THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. **151**, NO. 4 — 25 NOVEMBER 1966

Operator for Time Delay Induced by Scattering*

B. A. LIPPMANN
DRC Incorporated, Santa Barbara, California
(Received 6 July 1966)

The operator that gives the time delay induced by a scattering process is exhibited explicitly.

IT is well known that the time delay induced by a scattering process is given by the energy derivative of the phase shift.[1] Some time ago,[2] we observed that this theorem can be put in the "suggestive" form $S^{-1}\tau S$, where τ is interpreted as an "operator," equivalent, in an energy representation, to differentiation with respect to the energy. It is the purpose of this paper to make this relationship precise by exhibiting the time-delay operator explicitly.

That such an operator might exist is suggested by the action and angle variables of classical theory. For, if the action variable J_1 is identified with the energy, the conjugate angle variable ϕ_1 has the equation of motion

$$\dot{\phi}_1=\frac{\partial H}{\partial J_1}=1\,, \tag{1}$$

which integrates to

$$\phi_1(t)=t+\phi_1(0)\,. \tag{2}$$

One would expect that the first equation would have as its quantum-mechanical analog the conventional commutation relation that holds for conjugate variables, while the second equation, transcribed to quantum mechanics, would relate one of the quantum-mechanical operators to the time parameter. It is shown below that these expectations can be realized.

We first introduce the operator for the magnitude of the particle momentum

$$p=(p_x^2+p_y^2+p_z^2)^{1/2} \tag{3}$$

and the operator representing the projection of the radial coordinate upon the momentum

$$r_p=(1/p)(\mathbf{p}\cdot\mathbf{r})\,. \tag{4}$$

These are analogous to the operators introduced by Dirac[3] except that the roles of position and momentum have been exchanged.

As in Ref. 3, we easily find that these operators satisfy the commutation relation

$$[r_p,p]=i \tag{5}$$

and that the operator r_p+i/p, which is Hermitian, is canonically conjugate to p.

The time-delay operator τ is defined by

$$\tau=\frac{m}{2p}\left(r_p+\frac{i}{p}\right)+\left(r_p+\frac{i}{p}\right)\frac{m}{2p}\,. \tag{6}$$

For the free particle Hamiltonian, $H_0=p^2/2\mathrm{m}$, we find

$$[\tau,H_0]=i\,. \tag{7}$$

Although τ, as constructed in Eq. (4) above is formally Hermitian, some care must be taken in developing its properties, because of the fact that H_0 has a spectrum that is limited to positive values $(0,\infty)$.[4] However, this circumstance does not impair the interpretation, which follows directly from Eq. (7), of τ as an energy derivative in a representation in which H_0 is diagonal: $\tau\rightarrow i\partial/\partial E$.

* Work supported by Advanced Research Projects Agency.

[1] This appears to have first been shown quantum-mechanically by E. P. Wigner, Phys. Rev. **98**, 145 (1955), although relationships of this sort are well known in filter theory, where the frequency derivative of the phase characteristic gives the time required for a pulse of energy to pass through the filter [*Radiation Laboratory Series* (McGraw-Hill Book Company, Inc., New York, 1948), Vol. 8, p. 155]. A similar relation describes the spatial displacement of a wave packet upon reflection by a plane interface whose reflection coefficient is a function of the wave number parallel to the interface, L. M. Brekhovskikh, *Waves in Layered Media* (Academic Press Inc., New York, 1960), p. 105.

[2] See F. T. Smith, Phys. Rev. **118**, 349 (1960), Eq. (45).

[3] P. A. M. Dirac, *The Principles of Quanum Mechanics* (Oxford University Press, New York, 1958), p. 152.

[4] For example, the operator τ has no eigenfunctions; see W. Pauli, *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. 5/1, p. 63.

Calling the operator defined in Eq. (6) $\tau(0)$, we can also introduce the time-dependent operator

$$\tau(t)=e^{iH_0t}\tau(0)e^{-H_0t} \quad (8)$$
$$=t+\tau(0).$$

The relations given in Eqs. (7) and (8) above are the quantum mechanical analogs of the classical Eqs. (1) and (2).

The application to the calculation of time delays is made by considering the matrix element

$$(\Psi_a(t),\tau(0)\Psi_a(t)), \quad (9)$$

where the state vector $\Psi_a(t)$ is the *scattered* wave packet, evaluated for times after the scattering process has been completed. The time dependence of this state vector is then given by the free-particle Hamiltonian alone: $\Psi_a(t)=e^{-iH_0t}\Psi_a(0)$; here, the last factor represents the scattered-state vector extrapolated back to zero time.

In the following, both the incident and scattered wave packets are assumed to be normalized to unity. Since the scattered state is connected with the initial state vector $\phi_a(0)$ by the S matrix, (9) can be evaluated as below:

$$(\Psi_a(0),e^{iH_0t}\tau(0)e^{-iH_0t}\Psi_a(0))$$
$$=(\Psi_a(0),\{t+\tau(0)\}\Psi_a(0))$$
$$=t+(\phi_a(0),S^{-1}\tau(0)S\phi_a(0)). \quad (10)$$

If the S matrix has the form $S=e^{2i\delta(E)}$, in an energy representation the time delay operator becomes equivalent to energy differentiation, and we find, as a final result

$$(\Psi_a(t),\tau(0)\Psi_a(t))=t-2\frac{\partial\delta(E)}{\partial E}+(\phi_a(0),\tau(0)\phi_a(0)). \quad (11)$$

In this form the interpretation of the energy derivative of the phase shift as the time delay induced by the scattering process is apparent.

I am indebted to Professor Massey of University College, London, for his generous hospitality extended over part of the period during which these ideas were clarified.

PHYSICAL REVIEW VOLUME 151, NUMBER 4 25 NOVEMBER 1966

Gauge Invariance and Current Definition in Quantum Electrodynamics*

DAVID G. BOULWARE
University of Washington, Seattle, Washington
(Received 6 June 1966)

In order to study the problems of gauge invariance, Lorentz covariance, and the operator properties of the "Schwinger terms" in the current commutators, spin-$\frac{1}{2}$ quantum electrodynamics is written as the limit of a nonlocal theory. The conditions on such a theory are discussed and the nonlocal equations in the case of an external vector potential derived. The gauge-invariant, Lorentz-covariant limit of these equations is then discussed, and it is found that (in the case of spin $\frac{1}{2}$) the "Schwinger terms" are purely c-number. The quantized vector potential is considered by means of a Feynman path integral and its gauge structure determined. It is found that an automatically gauge-covariant theory results and that the c-number character of the Schwinger terms apparently persists.

I. INTRODUCTION

IN quantum electrodynamics, if the canonical commutation relations of its constituent fermion fields are used to calculate the commutator $[j^0(\mathbf{r}), j^k(\mathbf{r}')]$, it vanishes identically. This is in direct contradiction to the general theorem[1]

$$\langle 0|[j^l(\mathbf{r}), j^k(\mathbf{r}')]|0\rangle=-i\nabla^k\delta(\mathbf{r}-\mathbf{r}')c,$$

where c is non-negative and vanishes only if the vacuum is an eigenstate of the current j^μ. Hence, its vanishing implies that the current is a constant c-number current and that the electromagnetic field is free. Schwinger, in the same paper,[1] gave a partial solution to the problem by pointing out that the current should be defined as a limit of separated points. This, then, would not yield a gauge-invariant current unless there were some explicit dependence on the vector potential to cancel the gauge transformations of the charged fields. The relation of the additional dependence to Lorentz covariance and current conservation has also been discussed by Johnson[2] and by Brown.[2]

This device, while resolving the paradox of the commutation relations, raises the question of the proper equations of motion for the fields. In general one would expect both the current definition and the field equations for the charged fields to be changed. Also, the Lorentz

* Supported in part by the U. S. Atomic Energy Commission under RLO-1388B.

[1] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[2] K. Johnson, Nucl. Phys. **25**, 431 (1961); L. S. Brown, Phys. Rev. (to be published). The relation between explicit field dependence and current commutators is also discussed in J. Schwinger, *ibid.* **130**, 406 (1963); and by D. Boulware and S. Deser, *ibid.* this issue, **151**, 1278 (1966).