rapidly than $\lambda(0,d)$ with the decreasing thickness. For their tin films Douglass and Blumberg find that for their 4330-Å- and 573-Å-thick films the value of $\lambda(0,d)$ is 645 Å and 1150 Å, respectively. This is in agreement with our estimates in Table I.

SUMMARY AND CONCLUSIONS

In the vicinity of the transition temperature $0 \leqslant T_c - T \leqslant 0.3$°K, the critical fields of In films ranging in thickness from 585 to 3540 Å, are found to vary as $(T_c-T)^{1/2}$ in agreement with the predictions of Ginzburg and Landau, of Bardeen, and of Rickayzen; the observed and predicted critical field magnitudes are in reasonable agreement, considering the experimental uncertainties. At lower temperatures, our data are in good agreement with Rickayzen's model; the temperature dependence predicted by Maki is not borne out by our experiments.

We have also found evidence for the increase in the penetration depth with decreasing film thickness, as the results in Table I show.

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Vortices in an Imperfect Bose Gas. IV. Translational Velocity*

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The translational velocity of a vortex with circulation h/m in an imperfect Bose gas is calculated by considering the time dependence of the condensate wave function corresponding to a given initial configuration of vortices. Each vortex in a system of rectilinear vortices is shown to move with the local fluid velocity at its core; a vortex ring of radius R is shown to move with velocity $(h/4\pi mR)\ln(8R/\alpha)$, where α is a length the order of the core size. Both results agree with the predictions of classical hydrodynamics.

I. INTRODUCTION

RECENT studies of superfluid vortices in liquid He II and in type-II superconductors have led to renewed interest in the nineteenth century problem of the motion of classical vortices in an incompressible inviscid fluid. A rigorous derivation based on the nonlinear equations of classical hydrodynamics shows that a region of concentrated vorticity, such as a vortex core, moves with the local fluid velocity at that point.[1] The dynamical behavior of a system of classical vortices may therefore be calculated if the velocity field is known at each point of the fluid. In the special case of rectilinear vortices, the self-induced velocity of each vortex vanishes by symmetry, and the motion arises solely from the velocity field of the other vortices in the system.[2] For more general configurations, such as a vortex ring in unbounded fluid,[3] the self-induced motion may be the dominant effect.

The corresponding dynamics of quantum vortices has not yet been developed. Nevertheless, it has been generally assumed that the translational velocities are just the classical values.[4–7] In support of this view, it has been shown[4,5] for certain configurations that the classical translational velocity is equal to group velocity v_g computed from the energy E and momentum P according to the usual prescription $v_g=\partial E/\partial P$. The quantities E and P are either taken from classical hydrodynamics[4] or calculated from simple quantum models, such as an imperfect Bose gas.[5–7] Although this identification of the translational velocity with the group velocity is plausible, it remains an indirect approach and seems incapable of treating the motion of individual vortices in a large group. For this reason, we have developed a dynamical theory of the motion of vortices in an imperfect Bose gas, which provides a simple model of liquid He II. The calculated translational velocity agrees almost exactly with the prediction of classical hydrodynamics: a rectilinear vortex moves with the velocity of the fluid at its core, and a vortex ring of radius R in unbounded fluid moves perpendicular to its plane with a velocity $u=(h/4\pi mR)\ln(8R/\alpha)$, where α is a cutoff approximately equal to the radius of the vortex core.

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1 A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1950), p. 132.

2 Reference 1, pp. 154–155.

3 Reference 1, pp. 165–167.

4 G. W. Rayfield and F. Reif, Phys. Rev. **136**, A1194 (1964).

5 A. L. Fetter, Phys. Rev. **138**, A429 (1965).

6 K. Huang and A. C. Olinto, Phys. Rev. **139**, A1441 (1965).

7 D. Amit and E. P. Gross, Phys. Rev. **145**, 130 (1966).

Section II contains a derivation of the translational velocity of a single rectilinear vortex in a uniform stream, which is the simplest case of a moving vortex. As examples of more complicated configurations, the translational velocity is calculated for a system of rectilinear vortices (Sec. III) and for a vortex ring of radius R (Sec. IV).

II. SINGLE VORTEX IN A UNIFORM STREAM

The condensate of an imperfect Bose gas may be described by a one-particle wave function ψ that obeys a nonlinear field equation[8–10]

$$i\hbar\partial\psi/\partial t = -(2m)^{-1}\hbar^2\nabla^2\psi - \mu\psi + V_0|\psi|^2\psi, \qquad (1)$$

where the short-range repulsive interparticle potential $v(\mathbf{r}-\mathbf{r}')$ has been approximated by a delta-function interaction

$$v(\mathbf{r}-\mathbf{r}') = V_0\delta(\mathbf{r}-\mathbf{r}'), \quad (V_0>0). \qquad (2)$$

Here μ is the chemical potential and m is the atomic mass. The velocity of the fluid may be calculated from the condensate wave function

$$\begin{aligned}\mathbf{v} &= (2mi|\psi|^2)^{-1}\hbar[\psi^*\nabla\psi - (\nabla\psi^*)\psi] \\ &= (\hbar/m)\nabla S,\end{aligned} \qquad (3)$$

where S is the phase of the wave function

$$\psi(\mathbf{r},t) = [n(\mathbf{r},t)]^{1/2}\exp[iS(\mathbf{r},t)]. \qquad (4)$$

In cylindrical coordinates, Eq. (1) has a time-independent solution of the form

$$\psi_0(\mathbf{r},t) = n_0^{1/2}e^{i\theta}f(r), \qquad (5)$$

where n_0 is a constant. Equation (5) represents a vortex with quantized circulation h/m situated at the origin. A detailed calculation based on Eqs. (1) and (5) shows that the radial function has the following limiting behavior[8,9]:

$$\begin{aligned} &f(r) \propto r/a, && (r\ll a) \\ &f(r) \sim 1-\tfrac{1}{2}(a/r)^2, && (r\gg a)\end{aligned} \qquad (6)$$

where $a[=\hbar(2mn_0V_0)^{-1/2}]$ is the deBroglie wavelength. The radius of the core is approximately equal to a, while the asymptotic form of $f(r)$ determines the chemical potential: $\mu=n_0V_0$. The flow pattern associated with ψ_0 is

$$\mathbf{v}_0(\mathbf{r}) = (\hbar/mr)\hat{\theta}. \qquad (7)$$

This velocity field is just that of a classical vortex, and the stream lines are concentric circles. Since Eq. (5) satisfies Eq. (1) for all time, a single rectilinear vortex remains stationary at the origin, which agrees with the classical result.

[8] E. P. Gross, Nuovo Cimento **20**, 454 (1961).
[9] L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **40**, 646 (1961) [English transl.: Soviet Phys.—JETP **13**, 451 (1961)].
[10] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N. Y.) **34**, 291 (1965).

Suppose, however, that we are given the following initial wave function at $t=0$:

$$\psi_{\mathbf{u}}(\mathbf{r},0) = \exp(i\mathbf{u}\cdot\mathbf{r}m/\hbar)\psi_0(\mathbf{r}). \qquad (8)$$

It is easy to see from Eq. (3) that the initial velocity pattern $\mathbf{v}(\mathbf{r})$ is that of a single vortex in a uniform stream of velocity $\mathbf{u}$,

$$\mathbf{v}(\mathbf{r}) = \mathbf{u} + \mathbf{v}_0(\mathbf{r}). \qquad (9)$$

The subsequent motion of the system may be calculated from the time-dependent wave function $\psi_{\mathbf{u}}(\mathbf{r},t)$ corresponding to the initial conditions Eq. (8). Since the self-consistent field equation (1) is of first order in the time derivative, the initial rate of change of $\psi_{\mathbf{u}}$ is obtained by direct substitution of Eq. (8) into Eq. (1),

$$\begin{aligned} i\hbar\partial\psi_{\mathbf{u}}(\mathbf{r},t)/\partial t|_{t=0} &= -[(\hbar^2/2m)\nabla^2+\mu]\psi_{\mathbf{u}}(\mathbf{r},0) \\ &\quad + V_0|\psi_{\mathbf{u}}(\mathbf{r},0)|^2\psi_{\mathbf{u}}(\mathbf{r},0) \\ &= \tfrac{1}{2}mu^2\psi_{\mathbf{u}}(\mathbf{r},0) - \exp(i\mathbf{u}\cdot\mathbf{r}m/\hbar) \\ &\quad \times i\hbar\mathbf{u}\cdot\nabla\psi_0(\mathbf{r}).\end{aligned} \qquad (10)$$

The second form of Eq. (10) has been simplified by noting that $\psi_0(\mathbf{r})$ is a time-independent solution of Eq. (1). Thus $\psi_{\mathbf{u}}$ changes in two distinct ways: The first and second terms on the right side of Eq. (10) represent, respectively, a change of phase of the wave function and a uniform translation of the vortex with velocity $\mathbf{u}$. The phase change is associated with the energy of streaming motion of the fluid at infinity. It can be verified by direct substitution that the general time-dependent solution is

$$\psi_{\mathbf{u}}(\mathbf{r},t) = \exp(-\tfrac{1}{2}imu^2t/\hbar)\exp(i\mathbf{u}\cdot\mathbf{r}m/\hbar)\psi_0(\mathbf{r}-\mathbf{u}t), \qquad (11)$$

which is equivalent to Eq. (10) in the limit of small time. Equation (11) represents a rigid translation of the vortex with velocity $\mathbf{u}$, which is just the velocity of the uniform stream at the vortex core. Hence the translational velocity of the vortex is equal to the fluid velocity at the position of the vortex.

It may be objected that this simple example merely proves the Galilean invariance of the theory, because Eq. (8) is equivalent to a transformation to moving coordinates. In the following sections, however, the same method is applied to more complicated vortex configurations, in which the fluid is stationary at infinity. The relative separation of the vortices changes with time in the general case, so that no single coordinate transformation can bring all the vortices simultaneously to rest.

III. SYSTEM OF RECTILINEAR VORTICES

A problem of interest in connection with rotating He II is a system of rectilinear vortices parallel to the z axis, situated at the points $\{\mathbf{r}_j\}=\{r_j,\theta_j\}$ in the x-y plane. Each vortex is assumed to be singly quantized, so that the circulation will be taken as $\pm h/m$; the freedom of sign allows us to treat a vortex pair as well as

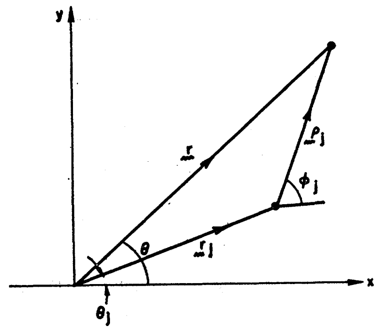

FIG. 1. Geometry of a rectilinear vortex situated at $\mathbf{r}_j$.

an array of identical vortices. It is convenient to introduce the following abbreviation (Fig. 1):

$$\boldsymbol{\rho}_j = \mathbf{r} - \mathbf{r}_j\,, \tag{12}$$

where $\boldsymbol{\rho}_j$ has components (ρ_j, φ_j). An approximate initial wave function for this system is a product of the wave functions given in Eq. (5), one for each of the separate vortices,

$$\psi(\mathbf{r},0) = n_0^{1/2} \prod_j g_j\,, \tag{13}$$

where

$$g_j = \exp(iS_j) f(\rho_j)\,, \tag{14}$$

and the product is over all the vortices. The phase of Eq. (14) is given by $S_j = \pm \varphi_j$, the sign being that of the circulation about the jth vortex. The validity of Eq. (13) has been considered previously,[5] where it is shown that the corrections are small if the distance between each vortex is large compared to the core size a.

The initial rate of change of the system may be found by substituting Eq. (13) into the right side of Eq. (1). A straightforward calculation shows that

$$\begin{aligned} i\hbar\partial\psi(\mathbf{r},t)/\partial t|_{t=0} &= n_0^{1/2}(\textstyle\prod_j g_j)\{\mu[\sum_k(1-f_k^2) + (\prod_k f_k^2) - 1] \\ &\quad - (\hbar^2/2m) \textstyle\sum_{kl}' (\nabla \ln g_k \cdot \nabla \ln g_l)\}\,, \end{aligned} \tag{15}$$

where the primed sum is over k and l separately, omitting the terms $k=l$. Consider the behavior of the different terms on the right side of Eq. (15). In the vicinity of a given vortex $j=1$ (say), the square bracket may be expanded as

$$\begin{aligned} \mu[\textstyle\sum_k (1-f_k^2) + (\prod_k f_k^2) - 1] &\approx \tfrac{1}{2}(\hbar^2/2m) \textstyle\sum_{kl}' a^2(\rho_k\rho_l)^{-2} \qquad (\rho_1 \gg a) \\ &\approx (\hbar^2/2m) \textstyle\sum_k' (r_{1k})^{-2}\,, \qquad (\rho_1 \ll a) \end{aligned} \tag{16}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ and the prime on the single sum means: omit the term $k=1$. The second term of Eq. (15) is given approximately as

$$\begin{aligned} (\hbar^2/2m) \textstyle\sum_{kl}' (\nabla \ln g_k \cdot \nabla \ln g_l) &\approx i\hbar \nabla \ln g_1 \cdot \{\textstyle\sum_k' [(\hbar/m)\nabla S_k|_{\mathbf{r}=\mathbf{r}_1} \\ &\quad + (\hbar/m) O(a^2/r_{1k}^3)]\} \\ &\quad + (\hbar^2/2m) \sum_{k(\neq 1)} \sum_{l(\neq 1)} O(1/r_{1k}r_{1l})\,. \end{aligned} \tag{17}$$

Inspection of the various terms shows that the dominant contribution to the initial rate of change of the wave function arises from the first term of Eq. (17); the corrections are small since the vortex cores are well separated.

It is now possible to prove that the time-dependent wave function corresponding to Eq. (15) may be written as

$$\psi(\mathbf{r},t) = n_0^{1/2} \textstyle\prod_j g(\mathbf{r} - \mathbf{r}_j - \mathbf{u}_j t)\,. \tag{18}$$

This solution represents a system of moving vortices, where $\mathbf{u}_j$ is the velocity of the jth vortex. The time derivative of Eq. (18) yields

$$\begin{aligned} i\hbar\partial\psi(\mathbf{r},t)/\partial t|_{t=0} &= i\hbar n_0^{1/2}(\textstyle\prod_j g_j) \\ &\quad \times \textstyle\sum_k \partial \ln g(\mathbf{r}-\mathbf{r}_k - \mathbf{u}_k t)/\partial t|_{t=0} \\ &= -i\hbar n_0^{1/2}(\textstyle\prod_j g_j) \sum_k \mathbf{u}_k \cdot \nabla \ln g_k\,. \end{aligned} \tag{19}$$

In the vicinity of the first vortex, the dominant term of Eq. (19) is

$$-i\hbar n_0^{1/2}(\textstyle\prod_j g_j) \mathbf{u}_1 \cdot \nabla \ln g_1\,, \tag{20}$$

and comparison with the first term of Eq. (17) gives the translational velocity of the first vortex:

$$\mathbf{u}_1 = \textstyle\sum_k' (\hbar/m) \nabla S_k|_{\mathbf{r}=\mathbf{r}_1}\,. \tag{21}$$

Since $(\hbar/m)\nabla S_k|_{\mathbf{r}=\mathbf{r}_1}$ is the fluid velocity at $\mathbf{r}_1$ due to the kth vortex at $\mathbf{r}_k$, Eq. (21) reproduces the classical result that the translational velocity of a given rectilinear vortex is equal to the total fluid velocity at its core arising from all the other vortices in this system.

The fluid velocity due to a system of rectilinear vortices vanishes at large distances like r^{-1}. Hence there is no kinetic energy associated with motion of the fluid at infinity, and the time-dependent phase factor that appears in Eq. (11) is absent in Eq. (18). This equation also provides an alternative derivation of the translational velocity $\mathbf{u}_i$ of the ith vortex. If $\ln\psi(\mathbf{r},t)$ is expanded in a Taylor series about the point $\mathbf{r}_i$ to leading order in (a/r_{ij}), the resulting expression takes precisely the form of Eq. (11) with $\mathbf{u}_i$ given by Eq. (21).

The sign of the circulation has not been specified, so that Eq. (21) may be applied to the motion of a vortex pair separated by a distance $2d$; the corresponding translational velocity is $u = (h/4\pi md)$, which agrees both with the classical expression[3] and with a previous quantum mechanical calculation based on the group velocity.[5] In principle, it should be possible to compute the quantum mechanical corrections to Eq. (21), but these are probably comparable with the errors introduced by the use of the product wave functions [Eq. (13)].

IV. VORTEX RING

The above method will now be applied to the motion of a vortex ring of radius R. A more approximate treatment is required than for rectilinear vortices, however, because Eq. (1) has never been proved to have exact solutions representing a vortex ring. We shall make some plausible assumptions concerning the initial wave function

$$\psi_R(\mathbf{r},0)=n_0^{1/2}f_R(\mathbf{r})\exp[iS_R(\mathbf{r})], \tag{22}$$

and calculate the time development from Eq. (1). The fluid velocity is related to the phase of the wave function, and it is a reasonable approximation to choose $S_R(\mathbf{r})$ to reproduce the classical velocity pattern $\mathbf{v}_R(\mathbf{r})$ of a vortex ring of radius R,[11]

$$(\hbar/m)\nabla S_R(\mathbf{r})=\mathbf{v}_R(\mathbf{r}). \tag{23}$$

The modulus function $f_R(\mathbf{r})$ in Eq. (22) will be taken as the radial function for a single rectilinear vortex, apart from small corrections due to the curvature of the vortex axis.

The geometry of the system is illustrated in Fig. 2, where the plane of the vortex ring is taken as the x-y plane. Throughout this section, three dimensional vectors will be resolved in cylindrical polar coordinates, so that $\mathbf{r}$ has components (ρ,θ,z). All quantities of physical interest are independent of the azimuthal angle θ, because of the symmetry of the ring. The core of the vortex ring lies on the circle $(R,\theta,0)$, and the sense of circulation is chosen so that the fluid at the center of the ring flows in the positive z direction. The velocity pattern of a classical vortex ring is most conveniently described in terms of the stream function $\Psi(\rho,z)$, from which the fluid velocity is computed with the equations

$$\mathbf{v}_R(\mathbf{r})=\hat{\rho}v_\rho(\mathbf{r})+\hat{z}v_z(\mathbf{r}), \tag{24}$$

$$v_\rho(\mathbf{r})=-\frac{1}{\rho}\frac{\partial\Psi}{\partial z},\quad v_z(\mathbf{r})=-\frac{1}{\rho}\frac{\partial\Psi}{\partial\rho}, \tag{25}$$

where $\hat{\rho}$ and $\hat{z}$ are unit vectors along the radial and axial directions. A detailed calculation[12] shows that

$$\Psi(\rho,z)=-(\hbar/m)(r_1+r_2)[K(\xi)-E(\xi)], \tag{26}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors in the plane of Fig. 2,

$$\mathbf{r}_1=\mathbf{r}-R\hat{\rho},\quad \mathbf{r}_2=\mathbf{r}+R\hat{\rho}, \tag{27}$$

and

$$\xi=(r_2-r_1)(r_2+r_1)^{-1}, \tag{28}$$

K and E are the complete elliptic integrals.[13]

The major difficulty in computing the translational velocity lies in separating the contributions from the nearby and distant portions of the ring. As an approximate method, we shall write

$$\begin{aligned}\psi_R(\mathbf{r},0)&=n_0^{1/2}\exp(iS_R+i\varphi)e^{-i\varphi}f_R(\mathbf{r})\\&=n_0^{1/2}\exp(iS_R+i\varphi)g_R(\mathbf{r}),\end{aligned} \tag{29}$$

[11] This approach has been developed by Amit and Gross, Ref. 7, in a calculation of the critical velocity associated with the creation of a vortex ring in a channel.

[12] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., p. 237.

[13] See, for example, H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (The Macmillan Company, New York, 1957), 3rd ed., pp. 170–173.

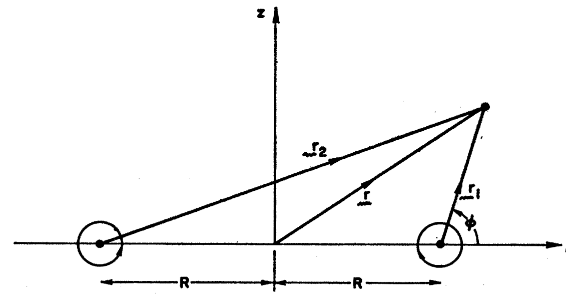

FIG. 2. Geometry of a vortex ring of radius R.

where φ is the angle between $\mathbf{r}_1$ and $\hat{\rho}$. The function $g_R(\mathbf{r})$ is essentially the wave function of a single rectilinear vortex bent into a ring of radius R. With Eqs. (29) and (1), the initial rate of change of the wave function ψ_R may be calculated to be

$$\begin{aligned}i\hbar\partial\psi_R(\mathbf{r},t)/\partial t|_{t=0}=n_0^{1/2}&\exp(iS_R+i\varphi)\\&\times\{-(\hbar^2/2m)\nabla^2g_R-\mu g_R\\&+V_0n_0|g_R|^2g_R-i(\hbar^2/m)\nabla\\&\times(S_R+\varphi)\cdot\nabla g_R-i(\hbar^2/2m)\\&\times[\nabla^2(S_R+\varphi)]g_R+(\hbar^2/2m)\\&\times|\nabla(S_R+\varphi)|^2g_R\}.\end{aligned} \tag{30}$$

The function g_R has not yet been specified completely, and it will now be chosen to satisfy the equation[14]

$$-(\hbar^2/2m)\nabla^2g_R-\mu g_R+V_0n_0|g_R|^2g_R=0. \tag{31}$$

This choice simplifies Eq. (30) considerably because the first three terms on the right side vanish identically:

$$\begin{aligned}i\hbar\partial\psi_R(\mathbf{r},t)/\partial t|_{t=0}=n_0^{1/2}&\exp(iS_R+i\varphi)\\&\times\{-i(\hbar^2/m)\nabla(S_R+\varphi)\cdot\nabla g_R\\&-i(\hbar^2/2m)[\nabla^2(S_R+\varphi)]g_R\\&+(\hbar^2/2m)|\nabla(S_R+\varphi)|^2g_R\}.\end{aligned} \tag{32}$$

As in Sec. III, the translational velocity of the ring will be calculated from the behavior of Eq. (32) near $\mathbf{r}=\hat{\rho}R$ ($\mathbf{r}_1$ small), so that terms which vanish as $\mathbf{r}_1\to 0$ will henceforth be neglected. A straightforward but tedious calculation based on Eqs. (23)–(28) yields the results

$$\begin{aligned}(\hbar/m)\nabla(S_R+\varphi)\approx(\hbar/4\pi mR)\{&[\ln(8R/r_1)-1]\\&\times(\hat{\rho}+\hat{z})+\cos\varphi\hat{\varphi}\},\end{aligned} \tag{33}$$

$$(\hbar/m)\nabla^2(S_R+\varphi)\approx-(\hbar/2\pi mR)(\sin\varphi/r_1), \tag{34}$$

where $\hat{\varphi}$ is the unit vector in the plane of Fig. 2 along the direction of increasing φ. Thus the gradient of $(S_R+\varphi)$ has only a weak logarithmic singularity as $\mathbf{r}_1\to 0$; the dominant $(r_1)^{-1}$ behavior associated with the vortex has been absorbed into the function g_R. Neither Eq. (33) nor (34) approaches a definite limit

[14] Equation (31) differs from that satisfied by Eq. (5) because of an additional term in the Laplacian. This difference leads to a correction of order $(\hbar/mR)$ in the translational velocity of the vortex ring. Since our result gives only the dominant term of order $(\hbar/mR)\ln(R/a)$, such an effect may be neglected.

as $\mathbf{r}_1 \to 0$ because of the angular dependence. It is therefore natural to replace each quantity by a spatial average over a small circle of radius $\alpha(=Ca)$, where C is a constant of order unity. With the definition

$$\langle f(\mathbf{r}_1)\rangle = (\pi\alpha^2)^{-1}\int_{r_1<\alpha} d^2r_1 f(\mathbf{r}_1)\,, \tag{35}$$

Eqs. (33) and (34) become

$$(\hbar/m)\langle\boldsymbol{\nabla}(S_R+\varphi)\rangle = \hat{z}(h/4\pi mR)\ln(8R/\alpha)\,, \tag{36}$$

$$(\hbar/m)\langle\nabla^2(S_R+\varphi)\rangle = 0\,. \tag{37}$$

In the limit $\mathbf{r}_1 \to 0$, Eq. (32) then reduces to

$$i\hbar\partial\psi_R(\mathbf{r},t)/\partial t|_{t=0} = n_0^{1/2}\exp(iS_R+i\varphi) \times\{-i\hbar(h/4\pi mR)\ln(8R/\alpha)\}\hat{z}\cdot\boldsymbol{\nabla}g_R\,, \tag{38}$$

since the last term on the right side of Eq. (32) vanishes at the vortex core. The corresponding time-dependent solution is

$$\psi_R(\mathbf{r},t) = n_0^{1/2}\exp[iS_R(\mathbf{r})+i\varphi]g_R(\mathbf{r}-\mathbf{u}t)\,, \tag{39}$$

where

$$\mathbf{u} = \hat{z}(h/4\pi mR)\ln(8R/\alpha) \tag{40}$$

is the translational velocity of the vortex ring. Equation (40) reproduces the dominant logarithmic behavior found in previous calculations of both the translational velocity of a classical vortex ring and the group velocity of a quantum-mechanical vortex ring.

V. DISCUSSION

This paper has demonstrated that it is possible to compute the translational velocity of vortex systems in an imperfect Bose gas from the dynamical field equation satisfied by the condensate wave function ψ. Given a particular initial wave function $\psi(\mathbf{r},0)$ representing some configuration of vortices, the dominant time dependence of $\psi(\mathbf{r},t)$ is that arising from the subsequent motion of the vortices. The velocity of each vortex is equal to that predicted by classical hydrodynamics. For a system of rectilinear vortices, the theory assumes a product wave function constructed from the quantum-mechanical wave function of each separate vortex; this is expected to be a good approximation as long as the vortex cores are well separated. The application to a large vortex ring is somewhat less satisfactory since the classical velocity field must be used in constructing the initial wave function. The corresponding translational velocity requires a cutoff, which is here chosen to be the order of the deBroglie wavelength a. This unphysical feature would be absent in a fully quantum-mechanical treatment. Such a calculation is very difficult, however, for it necessitates the integration of Eq. (1) subject to the boundary condition that $\psi_R(\mathbf{r}) \propto e^{i\varphi}r_1$ as $\mathbf{r}_1 \to 0$.

It is interesting to consider why the results of the present quantum-mechanical calculation are so similar to the classical predictions of the translational velocity of vortex systems. Gross[15] has shown how the theory of an imperfect Bose gas may be written in a hydrodynamic form. It differs from classical hydrodynamics only in the presence of an additional "quantum" pressure associated with rapid spatial variation of the condensate wave function ψ. If the quantum vortices are well separated, then $|\psi|$ is constant on the surface of a small cylinder (with a radius of several core radii) surrounding each vortex. It follows that the quantum pressure may be neglected in determining the motion of the fluid contained in the cylinder, so that classical hydrodynamics provides an adequate description of the dynamics of widely separated vortices in an imperfect Bose gas.

Similar questions arise in type-II superconductors, where the motion of the quantized flux lines has been studied intensively, both experimentally[16] and theoretically.[17] Unfortunately, a superconductor is an essentially more complicated physical system than an imperfect Bose gas. In particular, there appears to be no satisfactory time-dependent Ginzburg-Landau equation describing the motion of the condensed superelectrons, while Eq. (1) provides an exact description of the boson condensate in the limit of vanishing interactions. Thus there is no obvious extension of the present approach to a calculation of the motion of quantized flux lines.

ACKNOWLEDGMENT

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[16] See, for example, P. H. Borcherds, C. E. Gough, W. F. Vinen, and A. C. Warren, Phil. Mag. **10**, 349 (1964); Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **139**, A1163 (1965).
[17] See, for example, J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).