Calling the operator defined in Eq. (6) $\tau(0)$, we can also introduce the time-dependent operator

$$\tau(t)=e^{iH_0t}\tau(0)e^{-H_0t} \\ =t+\tau(0). \quad (8)$$

The relations given in Eqs. (7) and (8) above are the quantum mechanical analogs of the classical Eqs. (1) and (2).

The application to the calculation of time delays is made by considering the matrix element

$$(\Psi_a(t),\tau(0)\Psi_a(t)), \quad (9)$$

where the state vector $\Psi_a(t)$ is the *scattered* wave packet, evaluated for times after the scattering process has been completed. The time dependence of this state vector is then given by the free-particle Hamiltonian alone: $\Psi_a(t)=e^{-iH_0t}\Psi_a(0)$; here, the last factor represents the scattered-state vector extrapolated back to zero time.

In the following, both the incident and scattered wave packets are assumed to be normalized to unity. Since the scattered state is connected with the initial state vector $\phi_a(0)$ by the S matrix, (9) can be evaluated as below:

$$(\Psi_a(0),e^{iH_0t}\tau(0)e^{-iH_0t}\Psi_a(0)) \\ =(\Psi_a(0),\{t+\tau(0)\}\Psi_a(0)) \\ =t+(\phi_a(0),S^{-1}\tau(0)S\phi_a(0)). \quad (10)$$

If the S matrix has the form $S=e^{2i\delta(E)}$, in an energy representation the time delay operator becomes equivalent to energy differentiation, and we find, as a final result

$$(\Psi_a(t),\tau(0)\Psi_a(t))=t-2\frac{\partial\delta(E)}{\partial E}+(\phi_a(0),\tau(0)\phi_a(0)). \quad (11)$$

In this form the interpretation of the energy derivative of the phase shift as the time delay induced by the scattering process is apparent.

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Gauge Invariance and Current Definition in Quantum Electrodynamics*

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In order to study the problems of gauge invariance, Lorentz covariance, and the operator properties of the "Schwinger terms" in the current commutators, spin-½ quantum electrodynamics is written as the limit of a nonlocal theory. The conditions on such a theory are discussed and the nonlocal equations in the case of an external vector potential derived. The gauge-invariant, Lorentz-covariant limit of these equations is then discussed, and it is found that (in the case of spin ½) the "Schwinger terms" are purely c-number. The quantized vector potential is considered by means of a Feynman path integral and its gauge structure determined. It is found that an automatically gauge-covariant theory results and that the c-number character of the Schwinger terms apparently persists.

I. INTRODUCTION

In quantum electrodynamics, if the canonical commutation relations of its constituent fermion fields are used to calculate the commutator $[j^0(\mathbf{r}), j^k(\mathbf{r}')]$, it vanishes identically. This is in direct contradiction to the general theorem[1]

$$\langle 0|[j'(\mathbf{r}), j^k(\mathbf{r}')]|0\rangle=-i\nabla^k\delta(\mathbf{r}-\mathbf{r}')c,$$

where c is non-negative and vanishes only if the vacuum is an eigenstate of the current j^μ. Hence, its vanishing implies that the current is a constant c-number current and that the electromagnetic field is free. Schwinger, in the same paper,[1] gave a partial solution to the problem by pointing out that the current should be defined as a limit of separated points. This, then, would not yield a gauge-invariant current unless there were some explicit dependence on the vector potential to cancel the gauge transformations of the charged fields. The relation of the additional dependence to Lorentz covariance and current conservation has also been discussed by Johnson[2] and by Brown.[2]

This device, while resolving the paradox of the commutation relations, raises the question of the proper equations of motion for the fields. In general one would expect both the current definition and the field equations for the charged fields to be changed. Also, the Lorentz

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[1] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[2] K. Johnson, Nucl. Phys. 25, 431 (1961); L. S. Brown, Phys. Rev. (to be published). The relation between explicit field dependence and current commutators is also discussed in J. Schwinger, *ibid.* 130, 406 (1963); and by D. Boulware and S. Deser, *ibid.* this issue, 151, 1278 (1966).

invariance of the theory is no longer manifest, since a particular frame must be chosen in which to define the canonical commutation relations and to specify the altered current. It is not obvious that the limits taken in different frames yield the same expressions.

In Sec. II, we rederive the noncommutation relation in a way which does not depend on Lorentz covariance beyond the structure of Maxwell's equations. An extended discussion of possible nonlocal Hamiltonians is given in Sec. III. Heuristic arguments are given which determine the Hamiltonian uniquely and the consistency of the resultant currents with the requirements discussed in Sec. II is verified. The resultant $[F^{0k},j^l]$ commutator is an operator in the nonrelativistic case, but the operator terms are found to disappear in the relativistic limit. The relativistic limit can not be treated by simply looking at the fields and the field equations; hence, we look at propagator equations in Sec. IV. Also, to allow analytic considerations, we look first at the case of a c-number external vector potential. The equations for the propagators are derived, and a gauge-invariant propagator defined. These may, of course, be solved perturbatively and such a solution leads to the explicit dependence found in Sec. III. The result is not Lorentz covariant and there does not seem to be any way to define the limit of the solution so that it is Lorentz covariant. On the other hand, it is easy to define a Lorentz covariant limit of the gauge-invariant propagator equation. We define this as the solution to the Lorentz covariant theory. The current definition is then taken as the usual definition except that it is calculated in terms of the gauge-invariant propagator instead of the standard propagator. The nonlocal theory gives the current in terms of the gauge-invariant propagator but there are additional terms which do not behave properly in the local (relativistic) limit.

The solutions to the gauge-dependent and gauge-independent propagators are exhibited as a Fredholm solution. Then, given the propagators, the field dependence of the currents is calculated and it is found that there is only the dependence directly required by the Lehmann forms and that there is no operator dependence involved for this model.

The last section is devoted to the quantized vector potential and some indication of the results are given.

II. COMMUTATION RELATIONS

The simplest calculation of the commutator of different components of a vector follows directly from the Lehmann-Källén[3] spectral form. In the case of a Lorentz covariant theory,

$$\langle 0|j^\mu(x)j^\nu(x')|0\rangle=\int_0^\infty ds\{\rho_1(s)[g^{\mu\nu}-s^{-1}\partial^\mu\partial^\nu] -s^{-1}\rho_0(s)\partial^\mu\partial^\nu\}\Delta^{(+)}(x'-x',s)\,, \quad (1)$$

where $\Delta^{(+)}$ is the positive frequency function of mass $\sqrt{s}$, and ρ_1 and ρ_0 are each positive definite, arising from spin-1 and spin-0 states, respectively. Current conservation implies $\rho_0=0$.

Then all vacuum expectation values of equal-time commutators vanish except

$$\langle 0|[j^0(\mathbf{r}),j^k(\mathbf{r}')]|0\rangle = -i\int_0^\infty ds\, s^{-1}[\rho_1(s)+\rho_0(s)]\partial^k\delta(\mathbf{r}-\mathbf{r}')\,, \quad (2)$$

which can only vanish if $\rho_1\equiv 0\equiv\rho_0$. Then j^μ must vanish.[4]

In what follows, we will be concerned with non-Lorentz-invariant theories. Thus, we give a proof which only depends on rotational invariance and the fact that j^μ is the source of the electromagnetic field. We start from a radiation-gauge formulation of quantum electrodynamics. Then[5]

$$\langle 0|A^0(x)A^0(x')|0\rangle =\int_0^\infty d\omega\int d^3k\frac{e^{i(\mathbf{k}\cdot\xi-\omega\xi^0)}}{(2\pi)^3}\frac{1}{k^2}\rho_0(k,\omega)\,,$$

$$\langle 0|A^l(x)A^m(x')|0\rangle =\int_0^\infty d\omega\int d^3k\frac{e^{i(\mathbf{k}\cdot\xi-\omega\xi^0)}}{(2\pi)^3}(\delta^{lm}-k^lk^m/k^2)\rho_1(k,\omega)\,, \quad (3)$$

where $\xi^\mu=x^\mu-x'^\mu$. We have assumed only positive energies for the intermediate states, and we have ρ_0 and ρ_1 greater than or equal to zero as before. No A^0A^l term occurs because $\nabla\cdot\mathbf{A}=0$.

Now, we use Maxwell's equations; $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ on both fields and $\partial_\nu F^{\mu\nu}=j^\mu$ on the second field. Then

$$\langle 0|F^{0l}(x)j^0(x')|0\rangle =-i\int_0^\infty d\omega\int d^3k\frac{e^{i(\mathbf{k}\cdot\xi-\omega\xi^0)}}{(2\pi)^3}k^l\rho_0(k,\omega)\,, \quad (4a)$$

$$\langle 0|F^{0l}(x)j^m(x')|0\rangle =-i\int_0^\infty d\omega\int \frac{d^3k\, e^{i(\mathbf{k}\cdot\xi-\omega\xi^0)}}{(2\pi)^3}\omega\rho^{lm}(\mathbf{k},\omega)\,, \quad (4b)$$

where

$$\rho^{lm}(\mathbf{k},\omega)=(\delta^{lm}-k^lk^m/k^2)(\omega^2-k^2)\rho_1(k,\omega) +(k^lk^m/k^2)\rho_0(k,\omega)\,,$$

$$\langle 0|F^{lm}(x)j^0(x')|0\rangle=0\,, \quad (4c)$$

$$\langle 0|F^{lm}(x),j^n(x')|0\rangle=-i\int_0^\infty d\omega\int d^3k\frac{e^{i(\mathbf{k}\cdot\xi-\omega\xi^0)}}{(2\pi)^3} \times(k^l\delta^{mn}-k^m\delta^{ln})\rho_1(k,\omega)\,. \quad (4d)$$

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[3] H. Lehmann, Nuovo Cimento 11, 342 (1954); G. Källén, Helv. Phys. Acta 23, 201 (1950).

[4] P. Federbush and K. Johnson, Phys. Rev. 120, 1926 (1960).

[5] We use a space-like metric throughout, $(-1,1,1,1)$ and $\hbar=c=1$.

Then, the vacuum expectation values of the equal-time commutators all vanish except

$$\langle 0|[F^{0l}(\mathbf{r}),\, j^m(\mathbf{r}')]|0\rangle = -2i\int_0 d^3k\frac{e^{i\mathbf{k}\cdot\xi}}{(2\pi)^3}\int_0^\infty d\omega\,\omega\rho^{lm}(\mathbf{k},\omega). \quad (5)$$

The commutator is a nonvanishing, positive-definite function of $\mathbf{r}-\mathbf{r}'$ unless both ρ_0 and $(\omega^2-k^2)\rho_1$, are zero.

The first ($\rho_0=0$) implies that $A^0=0$, or $j^0=0$. Its contribution to the commutator is longitudinal; hence it derives directly from the $[j^0, j^l]$ commutator. In the relativistic case, this term comes from the definition of the current as the limit of a nonlocal function and is independent of the dependence on A_k. The second condition $[(\omega^2-k^2)\rho_1=0]$ is related to the explicit dependence of j^l on the transverse part of A_k. Gauge invariance can be maintained by defining j^l as a function of the (c-number) longitudinal part of A_k. But then the transverse part of Eq. (5) would vanish and we would have to have $\rho_1\propto\delta(\omega^2-k^2)$ or the electromagnetic field would be free. Proceeding one step further,

$$\langle 0|[j^0(\mathbf{r}),\, j^l(\mathbf{r}')]|0\rangle = -i\int d^3k\frac{e^{i\mathbf{k}\cdot\xi}k^l}{(2\pi)^3}\int_0^\infty d\omega\,\omega\rho_0(k,\omega),$$

is the rotationally invariant result. Although it need not have the locality of the Lorentz covariant result it is still definitely nonzero for any interacting system.

At this point, we must construct a theory which contains explicit dependence on A. It is very dangerous to do this in an *ad hoc* manner since one risks losing consistency between the various equations of motion. Rather than attempt to guess the right structure for j^μ, we will guess a Hamiltonian and commutation relations, then derive the currents.

III. THE NONLOCAL HAMILTONIAN

The primary requirement of any electrodynamic Hamiltonian is that it yield Maxwell's equations and, in the local limit, become Lorentz invariant. The relativistic Hamiltonian, in radiation gauge, may be written[6]

$$H=\int d\tau\left\{\tfrac{1}{2}F^{0k}F^0{}_k+\tfrac{1}{4}F^{kl}F_{kl} +\frac{1}{4}\left[\psi,\left(\boldsymbol{\alpha}\cdot\left(\frac{1}{i}\boldsymbol{\nabla}-eq\mathbf{A}\right)+m\beta\right)\psi\right]\right\}, \quad (6)$$

where

$$F^{kl}=\partial^kA^l-\partial^lA^k \quad\text{and}\quad \partial_kF^{0k}=j^0=\tfrac{1}{2}e\psi q\psi.$$

The current is then $j^k=\frac{1}{2}e\psi\alpha^kq\psi$. It is now necessary to modify H so that the coefficient of A_k is itself dependent on A; then the current will have the requisite commutation relations. Following Schwinger, we introduce nonlocality and additional A dependence into the coefficient of A^k. Rather than use a given separation, we will use a weight function and average over different separations. The crucial point is that $\rho(\mathbf{r},\mathbf{r})$ is finite. Then we replace $\frac{1}{2}e\psi\alpha^kq\psi$ by

$$\int d\tau'\rho(\mathbf{r},\mathbf{r}')\tfrac{1}{4}e[\psi(\mathbf{r}),\,\alpha^kqe(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')],$$

where e depends on A at least in such a way as to insure gauge invariance. Then, we add A^k to find

$$\int d\tau d\tau' A_k(x)\rho(\mathbf{r},\mathbf{r}')\tfrac{1}{4}[\psi(x),\,q\alpha^ke(x,x')\psi(x')]_{x^0=x'^0}$$

as the interaction term in the Hamiltonian. Under gauge transformations this term transforms like

$$\int d\tau d\tau'\rho(\mathbf{r},\mathbf{r}')(\partial_k\lambda(x))\frac{e}{4}[\psi(x),\,e(x,x')\alpha^kq\psi(x')]_{x^0=x'^0}.$$

Thus, the derivative term must also have the points separated so that the terms can cancel. At this point, there is no argument that the mass term must also have the points separated, but we will separate them for the uniformity and in order to simplify the equations. We leave the structure of the electromagnetic field equations unchanged. The Hamiltonian then becomes

$$H=\int d\tau\{\tfrac{1}{2}F^{0k}(x)F^0{}_k(x)+\tfrac{1}{4}F^{kl}(x)F_{kl}(x)\} +\int d\tau d\tau'\rho(\mathbf{r},\mathbf{r}')\left\{\frac{1}{4}\left[\psi(x),\,e(x,x')\times\left(\boldsymbol{\alpha}\cdot\left(\frac{1}{i}\boldsymbol{\nabla}'-eq\mathbf{A}(x')\right)+m\beta\right)\psi(x')\right]_{x^0=x'^0}\right\}. \quad (7)$$

The commutation relations and the functions ρ and e must still be specified. The only gauge which admits a canonical formalism with positive-definite inner products is the radiation or Coulomb gauge. Thus, we take radiation gauge and we require the field equation

$$\partial^0A^k(x)=F^{0kT}(x)=i[A^k(x),\,H]=F^{0k}(x)+\partial^kA^0(x). \quad (8)$$

This implies the canonical commutator

$$[F^{0k}(\mathbf{r}),\,A^l(\mathbf{r}')]=i[\delta^{kl}(\mathbf{r}-\mathbf{r}')]^T =i\left[\delta^{kl}(\mathbf{r}-\mathbf{r}')-\nabla^k\nabla'^l\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right], \quad (9)$$

for equal times. In order to maintain the gauge struc-

[6] We use Hermitian fields ψ and an antisymmetric charge matrix $q=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. The Dirac algebra must be written in a Majorana representation, $\alpha^\mu=\alpha^{\mu T}$, $\beta^T=-\beta$.

ture, we must have

$$\partial_0\psi(x)=(-i)[\psi(x),H]=ieqA_0(x)\psi(x) + \text{gauge-covariant terms}.$$

But, with the aid of the constraint $\partial_k F^{0k}=j^0$ we then have, from the $F^{0k}F^0{}_k$ term,

$$[\psi(\mathbf{r}),j^0(\mathbf{r}')]=eq\psi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'), \quad (10)$$

and, since j^0 is to be the generator of the gauge transformation on the charged fields, we require

$$[j^0,A^k]=0=[j^0,F^{\mu\nu}]=0=[j^0,e].$$

Further arguments by analogy with the Lorentz invariant theory can no longer be used, so to simplify ρ as much as possible, we assume

$$\{\psi(\mathbf{r}),\psi(\mathbf{r}')\}=\delta(\mathbf{r}-\mathbf{r}'). \quad (11)$$

j^0 may now be formally defined as $\frac{1}{2}e\psi q\psi$ and there can be no additional dependence on A^k or F^{0k} if j^0 is to simply generate the gauge transformation on the charged fields. There remain the terms ρ and e. In order to maintain a canonical formalism, the Hamiltonian can only refer to operators at a given time. This has already been included in writing ρ as a function of $\mathbf{r}$ and $\mathbf{r}'$. We now impose translational and rotational invariance separately on e and ρ. Thus $\rho(\mathbf{r},\mathbf{r}')=\rho(|\mathbf{r}-\mathbf{r}'|)$, and we further impose $\int d^3\xi\rho(\xi)=1$ as the normalization condition. There will be further conditions later to maintain the positive-definite structure of the Hamiltonian.

We must now discuss the function $e(x,x')$. Schwinger wrote

$$e(x,x')=\exp\left(ieq\int_{x'}^{x}dy_\mu A^\mu(y)\right),$$

which, under the gauge transformation $A^\mu\to A^\mu+\partial^\mu\lambda$ goes to $e(x,x')\exp ie[\lambda(x)-\lambda(x')]$, assuring gauge invariance. The integral is, however, path-dependent. In order to include the path dependence explicitly and the possibility of other forms, we will use

$$e(x,x')=\exp\left(ieq\int dy\, f^\mu(y,x,x')A_\mu(y)\right),$$

where f must have the property $\partial_\mu f^\mu(y,x,x')=\delta(y-x')-\delta(y-x)$. The further requirement of a canonical theory implies

$$f^\mu(y,x,x')=\delta(y^0-x^0)\tilde{f}^\mu(\mathbf{y},\mathbf{r},\mathbf{r}') \quad \text{for} \quad x^0=x'^0.$$

Then, $\tilde{f}^0$ must vanish to exclude time derivatives of a delta function. To see this, we observe that if, for $x^0=x'^0$, $f^0\neq 0$ then terms of the form $\partial_0 A^k$ must appear and we would have, dependence on F^{0k}, upsetting the equations

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu.$$

The longitudinal part of $\mathbf{f}$ for $x^0=x'^0$ is then completely determined. We could assume that the transverse part vanishes. Then, in the radiation gauge, we would have $e(x,x')=1$ and e would act to remove any c-number gauge transformation from the radiation gauge. Our commutation relations have restricted the gauge freedom to c-number gauge transformations in any case. This choice is not sufficient, however, since it does not yield the dependence on A^k necessary for $[F^{0l},j^m]$ to have a nonvanishing transverse part. Thus, $\mathbf{f}$ must have a transverse part yielding nontrivial dependence on A_k. In Appendix A, we show that the only form consistent with Lorentz and translational invariance is

$$f^\mu(y,x,x')=(x-x')^\mu\int_0^1 d\lambda\,\delta(y-x'-\lambda(x-x')).$$

The equations of motion are (8) and

$$A^0(x)=\int\frac{d\tau'}{4\pi|\mathbf{r}-\mathbf{r}'|}j^0(\mathbf{r}',x^0), \quad (12)$$

$$\begin{aligned}\partial_0F^{0k}(x)&=-i[F^{0k}(x),H]\\&=-\partial_lF^{lk}(x)+\tfrac{1}{4}e\int d^3\xi\,\rho(\xi)\Big\{[\psi(x),e(x,x-\xi)\alpha^kq\psi(x-\xi)]\\&\qquad-i\int dx'f^k(x,x'+\xi,x')\Big[\psi(x'+\xi),qe(x'+\xi,x')\Big(\boldsymbol{\alpha}\cdot\Big(\frac{1}{i}\nabla'-eq\mathbf{A}(x')\Big)+m\beta\Big)\psi(x')\Big]\Big\}, \end{aligned}\quad (13)$$

$$\begin{aligned}\frac{1}{i}\partial_0\psi(x)&=-[\psi,H]\\&=\tfrac{1}{2}eq\{A_0(x),\psi(x)\}-\int d^3\xi\,\rho(\xi)\Big\{\tfrac{1}{2}e(x,x-\xi)\Big[\boldsymbol{\alpha}\cdot\Big(\frac{1}{i}\nabla-eq\mathbf{A}(x-\xi)\Big)+m\beta\Big]\psi(x-\xi)\\&\qquad+\frac{1}{2}\Big(\boldsymbol{\alpha}\cdot\Big(\frac{1}{i}\nabla-eq\mathbf{A}(x)\Big)+m\beta\Big)e(x,x-\xi)\psi(x-\xi)\Big\}. \end{aligned}\quad (14)$$

The current is then

$$j^0(x)=\tfrac{1}{4}e[\psi(x),\,q\psi(x)]\,,$$

$$j^k(x)=\tfrac{1}{4}e\int d^3\xi\rho(\xi)\Big\{[\psi(x),\,e(x,\,x-\xi)\alpha^k q\psi(x-\xi)]$$

$$-i\int dx' f^k(x,\,x'+\xi,\,x')\Big[\psi(x'+\xi),\,qe(x'+\xi,\,x')\Big(\boldsymbol{\alpha}\cdot\Big(\frac{1}{i}\boldsymbol{\nabla}'-eq\mathbf{A}(x')\Big)+m\beta\Big)\psi(x')\Big]\Big\}\,; \quad (15)$$

it is manifestly gauge invariant and is conserved as it must be, since it is defined as $\partial_\nu F^{\mu\nu}=j^\mu$. The commutator $[F^{0l}(x),j^m(x')]$ is now, trivially,

$$[F^{0l}(\mathbf{r}),\,j^m(\mathbf{r}')]$$

$$=-\tfrac{1}{4}e^2\int d^3\xi\,\rho(\xi)\{\tilde{f}^l(\mathbf{r},\,\mathbf{r}',\,\mathbf{r}'-\xi)[\psi(x'),\,\alpha^m e(x',\,x'-\xi)\psi(x'-\xi)]+\tilde{f}^m(\mathbf{r}',\,\mathbf{r},\,\mathbf{r}-\xi)[\psi(x),\,\alpha^l e(x,\,x-\xi)\psi(x-\xi)]$$

$$-i\int d^3r''\tilde{f}^l(\mathbf{r},\,\mathbf{r}''+\xi,\,\mathbf{r}'')\tilde{f}^m(\mathbf{r}',\,\mathbf{r}''+\xi,\,\mathbf{r}'')\Big\{\psi(x''+\xi),\,e(x''+\xi,\,x'')\Big[\boldsymbol{\alpha}\cdot\Big(\frac{1}{i}\boldsymbol{\nabla}''-eq\mathbf{A}(x'')\Big)+m\beta\Big]\psi(x'')\Big\}\,, \quad (16)$$

where we have freely used $f^\mu(y,x,x')=-f^\mu(y,x',x)$.

It is now straightforward to calculate the dependence of j^k on A for a c-number vector potential, where the singular terms all result from the vacuum expectation value of the current. The explicit dependence is calculated in Appendix B and is given by Eqs. (B4).

The next problem is to find the solutions to these equations with a c number A, then to take the limit as $\rho\to\delta(\mathbf{r}-\mathbf{r}')$. The limit is not well defined, as is discussed in the next section, and there does not seem to be any way to define it except by requiring that the resultant theory be Lorentz-covariant. The problem of a quantized A presents all the difficulties of a relativistic quantum field theory as well, so we will first treat the external A problem.

IV. GREEN'S FUNCTIONS WITH AN EXTERNAL A

In this section, we define a gauge-invariant Green's function and discuss the $\rho(\xi)\to\delta(\xi)$ limit of the equation. The resultant Green's function produces a conserved current which contains all the requisite A dependence for a consistent theory. Our gauge-invariant Green's function is not equivalent to that of Mandelstam[7] in which a path-dependent gauge was chosen. The gauge factor, which is explicitly written and in general contains nontrivial dependence on $F^{\mu\nu}$, is, in fact, the WKB approximation to the Green's function. We define the gauge-dependent Green's function

$$G(x,x';A)\beta=\frac{i\langle 0|T(\psi(x)\psi(x'))|0\rangle^A}{\langle 0|0\rangle^A}. \quad (17)$$

The gauge-independent function is defined by

$$\bar{G}(x,x';F)=e(x',x)G(x,x';A). \quad (18)$$

Since $e(x,x')$ is an explicitly known function, we can solve directly for $\bar{G}$ instead of G.[8] First, however, the field equation for ψ, Eqs. (14), must be rewritten as an equation for G:

$$\gamma^0\Big(\frac{1}{i}\partial_0-eqA_0(x)\Big)G(x,x';A)+\frac{1}{2}\int d^3\xi\,\rho(\xi)$$

$$\times\Big\{\Big[\boldsymbol{\gamma}\cdot\Big(\frac{1}{i}\boldsymbol{\nabla}-eq\mathbf{A}(x)\Big)+m\Big]e(x,\,x-\xi)+e(x,\,x-\xi)$$

$$\times\Big[\boldsymbol{\gamma}\cdot\Big(\frac{1}{i}\boldsymbol{\nabla}-eq\mathbf{A}(x-\xi)\Big)+m\Big]\Big\}G(x-\xi,\,x';A)$$

$$=\delta(x-x'). \quad (19)$$

In the zero-field limit, the Fourier-transform equation is, for $A=0$,

$$[\gamma^0p_0+\bar{\rho}(p)(\boldsymbol{\gamma}\cdot\mathbf{p}+m)]G_\rho{}^0(p)=1\,, \quad (20)$$

where

$$0\leqslant\bar{\rho}(p)=\int d^3\xi\,e^{-i\mathbf{p}\cdot\xi}\rho(\xi)\,,\quad \bar{\rho}(0)=1\,,\quad \bar{\rho}\underset{p\to\infty}{\longrightarrow}0.$$

Then

$$G_\rho{}^0(p)=\frac{1}{\gamma^0p_0+\bar{\rho}(p)(\boldsymbol{\gamma}\cdot\mathbf{p}+m)}=\frac{\bar{\rho}(p)(m-\boldsymbol{\gamma}\cdot\mathbf{p})-\gamma^0p_0}{\omega^2-p^{02}}\,,$$

where $\omega(p)=\bar{\rho}(p)(p^2+m^2)^{1/2}$ is the modified energy of an electron with momentum $\mathbf{p}$. The configuration-space

[7] S. Mandelstam, Ann. Phys. **19**, 1 (1962).

[8] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964), used a similar technique in solving the problem of an electron in a plane wave. There the exponential factor comes out of the solution in a completely natural way. Their second Appendix is particularly close to the approach adopted in the present paper.

function is then

$$G_\rho^0(\xi)=\int \frac{d^4p}{(2\pi)^4}G_\rho^0(p)e^{ip\xi}$$
$$=i\int \frac{d^3p}{2(p^2+m^2)^{1/2}}[m-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma^0(m^2+p^2)^{1/2}\epsilon(\xi^0)]$$
$$\times\frac{\exp\{i[\mathbf{p}\cdot\boldsymbol{\xi}-\omega(p)|\xi^0|]\}}{(2\pi)^3}. \quad (21)$$

The modified propagator is the same as the usual Lorentz-invariant propagator except in its time dependence. For $\xi^0=0$,

$$G_\rho^0(\xi,0)=S^F(\xi,0)=\left(m-\boldsymbol{\gamma}\cdot\frac{1}{i}\boldsymbol{\nabla}\right)\Delta^F(\xi,0)+\tfrac{1}{2}i\gamma^0\epsilon(\xi^0)\delta(\xi).$$

The equation for $G_\rho(x,x';A)$ can then be rewritten as an integral equation

$$G_\rho(x,x';A)$$
$$=G_\rho^0(x-x')+\int dy\, G_\rho^0(x-y)\Big\{eq\gamma^0A_0(y)G_\rho(y,x';A)$$
$$+\int d^3\xi\,\rho(\xi)\Big[[eq\boldsymbol{\gamma}\cdot(\mathbf{A}(y-\xi)+\tfrac{1}{2}\mathbf{F}(y,y-\xi))]$$
$$\times e(y,y-\xi)-(e(y,y-\xi)-1)$$
$$\times\left(\boldsymbol{\gamma}\cdot\frac{1}{i}\boldsymbol{\nabla}+m\right)\Big]G(y-\xi,x';A)\Big\}, \quad (22)$$

where

$$F^\mu(x,x-\xi)=\int_0^1 d\lambda\,\xi_\nu F^{\mu\nu}(x-\lambda\xi).$$

The equation for $\bar{G}$ is, using Eqs. (18) and (19),

$$\gamma^0\left(\frac{1}{i}\partial_0-eq\mathbf{a}_0(x,x')\right)\bar{G}_\rho(x,x';F)+\int d^3\xi\,\rho(\xi)$$
$$\times\exp[ieq\mathfrak{F}(x,x-\xi,x')]\Big[\boldsymbol{\gamma}\cdot\Big(\frac{1}{i}\boldsymbol{\nabla}-eq\mathbf{a}(x-\xi,x')$$
$$-\frac{eq}{2}\mathbf{F}(x,x-\xi)\Big)+m\Big]\bar{G}_\rho(x-\xi,x',F)=\delta(x-x'), \quad (23)$$

where

$$\mathbf{a}^\mu(x,x')=\int_0^1\lambda d\lambda(x-x')_\nu F^{\nu\mu}(x'+\lambda(x-x')),$$

and

$$\mathfrak{F}(x,x-\xi,x')=\int_0^1 d\lambda\int_0^\lambda d\sigma(x-x')_\mu\xi_\nu$$
$$\times F^{\mu\nu}(x'+\lambda(x-x')-\sigma\xi);$$

or, rewriting Eq. (23) as an integral equation:

$$\bar{G}_\rho(x,x';F)$$
$$=G_\rho^0(x-x')+\int dyG_\rho^0(x-y)\Big\{eq\gamma^0a_0(y,x')\bar{G}_\rho(y,x';F)$$
$$+\int d^3\xi\,\rho(\xi)\Big[\exp[ieq\mathfrak{F}(y,y-\xi,x')]$$
$$eq\boldsymbol{\gamma}\cdot(\mathbf{a}(y-\xi,x')+\tfrac{1}{2}\mathbf{F}(y,y-\xi))$$
$$-(\exp[+ieq\mathfrak{F}(y,y-\xi,x')]-1)$$
$$\times\left(\boldsymbol{\gamma}\cdot\frac{1}{i}\boldsymbol{\nabla}+m\right)\Big]\bar{G}_\rho(y,-\xi,x';F)\Big\}. \quad (24)$$

Equations (22) and (24) can be solved perturbatively, yielding the nonlocal theories. The gauge-invariant propagator can then be used to calculate the current $\langle j^\mu\rangle^A$, using Eq. (15). The result is, naturally, not covariant; furthermore, the limit is not well defined. The resultant integrals contain factors of the form $\rho(\xi)(\xi)^n$ and there does not seem to be a generally consistent way to define such products in the limit $\rho\to\delta(\xi)$ so that the limit is Lorentz covariant.

Any arbitrary order of $\bar{G}$ exists and is Lorentz covariant as $\rho(\xi)\to\delta(\xi)$, for $x=x'$, assuming that A is sufficiently regular. However, when the currents $\langle j^\mu\rangle$ are calculated, and the result expressed in terms of a momentum space integral, integrals of derivatives of $\bar{\rho}(k)$ appear. These may then be integrated by parts so that the derivative appears on the other terms in the integrand; for example,

$$\int d^3k\frac{\partial}{\partial k}\bar{\rho}(k)k^m=?.$$

If the limit $\bar{\rho}\to 1$ is taken as it stands the result is zero, and integrating by parts yields a divergent result. If the integrals are evaluated with all derivatives off the $\bar{\rho}$'s, then $\langle j^\mu\rangle$ is not covariant. No general procedure was found which gave a covariant result. This should be expected, since the explicit dependence linear in A is given in Appendix B. Covariance only allows terms without derivatives.

Thus, we will take the limit of Eq. (24) to define the Lorentz covariant theory. The equation is already gauge invariant; we will define the Lorentz covariant limit by

$$\bar{G}(x,x';F)=G^0(x-x')+eq\int dy\,G^0(x-y)$$
$$\times\gamma^\mu a_\mu(y,x')\bar{G}(y,x';F), \quad (25)$$

and $G(x,x';A)$ is still given by Eq. (18). The solution to Eq. (25) defines the Lorentz covariant, gauge-invariant propagators. We can now take as an ansatz

$$\langle j^\mu(x)\rangle^A=\tfrac{1}{2}ie\,\mathrm{tr}q\gamma^\mu\bar{G}(x,x;F), \quad (26)$$

which is the naive limit of Eqs. (15). In fact, the resulting expression yields a Lorentz covariant, gauge-invariant, conserved current, so that all our criteria are satisfied.

It is also covenient to define $\bar{G}(x,x';F,z)$ which obeys

$$\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu - eqa_\mu(x,z)\right)+m\right]\bar{G}(x,x';F,z)=\delta(x-x'). \quad (27)$$

$\bar{G}(x,x';F,z)$ is just the Green's function for the vector potential $A_\mu(x)=a_\mu(x,z)$. The condition on A that the Green's function $G(x,x';A)$ exist is just sufficient to insure that the Green's function $\bar{G}(x,x';F,z)$ exists. Then a modified Fredholm theory applies and, as is shown in Appendix C,

$$\begin{aligned}\bar{G}(x,x';F,z) &= G^0(x-x')+\int dyK(x,y;F,z)G^0(y-x') \\ &\quad +\int dy\{K^2(x,y;F,z) \\ &\quad +[\det''(1-K^2(F,z))]^{-1}N''(x,y;F,z)\} \\ &\quad \times\left\{G^0(y-x')+\int dy'K(y,y';F,z)G^0(y'-x')\right\}, \quad (28)\end{aligned}$$

where $K(y,y';F,z)=eqG^0(y-y')\gamma^\mu a_\mu(y',z)$, and N'' and det$''$ are defined in Appendix C as the modified numerator and denominator functions when $\mathrm{tr}K^{3\dagger}K^3<\infty$.

The results of Appendix C may be summarized as follows:

(a) If $G(x,x';A)$ depends on $A_\mu(k)=\int dx\, e^{-ikx}A_\mu(x)$ through

$$\int\frac{dk}{(2\pi)^4}e^{ikx'}A_\mu(k)\Pi^\mu(k,x,x'),$$

then $\bar{G}(x,x',F)$ depends on $A_\mu(k)$ through

$$\int\frac{dk}{(2\pi)^4}e^{ikx'}[k_\nu A_\mu(k)-k_\mu A_\nu(k)]\frac{\partial}{\partial k_\nu}\int_0^1 d\lambda\,\Pi^\mu(\lambda k,x,x').$$

(b) All closed-loop diagrams with more than 4 vertices are already current-conserving and their A dependence may be calculated using either $\bar{G}$ or G. The closed loops with 2 or 4 vertices are not current conserving when calculated using G; $\bar{G}$ must be used and the corresponding dependence of $\langle j^\mu\rangle$ on A is

$$\begin{aligned}\langle j^\mu(x)\rangle^A = &-\int\frac{dk}{(2\pi)^4}e^{ikx}\left[\Pi^{\mu\nu}(k)-\frac{\partial}{\partial k_\nu}\int_0^1 d\lambda\,\Pi^{\mu\sigma}(\lambda k)k_\sigma\right]A_\nu(k)-\int\frac{dk_1dk_2dk_3}{(2\pi)^{12}}e^{i(k_1+k_2+k_3)x}[-iF_{\nu_1\mu_2}(k_1)] \\ &\times[-iF_{\nu_2\mu_2}(k_2)][-iF_{\nu_3\mu_3}(k_3)]\frac{\partial}{\partial k_{1\nu_1}}\frac{\partial}{\partial k_{2\nu_2}}\frac{\partial}{\partial k_{3\nu_3}}\int_0^1 d\lambda_1 d\lambda_2 d\lambda_3\,\Pi^{\mu\mu_1\mu_2\mu_3}(-\lambda_1k_1-\lambda_2k_2-\lambda_3k_3,\lambda_1k_1,\lambda_2k_2,\lambda_3k_3), \quad (29)\end{aligned}$$

where $\Pi^{\mu\nu}$ and $\Pi^{\mu\mu_1\mu_2\mu_3}$ are the closed-loop contributions calculated with G. The effect of the term linear in A is to remove the photon mass term and to enforce current conservation on the remainder. The resultant expression vanishes as $k\to 0$. The fourth-order loop is already current conserving with the exception of a finite constant times $g^{\mu\mu_1}g^{\mu_2\mu_3}+g^{\mu\mu_2}g^{\mu_1\mu_3}+g^{\mu\mu_3}g^{\mu_1\mu_2}$. This term is removed and the remainder vanishes when any one of the momenta vanishes.[9]

We can now calculate the explicit dependence of j^μ on A. The variation of $\langle j^\mu\rangle$ with respect to A is given by

$$\frac{\delta\langle j^\mu(x)\rangle^A}{\delta A_\nu(x')}=i\langle T(j^\mu(x)j^\nu(x'))\rangle^A+\left\langle\frac{\delta j^\mu(x)}{\delta A_\nu(x')}\right\rangle-i\langle j_\mu(x)\rangle^A\langle j^\nu(x)\rangle^A,$$

or

$$\left\langle\frac{\delta j^\mu(x)}{\delta A_\nu(x')}\right\rangle=\frac{\delta\langle j^\mu(x)\rangle^A}{\delta A_\nu(x')}-i\langle T(j^\mu(x)j^\nu(x'))\rangle^A+i\langle j^\mu(x)\rangle^A\langle j^\nu(x')\rangle^A. \quad (30)$$

We have that, for $A=0$,

$$\begin{aligned}\frac{\delta\langle j^\mu(x)\rangle}{\delta A_\nu(x')} &= -\int\frac{dk}{(2\pi)^4}e^{ik(x-x')}(k^2g^{\mu\nu}-k^\mu k^\nu)\frac{\alpha}{3\pi} \\ &\quad\times\int_{4m^2}^\infty\frac{ds}{s+k^2-i\epsilon}\left[1-\frac{4m^2}{s}\right]^{1/2}\left(1+\frac{2m^2}{s}\right) \\ &= i\langle T(j^\mu(x)j^\nu(x')^n\rangle+\left\langle\frac{\delta j^\mu(x)}{\delta A_\nu(x')}\right\rangle.\end{aligned}$$

Then, since the second term is local in time,

$$\langle 0|j^\mu(x)j^\nu(x')|0\rangle=\frac{\alpha}{3\pi}\int_{4m^2}^\infty ds\left[1-\frac{4m^2}{s}\right]^{1/2}\left(1+\frac{2m^2}{s}\right)\times(sg^{\mu\nu}-\partial^\mu\partial^\nu)\int\frac{d^3k}{2k^0}\frac{e^{ik(x-x')}}{(2\pi)^3},$$

[9] R. Karplus and M. Neuman, Phys. Rev. 80, 380 (1950).

where $k^0=[\mathbf{k}^2+s]^{1/2}$. But then

$$i\langle 0|T(j^\mu(x)j^\nu(x')^n|0\rangle$$
$$=-\frac{\alpha}{3\pi}\int\frac{dk}{(2\pi)^4}e^{ik(x-x')}(k^2g^{\mu\nu}-k^\mu k^\nu)\int_{4m^2}^\infty\frac{ds}{s+k^2-i\epsilon}$$
$$\times\left[1-\frac{4m^2}{s}\right]\left(1+\frac{2m^2}{s}\right)+(g^{\mu\nu}+\delta_0{}^\mu\delta_0{}^\nu)$$
$$\times\delta(x-x')\frac{\alpha}{3\pi}\int_{4m^2}^\infty ds\left[1-\frac{4m^2}{s}\right]^{1/2}\left(1+\frac{2m^2}{s}\right),$$

and

$$\left\langle\frac{\delta j^\mu(x)}{\delta A_\nu(x')}\right\rangle^A=-(g^{\mu\nu}+\delta_0{}^\mu\delta_0{}^\nu)\frac{\alpha}{3\pi}\int_{4m^2}^\infty ds\left(1-\frac{4m^2}{s}\right)^{1/2}$$
$$\times\left(1+\frac{2m^2}{s}\right)\delta(x-x'). \quad (31)$$

This term corresponds to the first term of Eq. (B4). There are no terms corresponding to the other two terms. The fact that the derivative term vanishes is immediately apparent from the form of Eq. (1), while the vanishing of the A^mA_m term requires a detailed study of the difference between the time-ordered product and the four-current propagator. Such an analysis has been attempted and does not seem to yield any terms. However, a completely rigorous treatment was not possible because any contributions would come from the situation where all four currents are coincident. On the other hand, we would expect that the dependence, if it were there, would be of the form of Eq. (B4), since it provides just the subtraction needed for current conservation. Then, if we consider $\langle\delta j^\mu(x)/\delta A^\nu(x')\rangle^A$ for constant A, we should find dependence. No such dependence is found; thus, any $(A)^3$ dependence of j must contain derivatives, in contradiction to Eq. (B4).

V. THE QUANTIZED ELECTROMAGNETIC FIELD

The remaining problem is to quantize the electromagnetic field. We have found a set of propagators of the charged fields, for an arbitrary classical electromagnetic field which satisfy all Lorentz-covariance, gauge-covariance and current-conservation properties.

The charged-particle propagators

$$G(x,x';A)=e(x,x')\bar{G}(x,x';F)$$

are given by essentially the same expressions as in the standard theory, except that gauge invariance is assured. These can now be used to calculate the quantum-theory propagator. Using a Feynman path integral to define the theory, we find formally, in Appendix D,

$$i^n\langle 0|T(\psi(x_1)\psi(x_2)\cdots\psi(x_n)\bar\psi(x_n')\cdots\bar\psi(x_1')^n|0\rangle^J$$
$$=e^{iW[J]}S(x_1,\cdots,x_n;x_n',\cdots,x_1';J)$$
$$=\sum_{\text{perm}}(-)\prod_i G\left(x_i,x_i';\frac{1}{i}\frac{\delta}{\delta J}\right)e^{iW[J]},$$

where $e^{iW[J]}$ is the generating functional for all the photon-field propagators.

A gauge-invariant function $\bar{S}$ which is analogous $\bar{G}$ to can also be defined.

$$e^{iW[J]}\bar{S}(x_1\cdots x_n,x_n'\cdots x_i';J)$$
$$=e^{iW[J]}\sum_{\text{perm}}(-)\prod_i\bar{G}\left(x_i,x_i';\frac{1}{i}\frac{\delta}{\delta J}+\frac{\delta W}{\delta J}\right),$$

where we have written $\bar{G}$ as a function of A rather than F and the gauge dependence of S can then be shown by expressing S as a function of $\bar{S}$

$$S(x,x';J)=e^{i(W[J+f(x_1x')]-W[J]}\bar{S}(x,x';J+f(x,x'))$$
$$f^\mu(x,x')=eqf^\mu(y,x,x').$$

The gauge dependence of the charged-field propagators is not at all simple in the Coulomb gauge. This may be understood by considering the effect of the function f^μ which gives the classical current of a point particle appearing at the point x' traveling in a straight line to the point x and disappearing. The Coulomb-gauge vector potential generated by this nonconserved current produces a longitudinal electric field with which the particle interacts as it propagates. The covariant vector potential $A^\mu(x)=\int dy'D(y-y')f^\mu(y',x,x')$, on the other hand, gives no contribution to $\bar{G}$. In any Lorentz gauge, the additional term in A coming from $\partial^\mu\partial^\nu$ terms in the photon propagator does not contribute to $F^{\mu\nu}$ or, therefore, to $\bar{G}$.

In the case of the multiple-particle propagators, the f's give a first approximation to scattering of the particles. Each particle feels the electromagnetic field of the straight line motion of the other charges and $\bar{G}$ responds to that field and the exponential terms give the WKB approximation to the motion. Then the general propagator is

$$S(x_1\cdots x_n,x_n'\cdots x_1';J)=\sum_{\text{perm}}(-)\prod_i G\left(x_1,x_i';\frac{\delta W[J]}{\delta J}+\frac{1}{i}\frac{\delta}{\delta J}\right)$$
$$=\sum_{\text{perm}}(-)e^{i(W[J+f]-W[J])}\prod_i\bar{G}\left(x_i,x_i';\frac{\delta W[J+f]}{\delta J}+\frac{1}{i}\frac{\delta}{\delta J}\right),$$
$$f^\mu(y)=e\sum_i q_if^\mu(x,x_i,x_i').$$

The current can also be calculated, in the absence of any charged particles:

$$\langle 0|j^\mu(x)|0\rangle^J=\tfrac{1}{2}ie\,\mathrm{tr}\gamma^\mu q\bar{G}\left(x,x;\frac{1}{i}\frac{\delta}{\delta J}\right)e^{iW[J]}=e^{iW[J]}\tfrac{1}{2}ir\,\mathrm{tr}\gamma^\mu q\bar{G}\left(x,x;\frac{\delta W}{\delta J}+\frac{1}{i}\frac{\delta}{\delta J}\right)=e^{iW[J]}\tfrac{1}{2}ie\,\mathrm{tr}\gamma^\mu q\bar{S}(x,x;J)_q$$

or

$$(-\partial^2\delta_\lambda{}^\mu+\partial^\mu\partial_\lambda)\frac{\delta W}{\delta J_\lambda(x)}=J_c{}^\mu(x)+\tfrac{1}{2}ie\,\mathrm{tr}\gamma^\mu q\bar{S}(x,x;J)\,,$$

where J_c is a conserved current whose expression in terms of J is gauge-dependent.[10] There is, of course, no equation for the gauge part of $\delta W/\delta J$.

The equations for S may be developed in the standard way and any calculational technique which is used in quantum electrodynamics may be used here. The only difference appears in the calculation of the various photon propagators. There, we are directed to use $\bar{S}$ instead S. But, we can now invoke the algorithm which we used in Sec. IV, and our results are: Do perturbation theory in the usual way except when considering closed-loop diagrams. In these, replace the usual contributions by the current-conserving part given by Eq. (C9).

Given $\langle 0|j^\mu(x)|0\rangle^J$, the $[F^{0k},j^l]$ and $[j^0,j^l]$ commutators can, of course, be calculated. Definitive results require a detailed analysis of the structure of the various propagators analogous to, but more difficult than, that required for the external vector potential. Heuristic arguments however, lead to the same result as in the c-number-A theory; the commutator is a c number. Again the anticipated structure is of the form $A^mA_mA^k$; and if we assume a constant external A and calculate $\delta j^k/\delta A_l$, the imposition of the low-energy theorem on the four-photon propagator assures its vanishing.

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APPENDIX A

We have

$$\partial_y f^\mu(y,x,x')=\delta(y-x')-\delta(y-x)\,. \qquad \text{(A1)}$$

The Fourier transform of f is

$$g^\mu(k,x,x')=\int dy\, e^{-ik(y-x')}f^\mu(y,x,x')\,;$$

translational invariance requires that $g^\mu=g^\mu(k,\,x-x')$, and, if the further requirement that g^μ be a vector under Lorentz transformations is imposed,

$$g^\mu(k,\,x-x')=(x-x')^\mu g_1+k^\mu g_2\,, \qquad \text{(A2)}$$

where g_1 and g_2 are scalar functions of k and $x-x'$. Eq. (A1) then transforms into

$$ik_\mu g^\mu=1-e^{-ik(x-x')}=ik(x-x')g_1+ik^2g_2$$

or

$$g^\mu(k,\xi)=\xi^\mu\frac{e^{-ik\xi}-1}{-ik\xi}+(\xi^\mu k^2-k^\mu k\xi)\rho\,.$$

For $\xi_0=0$, we must have $g^0=0$, or

$$g^0|_{\xi^0=0}=-k^0\mathbf{k}\cdot\boldsymbol{\xi}\,,.$$

Such a term gives dependence on $F^{\mu\nu}$ and j^μ, so we reject it, leaving

$$f^\mu(y,x,x')=(x-x')^\mu\int\frac{dk}{(2\pi)^4}e^{ik(y-x')}\frac{e^{-ik(x-x')}-1}{-ik(x-x')}$$
$$=(x-x')^\mu\int_0^1 d\lambda\,\delta(y-x'-\lambda(x-x'))\,. \qquad \text{(A3)}$$

If we have $(x-x')^\mu$ time-like, then ef^μ is just the classical current density of a point particle moving from x to x' along the straight line with velocity

$$\frac{\mathbf{x}-\mathbf{x}'}{|x^0-x'^0|}=\mathbf{v}\,,$$

$$j^0(y)=e\xi(x^0-x'^0)\delta(\mathbf{y}-\mathbf{x}'-(y^0-x'^0)\mathbf{v})\,,$$
$$j^k(y)=e\mathbf{v}\delta(\mathbf{y}-\mathbf{x}'-(y^0-x'^0)\mathbf{v})\,. \qquad \text{(A4)}$$

APPENDIX B

The space components of the current are given by Eq. (15). For small distances the asymptotic behavior of the fermion propagator is

$$S(\xi)\sim\frac{2}{\pi^2}\frac{\gamma\cdot\xi+\gamma^0[\xi_0+i\epsilon\xi^0(\xi^0)]}{[\xi^2-(|\xi^0|-i\epsilon)^2]}\,, \qquad \text{(B1)}$$

for $\xi^0=0$, since $G_p{}^0(\xi,0)=S(\xi,0)$,

$$\langle[\psi(\mathbf{r}),\,O\psi(\mathbf{r}')]\rangle=\frac{i}{\pi^2}\frac{\mathrm{tr}O\boldsymbol{\alpha}\cdot\boldsymbol{\xi}}{[\xi^2+\epsilon^2]^2}\,,\quad \boldsymbol{\xi}=\mathbf{r}-\mathbf{r}'$$

[10] B. Zumino, J. Math. Phys. **1**, 1 (1960); D. Boulware, *ibid.* **3**, 50 (1960).

and

$$\left\langle\left[\bar{\psi}(\mathbf{r}), O\mathbf{r}\cdot\frac{1}{i}\nabla'\psi(\mathbf{r}')\right]\right\rangle=\frac{1}{\pi^2}\frac{\mathrm{tr}O}{[\xi^2+\epsilon^2]^2}\left\{1-\frac{4\epsilon^2}{\xi^2+\epsilon^2}\right\};\tag{B2}$$

then$\langle j^k(x)\rangle$ has the explicit dependence

$$\tfrac{1}{4}e\int d^3\xi\,\rho(\xi)\bigg\{ie\int dy\,A_l(y)\Big[f^l(y,x,x-\xi)\langle[\bar\psi(x),\alpha^k q^2\psi(x-\xi)]\rangle+f^k(x,y+\xi,y)\langle[\bar\psi(y+\xi),\alpha^l q^2\psi(y)]\rangle$$

$$-i\int dx'f^k(x,x'+\xi,x')f^l(y,x'+\xi,x')\left\langle\left[\bar\psi(x'+\xi),q^2\alpha\cdot\frac{1}{i}\nabla'\psi(x')\right]\right\rangle\Big]+\frac{(ie)^3}{3!}\int dydy'dy''A_l(y)A_m(y')A_n(y'')$$

$$\times\Big[\langle[\bar\psi(x),q^4\alpha^k\psi(x-\xi)]\rangle f^l(y,x,x-\xi)f^m(y',x,x-\xi)f^n(y'',x,x-\xi)+3\langle[\bar\psi(y),q^4\alpha^l\psi(y-\xi)]\rangle f^k(x,y,y-\xi)$$

$$\times f^m(y',y,y-\xi)f^n(y'',y,y-\xi)-i\int dx'f^k(x,x'+\xi,x')f^l(y,x'+\xi,x')f^m(y',x'+\xi,x')$$

$$\times f^n(y'',x'+\xi,x')\left\langle\left[\bar\psi(x'+\xi),q^4\alpha\cdot\frac{1}{i}\nabla'\psi(x')\right]\right\rangle\Big]\bigg\},\tag{B3}$$

plus higher order terms. However, the A^5 term has a factor $\xi^5/\xi^3\to 0$ as $\rho(\xi)\to\delta(\xi)$; hence these are the only nonvanishing terms in the local limit. Then, $\langle 0|j^k(x)|0\rangle$ has the explicit dependence, for $\rho\to\delta(\xi)$,

$$-e^2\frac{2}{3\pi^2}\int d^3\xi\frac{\rho(\xi)\xi^2}{[\xi^2+\epsilon^2]^2}\left\{\left(1+\frac{4\epsilon^2}{\xi^2+\epsilon^2}\right)A^k(x)+\frac{\xi^2}{5}\frac{1}{12}\left(3+\frac{4\epsilon^2}{\xi^2+\epsilon^2}\right)[\nabla^2A^k(x)+2\nabla^k\nabla\cdot\mathbf{A}(x)]\right\}$$

$$+\frac{e^4 2}{5\pi^2}\int d^3\xi\frac{\rho(\xi)\xi^4}{[\xi^2+\epsilon^2]^2}\left(3+\frac{4\epsilon^2}{\pi^2}\right)A^k(x)\mathbf{A}(x)\cdot\mathbf{A}(x).\tag{B4}$$

APPENDIX C

We must consider the equation

$$\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu-a_\mu(x)\right)+m\right]G(x,x';a)=\delta(x-x')\tag{C1}$$

in the two cases

$$a_1{}^\mu(x)=A^\mu(x),\tag{C2}$$

and

$$a_2{}^\mu(x)=a^\mu(x,z)=\int_0^1\lambda d\lambda\,(x-z)_\nu F^{\nu\mu}(z+\lambda(x-z)),$$

where

$$F^{\mu\nu}(x)=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x).$$

Both equations can be written in the form

$$G(x,x';a)=G^0(x-x')+\int dy\,K(x,y;a)G(y,x';a),\tag{C3}$$

where

$$K(y,y';a)=eqG^0(y-y')\gamma^\mu a_\mu(y').$$

In either case a modified Fredholm theory applies, since under a similarity transformation, $\mathrm{tr}K^{2\dagger}K^2<\infty$ for any A which has a finite space-time integral of the energy density.[11] Then, iterating once,

$$G(x,x';a)=G^0(x-x')+(KG^0)(x,x';a)+(K^2G)(x,x';a),\tag{C4}$$

[11] See J. Schwinger, Phys. Rev. 93, 615 (1954); S. Weinberg, *ibid.* 135, B202 (1964).

and we must consider

$$\left(\frac{1}{1-K^2}\right)(x,x')=\delta(x-x')+K^2(x,x';a)+[\det''(1-K^2(a))]^{-1}N''(x,x';a)\,, \tag{C5}$$

where

$$\det''(1-K^2(a))=e^{\mathrm{tr}K^2+\frac{1}{2}\mathrm{tr}K^4}\det(1-K^2)$$

and

$$N''(x,x';a)=-\frac{\delta\det''(1-K^2(a))}{\delta K^2(x',x;a)}\,.$$

The function $\det''(1-K^2(a))$ is the standard Fredholm determinant with all 1×1 and 2×2 subdeterminants omitted. Thus, it may be expressed in terms of $\mathrm{tr}K^{2n}$ for $n>2$:

$$\mathrm{tr}K^{2n}=e^{2n}\int dy_1\cdots dy_{2n}a_{\mu1}(y_1)\cdots a_{\mu 2n}(y_{2n})\mathrm{tr}G^0(y_{2n}-y_1)\gamma^{\mu_1}qG^0(y_1-y_2)\gamma^{\mu_2}q\cdots q\gamma^{\mu_{2n-1}}G^0(y_{2n-1}-y_{2n})\gamma^{\mu_{2n}}q\,.$$

For K_1,

$$a_\mu(y)=\int\frac{dk}{(2\pi)^4}e^{iky}A_\mu(k)$$

and

$$\mathrm{tr}K_1^{2n}=2ie^{2n}\int\frac{dk_1\cdots dk_{2n}}{(2\pi)^{4(2n-1)}}A_{\mu1}(k_1)\cdots A_{\mu_{2n}}(k_{2n})\delta(\sum k_i)\frac{1}{i}\int\frac{dp}{(2\pi)^4}\,\mathrm{tr}G^0(p)\gamma_l{}^{\mu_1}G^0(p_1)\cdots\gamma^{\mu_{2n-1}}G^0(p_{2n-1})\gamma^{\mu_{2n}}, \tag{C6}$$

$$p_j=p-\sum_{l=1}^{j}k_l\,.$$

Then, for K_2,

$$a_\mu(y)=\int_0^1\lambda d\lambda\int\frac{dk}{(2\pi)^4}e^{ik(z+\lambda(y-z))}(y-z)^\nu i(k_\nu A_\mu(k)-k_\mu A_\nu(k))=\int\frac{dk}{(2\pi)^4}e^{ikz}(-iF_{\nu\mu}(k))\frac{\partial}{\partial k_\nu}\int_0^1 d\lambda\, e^{i\lambda k(y-z)}\,,$$

and

$$\mathrm{tr}K_2{}^{2n}=2ie^{2n}\int\frac{dk_1\cdots dk_{2n}}{(2\pi)^{4(2n-1)}}(\exp(i\sum_{j=1}^{2n}k_jz))(-iF_{\nu_1\mu_1}(k_1))\cdots(-iF_{\nu_{2n}\mu_{2n}}(k_{2n}))\frac{\partial}{\partial k_{1\nu}}\cdots\frac{\partial}{\partial k_{2n\nu_{2n}}}$$

$$\times\int_0^1 d\lambda_1\cdots d\lambda_{2n}\frac{1}{i}\int\frac{dp}{(2\pi)^4}\delta(\sum\lambda_jk_j)\mathrm{tr}G^0(p)\gamma^{\mu_1}G^0(p_1)\cdots\gamma^{\mu_{2n-1}}G^0(p_{2n-1})\gamma^{\mu_{2n}}\,, \tag{C7}$$

$$p_j=p-\sum_{l=1}^{j}\lambda_lk_l\,.$$

The integral which appears in both cases is

$$\delta(\sum k_j)\Pi^{\mu_1\cdots\mu_{2n}}(k_1\cdots k_{2n})\equiv\frac{e^{2n}}{(2n)!}\delta(\sum k_j)\sum_{\mathrm{perm}}\frac{1}{i}\int\frac{dp}{(2\pi)^4}\,\mathrm{tr}G^0(p)\gamma^{\mu_1}G^0(p_1)\cdots\gamma^{\mu_{2n-1}}G^0(p_{2n-1})\gamma^{\mu_{2n}}$$

$$p_j=p-\sum_{l=1}^{j}k_l\,,$$

which, for $n>2$, has the property

$$k_{j\mu_j}\delta(\sum_{l=1}^{2n}k_l)\Pi^{\mu_1\cdots\mu_{2n}}(k_1\cdots k_{2n})\equiv0 \tag{C8}$$

from current conservation and the convergence of the integrals.

Now

$$-iF_{\nu\mu}(k)\frac{\partial}{\partial k_\nu}\int_0^1 d\lambda\, f^\mu(\lambda k)=\left[A_\mu(k)k_\nu\frac{\partial}{\partial k_\nu}-A_\nu(k)k_\mu\frac{\partial}{\partial k_\nu}\right]\int_0^1 d\lambda f^\mu(\lambda k)$$

$$=A_\mu(k)\left\{\int_0^1 d\lambda\left[\lambda\frac{d}{d\lambda}f^\mu(\lambda k)+f^\mu(\lambda k)\right]-\frac{\partial}{\partial k_\mu}\int_0^1 d\lambda k_\nu f^\nu(\lambda k)\right\}$$

$$=A_\mu(k)\left\{f^\mu(k)-\frac{\partial}{\partial k_\mu}\int_0^1 d\lambda\, k_\nu f^\nu(\lambda k)\right\}. \tag{C9}$$

If $k_\mu f^\mu(k)=0$, we then have, assuming $|f^\mu(0)|<\infty$,

$$-iF_{\nu\mu}(k)\frac{\partial}{\partial k_\nu}\int_0^1 d\lambda f^\mu(\lambda k)\equiv A_\mu(k)f^\mu(k).$$

Thus, for $n>2$,

$$\mathrm{tr}K_1^{2n}=\mathrm{tr}K_2^{2n} \tag{C10}$$

and $\det''(1-K_2^2)=\det''(1-K_1^2)$, both being independent of z.

The same arguments cannot be made for $\mathrm{tr}K_2^2$ and $\mathrm{tr}K_2^4$ since each term contains nonconserved parts, but Eq. (C9) still defines the correct contribution.

The numerator function contains terms of the form $(K^nG^0)(x,x')$ as well.

The preceding results are based essentially on the gauge invariance of the denominator function. The numerator function is not gauge invariant, so the terms will not be equal. We are now in a position to calculate $j^\mu(x)$ and $\langle 0|0\rangle^A$. We require that the variation of the vacuum matrix element yield the current

$$\frac{1}{i}\frac{\delta\langle 0|0\rangle^A}{\delta A_\mu(x)}=\langle 0|j^\mu(x)|0\rangle^A=\langle 0|0\rangle^A\frac{ie}{2}\,\mathrm{tr}\gamma^\mu q\bar{G}(x,x;F) \tag{C11}$$

or

$$\frac{1}{i}\frac{\delta\ln\langle 0|0\rangle^A}{\delta A_\mu(x)}=\frac{ie}{2}\,\mathrm{tr}\gamma^\mu q\bar{G}(x,x;F,x). \tag{C12}$$

We have already calculated all the $\mathrm{tr}K^{2n}$ terms in $\bar{G}$, the remaining terms are of the form

$$(K_2{}^nG^0)(x,x';F,z)=e^nq^n\int dy_1\cdots dy_nG^0(x-y_1)\gamma^{\mu_1}a_{2\mu_1}(y_1)\cdots\gamma^{\mu_n}a_{2\mu_n}(y_n)G^0(y_n-x'), \tag{C13}$$

or

$$(K_2{}^nG^0)(x,x;F,x)=e^nq^n\int\frac{dk_1\cdots dk_n}{(2\pi)^{4n}}e^{i\Sigma k_ix}(-iF_{\nu_1\mu_1}(k_1))\cdots(-iF_{\nu_n\mu_n}(k_n))$$
$$\times\frac{\partial}{\partial k_{1\nu_1}}\cdots\frac{\partial}{\partial k_{n\nu_n}}\int_0^1 d\lambda_1\cdots d\lambda_nG^0(p)\gamma^{\mu_1}\cdots\gamma^{\mu_n}G^0(p_n), \tag{C14}$$

$$p_j=p-\sum_{l=1}^{j}\lambda_lk_l.$$

Then, the terms in the current are of the form

$$-\int\frac{dk_1\cdots dk_n}{(2\pi)^{4n}}e^{i\Sigma k_jx}(-iF_{\nu_1\mu_1}(k_1))\cdots(-iF_{\nu_n\mu_n}(k_n))\frac{\partial}{\partial k_{1\nu_1}}\cdots\frac{\partial}{\partial k_{n\nu_n}}\int_0^1 d\lambda_1\cdots d\lambda_n\Pi^{\mu\mu_1\cdots\mu_n}(-\sum_{l=1}^{n}\lambda_lk_l,\lambda_1k_1,\cdots,\lambda_nk_n)$$
$$=-\frac{i}{2(n+1)}\frac{\delta}{\delta A_\mu(x)}\,\mathrm{tr}K^{n+1}G^0, \tag{C15}$$

as may be seen by comparing the result with Eq. (C7). For $n=3$ or 1, however, there are additional terms. In the case of $n=3$, these are of the form

$$c(g^{\mu\mu_1}g^{\mu_2\mu_3}+g^{\mu\mu_2}g^{\mu_1\mu_3}+g^{\mu\mu_3}g^{\mu_1\mu_2}).$$

The k derivatives just remove them, justifying the simple dropping of the term. In the case $n=1$,

$$\Pi^{\mu\nu}(-k,k)=e^2\frac{1}{i}\int\frac{dp}{(2\pi)^4}\operatorname{tr}\gamma^\mu\frac{1}{m+\gamma(p-k)}\gamma^\nu\frac{1}{m+\gamma p}=\frac{\alpha}{4\pi}\int_0^1 d\lambda\{-4g^{\mu\nu}[\Lambda^2-(m^2+\lambda(1-\lambda)k^2)] +\text{conserved current terms}\}$$

where Λ^2 represents the quadratic photon mass-divergence.

The insertion of this expression into Eq. (C9) yields

$$\frac{\alpha}{3\pi}\tfrac{1}{3}(g^{\mu\nu}k^2-k^\mu k^\nu)+\text{conserved current terms}.$$

The photon-mass term has been eliminated, there is an additional finite charge renormalization, and the standard vacuum polarization remains. It should be noted that the photon-mass term does not involve any A dependence of j in the final analysis while the vacuum polarization does. Similarly, in the four-point function the cg terms do not yield any A dependence of j. These results are in contrast with the nonrelativistic case where the elimination of the terms implies A dependence.

Combining Eqs. (C5), (C10), and (C15),

$$-[\det''(1-K_2{}^2(F,x))]^{-1}\frac{e}{2}\int\operatorname{tr}\gamma^\mu qN_2''(x,y;F,x)\left[G^0(y-x)+\int dy'K_2(y,y';F,x)G^0(y'-x)\right]dy$$

$$=-[\det''(1-K_1{}^2(A))]^{-1}\frac{e}{2}\int dydy'\operatorname{tr}\gamma^\mu qN_1''(x,y;A)K_1(y,y';A)G^0(y'-x)$$

$$=\frac{1}{4}\frac{\delta}{\delta A_\mu(x)}\ln\det''[1-K_1{}^2(A)],\tag{C16}$$

or

$$\frac{\delta\ln}{\delta A_\mu(x)}\langle 0|0\rangle^A=\frac{1}{2}\frac{\delta}{\delta A_\mu(x)}\{\ln[\det''(1-K^2(A))]^{1/2}-\tfrac{1}{4}\text{“}trK^4\text{”}-\tfrac{1}{2}\text{“tr}K^2\text{”}\},\tag{C17}$$

where “trK^{2n}” means to take the current conserving part and “trK^2” is, of course, logarithmically divergent.

APPENDIX D

In the case of an external vector potential, we have been forced to give up a strict Hamiltonian formulation of the theory, and thus lost even this heuristic formulation for the fully quantized theory. An alternative formulation exists, however. In the usual theory, the knowledge of the fermion propagators for all possible external potentials can, subject to serious existence and definition questions, be used as a formal basis for the fully interacting theory by means of the Feynman path integrals.[12] This Appendix is essentially a sketch of such a definition, using our gauge covariant and invariant propagators as a basis for the development. We have propagators in the presence of an external vector potential

$$G(x,x';A)=e(x,x')\bar{G}(x,x';F).$$

Then, the quantum propagators, in the presence of a source current for the electromagnetic field, are given by

$$\langle 0|0\rangle^J S(x_1\cdots x_n,\,x_n'\cdots x_1';J)=\frac{\int d[A]\langle 0|0\rangle^A\sum_{\text{perm}}(-)\prod_{i=1}^n G(x_i,x_i';A)\exp\{i\int dx[L^0(x,A)+J_\mu A^\mu(x)]\}}{\int d[A]\langle 0|0\rangle^A\exp\{i\int dx[L^0(x,A)]\}}.\tag{D1}$$

The quantity $\langle 0|0\rangle^A$ does not exist as it stands, due to the logarithmic divergence in $\langle(Tjj)\rangle$. But counter terms can be inserted in L^0 to give a convergent integrand. We assume that further counter terms yield a convergent $\langle 0|0\rangle^J$, the generating functional for the photon propagators. This question goes beyond the scope of the present work and involves the question of an existent theory, so we shall not pursue it further.

[12] A list of references may be found in I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. 1, 48 (1960).

A set of gauge invariant propagators can be defined

$$\langle 0|0\rangle^J \bar{S}(x_1\cdots x_n,\, x_n'\cdots x_1';\, J)=\frac{\int d[A]\langle 0|0\rangle^A \sum_{\text{perm}}(-)\prod_{i=1}^{n}\bar{G}(x_i,x_i';F)\exp\{i\int dx[L^0+JA]\}}{\int d[A]\langle 0|0^A \exp\{i\int dx L^0(x,A)\}}. \tag{D2}$$

The factors $G(\;;A)\exp(i\int JA)$ or $\bar{G}(\;;A)\exp(i\int JA)$ occur in the expressions for S and $\bar{S}$. Since these may formally be rewritten as $G(\;;(1/i)\delta/\delta J)\exp(i\int JA)$, and $\bar{G}(\;;(1/i)\delta/\delta J)\exp(i\int JA)$, Eqs. (D2) can be rewritten as functional derivative expressions of

$$\langle 0|0\rangle^J=\frac{\int d[A]\langle 0|0\rangle^A \exp\{i\int dx[L^0(x,A)+J_\mu A^\mu(x)]\}}{\int d[A]\langle 0|0\rangle^A \exp\{i\int dx\, L^0(x,A)\}}=e^{iW[J]}, \tag{D3}$$

which we assume exists and has the proper form. Then W can be written in the usual linked cluster expansion:

$$W[J]=\frac{1}{2}\int dydy'J_\mu(y)D^{\mu\nu}(y-y')J_\nu(y')+\sum_{n=2}^{\infty}\frac{1}{2n}\int dy_1\cdots dy_{2n}dz_1\cdots dz_{2n}$$
$$\times[\prod_{i=1}^{2n} J_{\nu_1}(y_i)D^{\nu_i\mu_i}(y_i-z_i)]T^{\mu_1\cdots\mu_{2n}}(z_1\cdots z_{2n}), \tag{D4}$$

with

$$\partial_{\mu_i}T^{\mu_1\cdots{}_{2n}}(z_1\cdots z_{2n})=0,$$

and T is completely symmetric. The equation for $\delta W/\delta J$ is given in the text. $D^{\mu\nu}$ is the fully interacting propagator and it may be taken to contain gauge parts which are not given by the path integral. These would be inserted by a gauge transformation.[12] Then

$$S(x_1\cdots x_n,\, x_n'\cdots x_1';\, J)=e^{-iW[J]}\sum_{\text{perm}}(-)\prod_{i=1}^{n}\left(Gx_i,x_i',\frac{1}{i}\frac{\delta}{\delta J}\right)e^{iW[J]}=\sum_{\text{perm}}(-)\prod_{i}G\left(x_i,x_i';\frac{\delta W}{\delta J}[J]+\frac{1}{i}\frac{\delta}{\delta J}\right), \tag{D5}$$

and

$$\bar{S}(x_1\cdots x_n,\, x_n'\cdots x_1';\, J)=\sum_{\text{perm}}(-)\prod_{i}\bar{G}\left(x_i,x_i';\frac{\delta W}{\delta J}+\frac{1}{i}\frac{\delta}{\delta J}\right). \tag{D6}$$

We also have

$$S(x_1\cdots x_n,\, x_n'\cdots x_1';\, J)=\sum_{\text{perm}}(-)\prod_{i=1}^{n}e^{i(W[J+f]-W[J])}\bar{G}\left(x_i,x_i';-\frac{1}{i}\frac{\delta}{\delta J}W[J+f]+\frac{1}{i}\frac{\delta}{\delta J}\right), \tag{D7}$$

where

$$f^\mu=\sum_i eqif^\mu(y,x_i,x_i'), \tag{D8}$$

which, in the single particle case, yields

$$\bar{S}(x,x';J)=e^{i(W[J-f]-W[J])}S(x,x';J-f),\quad f^\mu=eqf^\mu(y,x,x'). \tag{D9}$$

The exponential term is in general not well defined except in the limit as $x\to x'$. All the gauge dependence, however, does cancel, as it must by the derivation.

We have formally derived expressions for S, $\bar{S}$ and the various photon propagators. If these are expanded in e, the usual perturbation series results, except that the closed-loop diagrams must be modified as before. Here, however, because of the additional divergent integrals, all closed-loop diagrams with ≤ 4 photon external lines must be modified.