Photon Counting Statistics of Gaussian Light*

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It is well known that the photocount distribution associated with the photoelectric detection of fluctuating light beams carries information about the probability density of the light intensity. In this paper, we present an exact solution to a problem of particular interest which arises in connection with the photoelectric detection of narrow-band Gaussian (thermal or pseudothermal) light, namely, the problem of determining the photocount distribution when the counting-time interval is not necessarily short compared to the coherence time of the light. The method developed in this paper is applied to the case where the spectral profile of the light is Lorentzian; it leads to an exact expression for the photocount generating function and it provides simple recurrence relations for the photocount distribution and for its factorial moments.

THE determination of statistical properties of fluctuating light beams from photoelectric measurements has recently become the subject of extensive investigations.[1–6] In this paper we present an exact solution to a problem of particular interest, which arises in connection with the detection of narrow-band Gaussian (thermal) light, namely, the problem of determining the photocount distribution when the counting-time interval T is not necessarily short compared with the coherence time τ_c of the light. The method developed in the present paper is applied to the particularly important case where the spectrum of the light is Lorentzian.

Attempts to solve this problem have in the past led either to approximate expressions for the photocount distribution,[7,8] or to general expressions for its cumulants[9] and its factorial moments,[10] which, for practical cases, could be evaluated only after lengthy algebraic calculations. Our present analysis provides the exact expression for the generating function $G(s)$ of the photocount distribution $p(n)$, defined as[11]

$$G(s)=\sum_{n=0}^{\infty} p(n)(1-s)^n, \tag{1}$$

and establishes simple recurrence relations for the $p(n)$ distribution and it factorial moments.

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[9] L. Mandel, in *Progress in Optics*, edited by E. Wolf (John Wiley & Sons, Inc., New York, 1963), Vol. II, p. 229.

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[11] See, for example, R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. deWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, Science Publishers, Inc., New York, 1965), p. 63.

It follows from the well-known expression[9,12]

$$p(n)=\int_0^{\infty}\frac{(\alpha E)^n}{n!}e^{-\alpha E}P(E)dE, \tag{2}$$

relating the photocount distribution $p(n)$ to the probability distribution $P(E)$ of the light intensity

$$E=\int_t^{t+T} I(t')dt',$$

(α being a measure of the photoefficiency of the detector), that the generating function $G(s)$ is the Laplace transform of $P(E)$. By the use of the Karhunen-Löeve expansion,[13] it can be shown that the generating function corresponding to the photocount distribution, associated with the detection of narrow-band Gaussian light of arbitrary spectral profile, is given by

$$G(s)=\prod_{k=0}^{\infty}\left[1+\frac{\lambda_k}{\langle n\rangle}s\right]^{-1}. \tag{3}$$

Here $\langle n\rangle=\alpha\langle E\rangle$ is the average number of photocounts in the time interval T, and λ_k are the eigenvalues of the homogeneous Fredholm integral equation, whose kernel is the normalized second-order autocorrelation function $\gamma(t,t')$ of the complex field amplitude, i.e.,

$$\lambda_k\varphi_k(t)=\int_{-T/2}^{T/2}\gamma(t,t')\varphi_k(t')dt', \quad |t|\leqslant T/2(k=0,1,2,\cdots). \tag{4}$$

Under well-known conditions[13] on the kernel $\gamma(t,t')$, the eigenfunctions $\varphi_k(t)$ form a complete orthonormal set in the domain $-T/2\leqslant t\leqslant T/2$.

Consider next the particularly important case in which the spectral profile of the narrow-band Gaussian light is Lorentzian of linewidth Γ. The corresponding kernal is

$$\gamma(t,t')=\exp[-\Gamma|t-t'|]. \tag{5}$$

[12] L. Mandel, Proc. Phys. Soc. (London) **72**, 1037 (1958).

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In this case, the solutions to this integral equation, well known in the theory of random noise,[14] are

$$\lambda_k = \langle n\rangle \Gamma T / [\Gamma^2 T^2 + \chi_k^2], \qquad (6)$$

where the χ_k are the non-negative roots of either of the equations

$$2\chi \tan\chi = \Gamma T, \qquad (7)$$
$$2\chi \cot\chi = -\Gamma T.$$

Making use of Hadamard's factorization theorem,[15] it readily follows, from Eqs. (3), (6), and (7), that the generating function $G(s)$ is

$$G(s) = \frac{\exp(\Gamma T)}{[\cosh z + \sinh z(\Gamma T/2z + z/2\Gamma T)]}, \qquad (8)$$

where $z = [2\Gamma T\langle n\rangle s + \Gamma^2 T^2]^{1/2}$.

It is clear from the definition of the generating function that the probabilities $p(k)$ and the factorial moments $\langle n^{[k]}\rangle$ are readily expressed in terms of the kth derivative of $G(s)$ at $s=1$ and $s=0$, respectively.[11] Upon application of Leibniz' differentiation rule to Eq. (8), one obtains the following recurrence relations:

$$k!p(k) = \sum_{r=0}^{k-1} (-1)^{k+r+1} \binom{k}{r} r!p(r) [D_{k-r}(s)]_{s=1}, \qquad (9)$$

$$\langle n^{[k]}\rangle = \sum_{r=0}^{k-1} (-1)^{k+r+1} \binom{k}{r} \langle n^{[r]}\rangle [D_{k-r}(s)]_{s=0}, \qquad (10)$$

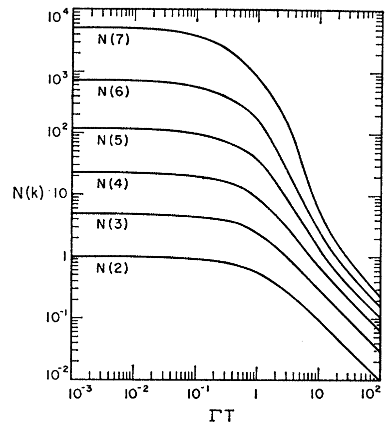

FIG. 1. Normalized factorial moments versus normalized counting-time interval.

where

$$D_l(s) = G(s)e^{-\Gamma T}[\langle n\rangle\Gamma T/z]^l \{\tfrac{1}{2}\Gamma T i_l(z) + z(1 + 1/2\Gamma T) i_{l-1}(z) + (z^2/2\Gamma T) i_{l-2}(z)\}, \qquad (11)$$

i_l being the modified spherical Bessel function[16] of the first kind of order l, and $z = [2\Gamma T\langle n\rangle s + \Gamma^2 T^2]^{1/2}$.

It appears from Eqs. (9), (10), and (11) that, in comparing experimental results with theory, it is convenient to make use of the factorial moments $\langle n^{[k]}\rangle$. These moments, readily evaluated from Eqs. (10) and (11), define the photocount distribution $p(n)$ and are equivalent to an explicit expression for $p(n)$. Figure 1, obtained from the present theory, shows the first few normalized factorial moments $N(k) = (\langle n^{[k]}\rangle / \langle n\rangle^k) - 1$ as functions of the dimensionless parameter ΓT. The method described in this paper may be readily applied to narrow-band Gaussian light of arbitrary spectral profile.

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[15] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954).

[16] See, for example, M. Abramowitz and I. A. Stegun, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1954).