2m^2A_e(m^2),$$

which can be verified directly. It is known that the additional term expresses a necessary dependence of the current operator upon an external potential, and that the whole combination is generated automatically by the quantum action principle.[3] That is of some importance, for the individual parts seem to require the existence of $\int dm^2m^2A_e(m^2)$, whereas it is only the existence of $\int dm^2A_e(m^2)<1$ that is required of the complete Green's-function structure.

[3] J. Schwinger, Phys. Rev. 130, 406 (1963).

When charge densities are considered, the additional delta-function term does not appear, and

$$i\langle(j^0(x)j^0(x'))_+\rangle=-\nabla^2\int dm^2m^2A_e(m^2)\Delta_+(_+-x',m^2).$$

An alternative version of this result is obtained with the aid of the static Green's function, defined in space-time by the differential equation

$$-\nabla^2\mathfrak{D}(x-x')=\delta(x-x'),$$

and having the explicit form

$$\mathfrak{D}(x-x')=\delta(x^0-x^{0\prime})\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

It is

$$\int(dx_1)\mathfrak{D}(x-x_1)i\langle(j_0(x_1)j^0(x'))_+\rangle=\int dm^2m^2A_e(m^2)\Delta_+(x-x',m^2).$$

A particularly useful consequence is found on integrating over all space-time. Since $(m^2>0)$

$$\int(dx)\Delta_+(x,m^2)=1/m^2,$$

we get

$$\int(dx)(dx_1)\mathfrak{D}(x-x_1)i\langle(j^0(x_1)j^0(x'))_+\rangle=\int dm^2A_e(m^2),$$

and similarly

$$\int(dx)(dx_1)\mathfrak{D}(x-x_1)i\langle({}^*j^0(x_1){}^*j^0(x'))_+\rangle=\int dm^2A_e(m^2).$$

Thus,

$$A_0=1-\int(dx)(dx_1)\mathfrak{D}(x-x_1)[i\langle(j^0(x_1)j^0(x'))_+\rangle+i\langle({}^*j^0(x_1){}^*j^0(x'))_+\rangle].$$

We add one disconnected remark about space-parity and the electromagnetic field. The only parity-violating term in $F_{\mu\nu,\lambda\kappa}(p)$ is the one with weight factor A_{eg}. This contribution is isolated by considering the pseudoscalar $\mathbf{E}\cdot\mathbf{H}$, for

$$-\tfrac{1}{3}\langle\mathbf{E}(x)\cdot\mathbf{H}(x')\rangle=-\tfrac{1}{3}\langle\mathbf{H}(x)\cdot\mathbf{E}(x')\rangle=\int dm^2m^2A_{eg}(m^2)\Delta^{(+)}(x-x',m^2).$$

An electromagnetic parity violation, in the specific sense of a nonvanishing vacuum expectation value of the pseudoscalar field combination, can occur only if magnetic charge exists, and if it is correlated with electrical charge.

CHARGE RENORMALIZATION

We consider a system composed of a spin-$\frac{1}{2}$ field $\psi(x)$, carrying electrical charge e (a subscript is omitted), a spin-$\frac{1}{2}$ field $\chi(x)$ that carries magnetic charge g, and externally specified quasi-static electric and magnetic charge distributions, $J^0(x)$ and ${}^*J^0(x)$. The energy density operator for this system is

$$\begin{aligned}T^{00}(x)=&\tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2)(x)+(3i/\boldsymbol{\epsilon}^2)\bar\psi(x+\tfrac{1}{2}\epsilon)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}\psi(x-\tfrac{1}{2}\epsilon)\\&\times\exp\left[ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}d\mathbf{x}_1\cdot\mathbf{A}(x_1)\right]+m_e\bar\psi(x)\psi(x)\\&+(3i/\boldsymbol{\epsilon}^2)\bar\chi(x+\tfrac{1}{2}\epsilon)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}\chi(x-\tfrac{1}{2}\epsilon)\\&\times\exp\left[ig\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}d\mathbf{x}_1\cdot\mathbf{B}(x_1)\right]+m_g\bar\chi(x)\chi(x),\end{aligned}$$

where each $\boldsymbol{\epsilon}$ is independently averaged over all directions before becoming of arbitrarily small magnitude. Here

$$\mathbf{A}(x)=\mathbf{A}^T(x)+\int(dx')\mathbf{a}_\mathbf{n}(x-x')({}^*j^0+{}^*J^0)(x'),$$

$$\mathbf{B}(x)=\mathbf{B}^T(x)-\int(dx')\mathbf{a}_\mathbf{n}(x'-x)(j^0+J^0)(x'),$$

and

$$\mathbf{H}(x)=\mathbf{H}^T(x)-\nabla\int(dx')\mathfrak{D}(x-x')({}^*j^0+{}^*J^0)(x')=\nabla\times\mathbf{A}(x)-\int(dx')\mathbf{h}_\mathbf{n}(x-x')({}^*j^0+{}^*J^0)(x'),$$

$$\mathbf{E}(x)=\mathbf{E}^T(x)-\nabla\int(dx')\mathfrak{D}(x-x')(j^0+J^0)(x')=-\nabla\times\mathbf{B}(x)+\int(dx')\mathbf{h}_\mathbf{n}(x'-x)(j^0+J^0)(x').$$

The vector functions $\mathbf{a}_\mathbf{n}(x-x')$ and $\mathbf{h}_\mathbf{n}(x-x')$ contain the factor $\delta(x^0-x^{0\prime})$. They obey the differential equation

$$\nabla\times\mathbf{a}_\mathbf{n}(x)=-\nabla\mathfrak{D}(x)+\mathbf{h}_\mathbf{n}(x),$$

where $\mathbf{h}_\mathbf{n}(\mathbf{x})$, the three-dimensional factor of $\mathbf{h}_\mathbf{n}(x)$, is a two-dimensional distribution localized on the semi-infinite line $\mathbf{x}=\mathbf{n}|\mathbf{x}|$ and having the direction of $\mathbf{n}$. It is completely specified by the surface integral enclosing the origin,

$$\int d\mathbf{S}\cdot\mathbf{h}_\mathbf{n}(\mathbf{x})=-1.$$

These properties of $\mathbf{h}_\mathbf{n}(x)$, combined with the gauge condition

$$\nabla\cdot\mathbf{a}_\mathbf{n}(x)=0,$$

fully determine $\mathbf{a}_\mathbf{n}(x)$.

In addition to the lowest energy value of the system we shall be interested in the additional angular momentum of this system in the direction of the singularity line. It is given by (throughout the relativistic field discussion, charge is measured in rationalized units and $\hbar=c=1$)

$$K=(1/4\pi)\int(d\mathbf{x})(j^0+J^0)(x)\int(d\mathbf{x}')({}^*j^0+{}^*J^0)(x').$$

Accordingly we discuss a system with effective Hamiltonian

$$H=P^0+4\pi\omega K,$$

where ω is an arbitrary parameter, and examine the dependence of the vacuum transformation function $\langle 0_+|0_-\rangle^J$ upon the external charge distribution. It is supposed that the system is initially in the vacuum state $|0_-\rangle$, and that an arbitrarily prescribed charge distribution is established adiabatically by suitable separation of initially compensating charge distributions. After a time sufficient to register the static properties of this system has elapsed, the charge distribution is adiabatically recombined, thereby regaining the vacuum state $\langle 0_+|$. Alternatively, we can concentrate upon a macroscopic but finite region of space, and move charges into and then out of this region, always sufficiently slowly that the quasi-static character of the situation is maintained.

The quantum action principle describes the dependence of the transformation function upon the external charge distributions, contained in the effective Hamiltonian, by

$$\delta_J\langle 0_+|0_-\rangle^J=-i\langle 0_+|\int dx^0\delta_J H|0_-\rangle^J$$

or

$$\delta_J\ln\langle 0_+|0_-\rangle^J=-i\int dx^0\langle\delta_J H\rangle^J,$$

where

$$\langle\delta H\rangle^J=\langle 0_+|\delta H|0_-\rangle^J/\langle 0_+|0_-\rangle^J.$$

The explicit form is

$$\begin{aligned}i\delta_J\ln\langle 0_+|0_-\rangle^J
&=\int(dx)(dx')[\delta J^0(x)\mathfrak{D}(x-x')(\langle j^0\rangle+J^0)(x')\\
&\quad+\delta{}^*J^0(x)\mathfrak{D}(x-x')(\langle{}^*j^0\rangle+{}^*J^0)(x')\\
&\quad+\delta J^0(x)\mathbf{a}_{\mathbf{n}}(x-x')\cdot\langle{}^*\mathbf{j}(x')\rangle-\langle\mathbf{j}(x)\rangle\cdot\mathbf{a}_{\mathbf{n}}(x-x')\\
&\quad\times\delta{}^*J^0(x')+\omega\delta J^0(x)\delta(x^0-x^{0\prime})(\langle{}^*j^0\rangle+{}^*J^0)(x')\\
&\quad+\omega(\langle j^0\rangle+J^0)(x)\delta(x^0-x^{0\prime})\delta{}^*J^0(x')],\end{aligned}$$

where the current operators are identified as

$$\mathbf{j}(x)=e(3\boldsymbol{\varepsilon}/\boldsymbol{\varepsilon}^2)\bar\psi(x+\tfrac12\epsilon)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}\psi(x-\tfrac12\epsilon)\times\exp\left[ie\int_{x-\frac12\epsilon}^{x+\frac12\epsilon}d\mathbf{x}_1\cdot\mathbf{A}(x_1)\right],$$

$${}^*\mathbf{j}(x)=g(3\boldsymbol{\varepsilon}/\boldsymbol{\varepsilon}^2)\bar\chi(x+\tfrac12\epsilon)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}\chi(x-\tfrac12\epsilon)\times\exp\left[ig\int_{x-\frac12\epsilon}^{x+\frac12\epsilon}d\mathbf{x}_1\cdot\mathbf{B}(x_1)\right]$$

and the ϵ limiting process is understood. In writing this, we have introduced a simplification concerning the line integral of $\mathbf{a}_{\mathbf{n}}(x-x')$, which is permissible only if $\mathbf{x}-\mathbf{x}'$ and the singularity line never coincide in direction. The two points x and x' refer, in application, to the neighborhoods of two distinct charge distributions which are widely separated. Accordingly, we have only to ensure that $\mathbf{n}$ is not parallel to the line connecting the two charges.

To reduce the length of this transformation function formula, we define a modified quantity

$$\begin{aligned}\langle 0_+||0_-\rangle^J=\langle 0_+|0_-\rangle^J\exp\Big\{i\int(dx)(dx')[\tfrac12 J^0(x)\mathfrak{D}(x-x')
&\times J^0(x')+\tfrac12{}^*J^0(x)\mathfrak{D}(x-x'){}^*J^0(x')\\
&+\omega J^0(x)\delta(x^0-x^{0\prime}){}^*J^0(x')]\Big\}\end{aligned}$$

which obeys a similar variational equation, from which all terms that refer only to the external charges are removed. Then, in order to make explicit that $\langle j^\mu(x)\rangle^J$ and $\langle{}^*j^\mu(x)\rangle^J$ are currents induced by the presence of the external charges, we apply a second variation:

$$\begin{aligned}i\delta_J{}^2\ln\langle 0_+||0_-\rangle^J
&=\int(dx)(dx')[\delta J^0(x)\mathfrak{D}(x-x')\delta\langle j^0(x')\rangle\\
&\quad+\delta{}^*J^0(x)\mathfrak{D}(x-x')\delta\langle{}^*j^0(x')\rangle+\delta J^0(x)\mathbf{a}_{\mathbf{n}}(x-x')\cdot\delta\langle{}^*\mathbf{j}(x')\rangle\\
&\quad-\delta\langle\mathbf{j}(x)\rangle\cdot\mathbf{a}_{\mathbf{n}}(x-x')\delta{}^*J^0(x')\\
&\quad+\omega\delta J^0(x)\delta(x^0-x^{0\prime})\delta\langle{}^*j^0(x')\rangle\\
&\quad+\omega\delta\langle j^0(x)\rangle\delta(x^0-x^{0\prime})\delta{}^*J^0(x')].\end{aligned}$$

We shall assume that the induced charges and currents are linear functionals of the external charge distributions, supposing that the dominance of this part can be realized by suitable restrictions on the charges.

The action principle supplies time-ordered operator expressions for these induced quantities. Thus

$$\delta\langle j^0(x)\rangle^J=\delta[\langle 0_+|j^0(x)|0_-\rangle^J/\langle 0_+|0_-\rangle^J],$$

which will differ from $\delta[\ln\langle 0_+|0_-\rangle^J]$ only by the insertion of the operator $j^0(x)$ in the appropriate time position,[4]

$$\delta\langle j^0(x)\rangle^J=-\int(dx')(dx'')i\langle(j^0(x)j^0(x'))_+\rangle\times\mathfrak{D}(x'-x'')\delta J^0(x'').$$

The linear approximation is introduced by using vacuum expectation values for time-ordered products. An inessential simplification permitted by our model is the neglect of electric and magnetic charge correlations. We have omitted the term containing $\langle(j^0(x)\mathbf{j}(x'))_+\rangle$ since it implies the combination $\nabla\cdot\mathbf{a}_n$, which vanishes. Also omitted is the ω-proportional contribution since the three-dimensional volume integral of $\langle(j^0(x)j^0(x'))_+\rangle$ is zero. An analogous statement is

$$\delta\langle{}^*j^0(x)\rangle^J=-\int(dx')(dx'')i\langle({}^*j^0(x)\,{}^*j^0(x'))_+\rangle\times\mathfrak{D}(x'-x'')\delta\,{}^*J^0(x').$$

In extending this treatment to induced currents, we must take into account that the current operators are explicit functions of the external charges. The effect of this is to supplement the time-ordered products by terms designed to maintain the current conservation conditions. Since these structures are known from the work of the previous section, we shall exhibit only the time-ordered products but understand the complete combinations. Then

$$\delta\langle\mathbf{j}(x)\rangle^J=\int(dx')(dx'')i\langle(\mathbf{j}(x)\mathbf{j}(x'))_+\rangle\cdot\mathbf{a}_n(x'-x'')\delta\,{}^*J^0(x'')$$

and

$$\delta\langle{}^*\mathbf{j}(x)\rangle^J=-\int(dx')(dx'')\delta J^0(x'')\mathbf{a}_n(x''-x')\times i\langle({}^*\mathbf{j}(x')\,{}^*\mathbf{j}(x))_+\rangle.$$

There are other contributions involving vacuum expectation values of the product of charge density and current operators, but again these will not appear in the final result.

The integrated form of the modified transformation function is

$$\langle 0_+||0_-\rangle^J=\exp\left\{i\int(dx)(dx')[\tfrac{1}{2}J^0(x)R_e(x,x')J^0(x')+\tfrac{1}{2}\,{}^*J^0(x)R_g(x,x')\,{}^*J^0(x')+\omega J^0(x)R_{eg}(x,x')\,{}^*J^0(x')]\right\},$$

where

$$R_e(x,x')=\int(dx'')(dx''')\mathfrak{D}(x-x'')i\langle(j^0(x'')\,j^0(x'''))_+\rangle\mathfrak{D}(x'''-x')+\int(dx'')(dx''')\mathbf{a}_n(x-x'')\cdot i\langle({}^*(\mathbf{j}x'')\,{}^*\mathbf{j}(x'''))_+\rangle\cdot\mathbf{a}_n(x'-x'''),$$

$$R_g(x,x')=\int(dx'')(dx''')\mathfrak{D}(x-x'')i\langle({}^*j^0(x'')\,{}^*j^0(x'''))_+\rangle\mathfrak{D}(x'''-x')+\int(dx'')(dx''')\mathbf{a}_n(x''-x)\cdot i\langle(\mathbf{j}(x'')\mathbf{j}(x'''))_+\rangle\cdot\mathbf{a}_n(x'''-x'),$$

and

$$R_{eg}(x,x')=\int(dx'')(dx''')\mathfrak{D}(x-x'')i\langle(j^0(x'')j^0(x'''))_+\rangle\delta(x^{0\prime\prime\prime}-x^{0\prime})+\int(dx'')(dx''')\delta(x^0-x^{0\prime\prime})i\langle({}^*j^0(x'')\,{}^*j^0(x'''))_+\rangle\mathfrak{D}(x'''-x').$$

With its neglect of contributions that are quadratic in the external current distributions, as distinguished from the charge distributions, this result can claim physical validity only in the limit of adiabatically slow motion of the charges. The quasi-static situation is represented by simplifications of the type

$$\int dx^{0\prime}R_e(x,x')J^0(\mathbf{x}',x^{0\prime})\cong J^0(\mathbf{x}',x^0)\int_{-\infty}^{\infty}dx^{0\prime}R_e(x,x').$$

When this procedure is applied to $R_{eg}(x,x')$, keeping in mind the translational invariance of the various functions, the immediate result is

$$\int dx^{0\prime}R_{eg}(x,x')=1-A_0,$$

according to the relation established in the preceding section. In the quasi-static limit, then,

$$\int(dx)(dx')J^0(x)R_{eg}(x,x')\,{}^*J^0(x')\cong(1-A_0)\int(dx)(dx')J^0(x)\delta(x^0-x^{0\prime})\,{}^*J^0(x').$$

[4] To be accurate, it is $j^0-\langle j^0\rangle^J$ that appears here, but the distinction disappears in the linear approximation.

The quasi-static condition also selects the three-dimensional projections of $i\langle(jj)_+\rangle$ and $i\langle(^*j^*j)_+\rangle$, the residues after integration over relative time coordinates. We can then relate the vector and scalar structures, as indicated by

$$i\langle(j_k(x)j_l(x'))_+\rangle\rightarrow(\delta_{kl}\nabla^2-\partial_k\partial_l)(1/-\nabla^2)i\langle(j^0(x)j^0(x'))_+\rangle$$

or, in dyadic notation,

$$i\langle(\mathbf{j}(x)\mathbf{j}(x'))_+\rangle\rightarrow(\times\nabla)\cdot(\nabla'\times)(1/-\nabla^2)i\langle(j^0(x)j^0(x'))_+\rangle.$$

The vector operations can be transferred to the two $\mathbf{a}_\mathrm{n}$ factors, which gives the scalar product of two $\nabla\times\mathbf{a}_\mathrm{n}$ vectors. By inserting the relation

$$\nabla\times\mathbf{a}_\mathrm{n}=-\nabla\mathfrak{D}+\mathbf{h}_\mathrm{n}$$

successively, we obtain the static equivalence

$$\int(dx'')(dx''')\mathbf{a}_\mathrm{n}(x''-x)\cdot i\langle(\mathbf{j}(x'')\mathbf{j}(x'''))_+\rangle\cdot\mathbf{a}_\mathrm{n}(x'''-x')$$
$$\rightarrow\int(dx'')(dx''')\mathfrak{D}(x-x'')i\langle(j^0(x'')j^0(x'''))_+\rangle$$
$$\times\mathfrak{D}(x'''-x')-\int(dx'')(dx''')\mathbf{h}_\mathrm{n}(x''-x)$$
$$\times[(1/-\nabla''^2)i\langle(j^0(x'')j^0(x'''))_+\rangle]\cdot\mathbf{h}_\mathrm{n}(x'''-x')$$

with an analogous result for magnetic currents. We have seen that

$$(1/-\nabla''^2)i\langle(j^0(x'')j^0(x'''))_+\rangle$$
$$=\int(dx_1)\mathfrak{D}(x''-x_1)i\langle(j^0(x_1)j^0(x'''))_+\rangle$$
$$=\int dm^2m^2A_e(m^2)\Delta_+(x''-x''',\,m^2).$$

Only the time integral is of concern here. It is given for each m by

$$\int_{-\infty}^{\infty}dx^0\Delta_+(x,m^2)=e^{-m|\mathbf{x}|}/4\pi|\mathbf{x}|\,.$$

Accordingly, this function is of short range, as specified by the inverse of a mass that is characteristic of the vacuum polarization process. We are interested in the interaction of charges separated by much larger distances. Since the two points x'' and x''' in the product $\mathbf{h}_\mathrm{n}(x''-x)\cdot\mathbf{h}_\mathrm{n}(x'''-x')$ are practically identical the product vanishes, unless the line connecting the specified charges has practically the same direction as the singularity line. It is just this possibility that has has already been excluded.

As a consequence of these substitutions, it appears that R_e and R_g are identical,

$$R_e(x,x')=R_g(x,x')=\int(dx''')\int(dx'')$$
$$\times\mathfrak{D}(x-x'')[i\langle(j^0(x'')j^0(x'''))_+\rangle$$
$$+i\langle(^*j^0(x'')\,^*j^0(x'''))_+\rangle]\mathfrak{D}(x'''-x').$$

Furthermore, we can exploit the short-range nature of the function of $x-x'''$ that emerges from the x''' integration to replace x''' by x, in the function $\mathfrak{D}(x'''-x')$. The result is simply

$$R_e(x,x')=R_g(x,x')\cong(1-A_0)\mathfrak{D}(x-x')$$

and we recall that

$$R_{eg}(x,x')\cong(1-A_0)\delta(x^0-x^{0\prime}).$$

The final form of the quasi-static transformation function is now obvious:

$$\langle 0_+|0_-\rangle^J\cong\exp\left\{-iA_0\int(d_gx)(dx')[\tfrac{1}{2}J^0(x)\mathfrak{D}(x-x')\right.$$
$$\times J^0(x')+\tfrac{1}{2}\,^*J^0(x)\mathfrak{D}(x-x')\,^*J^0(x')$$
$$\left.+\omega J^0(x)\delta(x^0-x^{0\prime})\,^*J^0(x')]\right\}.$$

The single time integral actually contained in the exponential function identifies the lowest eigenvalue of the Hamiltonian operator, $P^0+4\pi\omega K$. The Coulomb interaction energies and the additional angular momentum of electric and magnetic charges are evident, with the phenomenological measure of all charges exhibiting the additional factor

$$C=A_0^{1/2}.$$

This derivation of the universal charge-renormalization constant completes our discussion.[5]

[5] I should also emphasize the importance of the operational definitions of charges that have been used. The expectation values of total charge operators would be quite misleading as a guide to renormalization.