Theory of the Reaction $n+d \rightarrow n+n+p$

R. AARON*
Laboratory for Theoretical Studies, National Aeronautics and Space Administration, Goddard Space Flight Center, Greenbelt, Maryland

AND

R. D. AMADO†
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received 25 May 1966)

We calculate the proton energy spectrum at various angles for the process $n+d \rightarrow n+n+p$ at 14.4-MeV incident-neutron laboratory energy, using our exact three-body theory with separable, S-wave, spin-dependent, two-body forces. The major features of the experimental data are reproduced, but not the precise magnitudes. The integral equations we solve are quite singular and we find it necessary to deform the integration path. This involves the careful treatment of some branch points. Our results are not good enough to resolve the problem of the neutron-neutron scattering length, but by comparing Watson theory with our exact calculations we show that Watson theory is inadequate in this problem.

I. INTRODUCTION

We have calculated the cross section for the reaction $n+d \rightarrow n+n+p$ at 14.4 MeV using our exact three-body formalism with separable, S-wave, spin-dependent, nucleon-nucleon forces.[1] In such a formalism one sacrifices sophistication in the two-body force for careful treatment of three-body effects. As we should expect from our calculations of elastic neutron-deuteron scattering[1] this approach gives the major features of the breakup process, although agreement with experiment is not as good as in the elastic-scattering case.

In addition to its general interest as a three-body problem, the breakup reaction is interesting as a source of information on the neutron-neutron scattering length, the effects of which show up clearly as a strong final-state enhancement in this reaction. Attempts to use a form of Watson theory[2] to extract a scattering length from the data have yielded -21 ± 2 F[3] and $-23.6_{-1.6}^{+2.0}$ F[4] which is consistent with a neutron-proton singlet scattering length of -23.78 F.[5] Similar analysis of the data from the theoretically much cleaner reaction $\pi^-+d \rightarrow \gamma+2n$ gives -16.4 ± 1.3 F.[6] A microscopic theory of charge independence gives about -17 F for the neutron-neutron scattering length and excludes values near -24 F.[7] Our theory, although giving the qualitative features of the deuteron breakup data, is not good enough to fit it in detail and choose between the scattering lengths. However, analyzing our exact theoretical results by Watson theory shows that Watson theory would lead one astray in precisely the direction of too large a scattering length.

In Sec. II we outline the theory of the reaction in our formalism. In particular we express the breakup amplitude in terms of the elastic scattering amplitudes off the energy shell, which are in principle a by-product of our previous elastic scattering calculation. In fact, singularities in the kernel of the integral equation prove most troublesome in this case and we are forced to deform integration contours to avoid them. In Sec. III we discuss the analysis necessary for doing this for off-energy-shell amplitudes. The results are presented in Sec. IV and in Sec. V the problem of the neutron-neutron scattering length and of Watson theory is discussed. Some conclusions are presented in Sec. VI.

II. THEORY

In our theory of the three-nucleon system, the force between nucleons is taken to arise from a sum of separable interactions. Such a simplified interaction reduces the coordinate complexities of the three-body problem to manageable proportions. It is convenient to represent each correlated or interacting pair by a "particle." In this theory the breakup amplitude is given in terms of the off-the-energy-shell amplitude for $n+d \rightarrow N+$(correlated pair). These amplitudes are the solutions of the set of coupled linear integral equations we have previously discussed.[1] To obtain the breakup from these, one allows the correlated pair to propagate and then disassociate. This is done by appending a propagator and vertex to the off-energy-shell amplitudes, as indi-

* National Academy of Sciences—National Research Council Post-doctoral Resident Research Associate. Permanent address: Department of Physics, Northeastern University, Boston, Massachusetts.

† Supported in part by the National Science Foundation.

[1] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **140**, B1291 (1965). References to earlier work are contained here.

[2] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[3] M. Cerineo, K. Ilakovac, I. Šlaus, P. Tomaš, and V. Valković, Phys. Rev. **133**, B948 (1964).

[4] V. K. Voitovetskii, I. L. Korsunskii, and Y. F. Pazhin, Phys. Letters **10**, 109 (1964); V. K. Voitovetskii, I. L. Korunskii, and Y. F. Pazhin, Nucl. Phys. **64**, 513 (1965).

[5] M. J. Moravcski, *The Two-Nucleon Interaction* (Oxford University Press, New York, 1963).

[6] R. P. Haddock, R. M. Salter, Jr., M. Zeller, J. B. Czirr, and D. R. Nygren, Phys. Rev. Letters **14**, 318 (1965). A similar result in the reaction T(d,He[3])$2n$ has been reported by E. Baumgartner, H. E. Conzett, E. Shield, and R. J. Slobodrian, Phys. Rev. Letters **16**, 105 (1966).

[7] Cf. L. Heller, P. Signell, and N. R. Yoder, Phys. Rev. Letters **13**, 577 (1964).

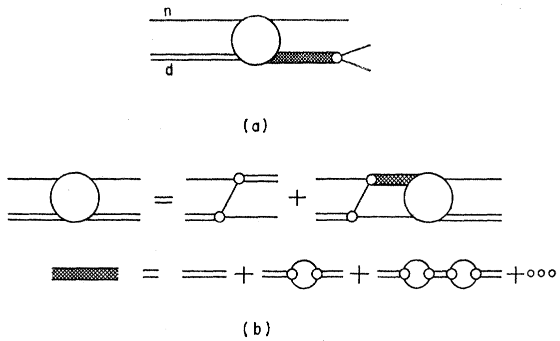

FIG. 1. (a) Schematic representation of a term in the breakup amplitude. The large circle is the (off-energy-shell) amplitude for neutron plus deuteron goes to nucleon plus correlated pair. The cross-hatched line is the full propagator for this pair, and the small circle the vertex for its disassociation into two nucleons. (b) Schematic representation of the integral equation for the off-energy-shell amplitude and for the full propagator. The intermediate correlated pairs in the integral equation are to be summed over.

cated in Fig. 1. This must be done in all possible ways, with due attention to spin and antisymmetry. That the term shown in Fig. 1 is indeed all there is in this theory, and that in particular there are no new equations to solve, may be seen by making a graphical expansion of the breakup process as we have in Fig. 2. Alternatively, we can derive this result starting with a Hamiltonian along lines very similar to those we used for the triton wave function.[8]

For the three-nucleon system we take two separable S-wave interactions, one in the spin-zero, isospin-one state, which we call φ, and one in the spin-one, isospin-zero state, which we call d. The equations for $n+d\to n+d$ and $n+d\to n+\varphi$ in the center-of-mass system are then ($\hbar=2m=1$)

$$(\mathbf{k},c|t_S(E)|\mathbf{k}',d)=\chi_{S,cd}(\mathbf{k},c|B(E)|\mathbf{k}',d)+\frac{1}{(2\pi)^3}\sum_b\chi_{S,cb}\int d^3p\times(\mathbf{k},c|B(E)|\mathbf{p},b)P_b(E,p^2)(\mathbf{p},d|t(E)|\mathbf{k}',d). \quad (1)$$

The momentum labels the nucleon momentum; b and c stand for d or φ; and E is the total energy. S is the total spin and may be $\frac{3}{2}$ or $\frac{1}{2}$. Since we always start with $n+d$, the total isotopic spin is $\frac{1}{2}$. The χ's are spin factors; $\chi_{3/2,dd}=-1$ and $\chi_{3/2,d\phi}=\chi_{3/2,\phi d}=\chi_{3/2\phi\phi}=0$; $\chi_{1/2,dd}=\chi_{1/2,\phi\phi}=\frac{1}{2}$; $\chi_{1/2,d\phi}=\chi_{1/2,\phi d}=-\frac{3}{2}$. They have been derived previously.[1] Using a Hulthén form $1/(k^2+\beta^2)$ for

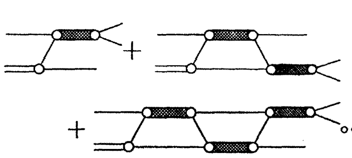

FIG. 2. Sum of graphs contributing to the breakup. Intermediate correlated pairs are to be summed over.

[8] R. D. Amado, Phys. Rev. **141**, 902 (1966).

the interaction, the Born terms are given by

$$(k,a|B(E)|k',b)=\gamma_a\gamma_b[((\mathbf{k}+\tfrac{1}{2}\mathbf{k}')^2+\beta_b^2)\times((\mathbf{k}'+\tfrac{1}{2}\mathbf{k})^2+\beta_a^2)(E-k^2-k'^2-(\mathbf{k}+\mathbf{k}')^2)]^{-1}. \quad (2)$$

γ_a is the coupling constant for the nucleons to the correlated pair a. The propagators are[9]

$$P_d(E,p^2)=-\left[(\sigma+\epsilon)\frac{\gamma_d^2}{(2\pi)^3}\times\int\frac{d^3n}{(n^2+\beta_d^2)^2(\epsilon+2n^2)^2(\sigma-2n^2)}\right]^{-1}, \quad (3)$$

$$P_\phi(E,p^2)=-\left[1+\frac{\gamma_\phi^2}{(2\pi)^3}\int\frac{d^3n}{(n^2+\beta_\phi^2)^2(\sigma-2n^2)}\right]^{-1},$$

$$\sigma=E-\tfrac{3}{2}p^2.$$

ϵ is the deuteron binding energy. The relation between the parameters β and γ and the low-energy nucleon-nucleon singlet scattering data is

$$\beta_\phi=(3/2r_s)[1+(1-16r_s/9a_s)^{1/2}]$$
$$\gamma_\phi^2=16\pi\beta_\phi^4a_s/(a_s\beta_\phi-2). \quad (4)$$

We take a singlet scattering length of $a_s=-23.78$ F and an effective range of $r_s=2.67$ F.[5] In the triplet channel we fit to a deuteron binding energy of -2.226 MeV and a scattering length of $a_t=5.411$ F according to

$$a_t=2(\beta_d+\alpha_d)^2/[\gamma_d\beta_d(\alpha_d+2\beta_d)],$$
$$\gamma_d=32\pi\alpha_d\beta_d(\alpha_d+\beta_d)^3, \quad (5)$$
$$\alpha_d^2=\tfrac{1}{2}\epsilon.$$

We can now express the breakup amplitude in terms of the solution of (1). Let us label the final momenta of the three nucleons $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ subject to $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$ and the final spins and isospins $m_1m_2m_3$; $i_1i_2i_3$. Then the amplitude in the total spin S ($m_S=\frac{1}{2}$) channel corresponding to the term in which 1 and 2 emerge from the correlated pair b can be written

$$(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;m_1,m_2,m_3;i_1,i_2,i_3|M_{S,b}(E)|\mathbf{k}',d)=A_{S,b}(1,2;3)\langle S\tfrac{1}{2}|S_bm_b,\tfrac{1}{2}m_3\rangle\langle S_bm_b|\tfrac{1}{2}m_1,\tfrac{1}{2}m_2\rangle\times\langle\tfrac{1}{2}-\tfrac{1}{2}|T_b\tau_b,\tfrac{1}{2}i_3\rangle\langle T_b\tau_b|\tfrac{1}{2}i_1,\tfrac{1}{2}i_2\rangle, \quad (6)$$

where $\langle JM|j_1m_1,j_2m_2\rangle$ is the usual Clebsch-Gordan coefficient, S_b and T_b are the spin and isospin of b, and m_b, and τ_b are its spin and isospin projections. We use an isospin convention in which the neutron has third component $-\frac{1}{2}$, so the total isospin is $\frac{1}{2}$ and the third component $-\frac{1}{2}$ of the three-body system. A is the am-

[9] In our previous work (Ref. 1), the wave-function renormalization constant of the deuteron Z appeared in the propagator as parameter. We set it equal to zero here since it has little effect at these relatively high energies. It has been claimed by Phillips [A. C. Phillips, Phys. Rev. **142**, 984 (1966)] that with $Z\neq0$, the theory does not satisfy two- or three-particle unitarity. In our opinion this claim is false.

plitude without spin factors and is given by

$$A_{S,b}(1,2;3)=\gamma_b/\{[\tfrac{1}{2}(\mathbf{k}_2-\mathbf{k}_1)]^2+\beta_b^2\}$$
$$\times P_b(E,k_3^2)(\mathbf{k}_3,b\,|\,T_S(E)\,|\,\mathbf{k}',d)\,. \quad (7)$$

A is symmetric in 1 and 2; the antisymmetry is provided by the Clebsch-Gordan coefficients. If we call (6) $M_{S,b}(12,3)$, the entire properly antisymmetric amplitude is

$$M_{S,b}=M_{S,b}(12,3)+M_{S,b}(31,2)+M_{S,b}(23,1) \quad (8)$$

and the cross section is proportional to $|M|^2$, where

$$|M|^2=\tfrac{1}{3}|M_{1/2}|^2+\tfrac{2}{3}|M_{3/2}|^2 \quad (9)$$

with

$$|M_S|^2=\sum_{\substack{a,b\\ \text{final spins}}} M_{S,a}M_{S,b}^*\,, \quad (10)$$

assuming we do not wish to study final-state polarizations. If we assume the proton is particle 3, we can do all the spin sums, etc., by the standard methods of Racah algebra. We get for the quartet contribution

$$|M_{3/2}|^2=\tfrac{1}{2}|A_{3/2,d}(3,1;2)|^2+\tfrac{1}{2}|A_{3/2,d}(2,3;1)|^2$$
$$-\mathrm{Re}[A_{3/2,d}(3,1;2)A_{3/2,d}^*(2,3;1)] \quad (11)$$

and for the doublet

$$|M_{1/2}|^2=\tfrac{2}{3}|A_{1/2,\phi}(1,2;3)|^2+\tfrac{1}{2}|(A_{1/2,d}(3,1;2)|^2$$
$$+\tfrac{1}{2}|A_{1/2,d}(2,3;1)|^2+\tfrac{1}{6}|(A_{1/2,\phi}(3,1;2)|^2$$
$$+\tfrac{1}{6}|A_{1/2,\phi}(2,3;1)|^2+\mathrm{Re}[\tfrac{1}{3}A_{1/2,\phi}(1,2;3)$$
$$\times A_{1/2,\phi}^*(3,1;2)+\tfrac{1}{3}A_{1/2,\phi}(1,2;3)A_{1/2,\phi}^*(2,3;1)$$
$$-A_{1/2,\phi}(1,2;3)A_{1/2,d}^*(3,1;2)-A_{1/2,\phi}(1,2;3)$$
$$\times A_{1/2,d}(2,3;1)+\tfrac{1}{2}A_{1/2,d}(3,1;2)A_{1/2,d}^*(2,3;1)$$
$$-\tfrac{1}{2}A_{1/2,d}(3,1;2)A_{1/2,\phi}^*(2,3;1)-\tfrac{1}{2}A_{1/2,\phi}(3,1;2)$$
$$\times A_{1/2,d}^*(2,3;1)-\tfrac{1}{6}A_{1/2,\phi}(3,1;2)$$
$$\times A_{1/2,\phi}^*(2,3;1)]\,. \quad (12)$$

If one wants only the proton spectrum at a fixed angle, one must then integrate over the neutron momenta. This considerably simplifies (11) and (12), using $A(1,2;3)=A(2,1;3)$.

Thus far we have outlined the theory, assuming isotopic spin invariance. Since we wish to study the possibility that the singlet neutron-neutron scattering length is about -17 F while the singlet neutron-proton one is -23.78 F, we should calculate without isotopic spin invariance. To do this would require a third correlated-pair interaction and would be quite complex. We have checked that changing the singlet scattering length from -24 to -17 F for *all* singlet S-wave pairs (a considerable over-estimate) produces no significant effect on the off-shell amplitudes. On the other hand, this change will have considerable effect on the final propagator we append to the amplitudes, since it is exactly this propagator which takes account of the final rescattering of the particles before they are detected. It is, of course, just the sensitivity of this to the scattering length that makes the break-up reaction interesting

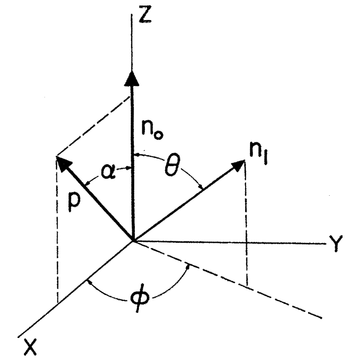

FIG. 3. Coordinate system used in Eq. (14).

to study. Therefore our procedure is to calculate the off-shell amplitudes with isospin invariance and a singlet scattering length of -23.78 F and to use -23.78 F in the final neutron-proton propagators as well, but in those terms in which the final correlated pair is two neutrons we take either -23.78 F or -17 F for the scattering length used in (4) to determine the parameters for the propagator and vertex.

In terms of the amplitudes of (11) and (12) the total cross section σ is then

$$\sigma=\frac{1}{v(2\pi)^5}\int d^3p\,d^3n_1d^3n_2\,\delta(\mathbf{n}_1+\mathbf{n}_2+\mathbf{p})\delta(E-n_1^2-n_2^2-p^2)$$
$$\times\{\tfrac{1}{3}|M_{1/2}|^2+\tfrac{2}{3}|M_{3/2}|^2\}\,, \quad (13)$$

where v is the relative velocity of the incident particles, and where we now call the proton momentum $\mathbf{p}$ and the final neutron momenta $\mathbf{n}_1$, $\mathbf{n}_2$. We choose a coordinate system in which the incident momentum $\mathbf{n}_0$ is in the positive z direction and the final proton momentum $\mathbf{p}$ is in the x-z plane. (See Fig. 3.) The partial cross section $d^3\sigma/d\Omega_{pL}dE_{pL}$ is

$$\frac{d^3\sigma}{d\Omega_{pL}dE_{pL}}=\frac{p_L}{(2\pi)^5v}\int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi\int_0^\infty n_1^2dn_1\,,$$
$$\delta(2n_1^2+2n_1px+2p^2-E)|M|^2$$
$$=\frac{1}{(2\pi)^5}\frac{p_L}{v}\int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi\frac{n_1'^2}{4n_1'+2px}|M|^2. \quad (14)$$

The subscript L refers to laboratory quantities,

$$x=\cos\alpha\,\cos\theta+\sin\alpha\,\sin\theta\,\cos\phi\,, \quad (15)$$

and

$$n_1'=\tfrac{1}{2}\{-px\pm[p^2x^2+2(E-2p^2)]^{1/2}\}\,, \quad (16)$$

is real and non-negative. We evaluate the above double integral numerically on an IBM 7094, carefully treating the limitations imposed by the energy δ function.

The above method for evaluating the partial cross section is the most straightforward, but not the most efficient in terms of computer time. For example, the term in the cross section involving $|A_{1/2,\phi}(1,2;3)|^2$ can be evaluated without integration by a proper choice of coordinate system and the two-dimensional integral defined above can be reduced to a one-dimensional integral. Then, however, the partial-wave decomposition of the off-energy-shell amplitudes involve Y_{lm}'s with sums over both l and m. In order to avoid this, and to have a uniform scheme for all terms we have used Eq. (14).

III. DEFORMATION OF CONTOURS

The coupled integral equations (1) are solved numerically by replacing the integrals by sums using Gaussian quadratures and inverting the resulting matrix equations on a high-speed computer.

A major difficulty in doing this is the treatment of the singularities of the kernel. These arise both from the propagator and the Born term. The former are easily dealt with since their position depends only on p, but the location of the singularities of the Born term depend on both p and k and are therefore not easily managed. The effects of these singularities (after some averaging) on straightforward numerical solutions of the equations is particularly disturbing on the off-energy-shell amplitudes. In calculating K-d and Λ-d elastic scattering, Hetherington and Schick[10] circumvented this problem by deforming the integration path in (1) away from the singularities. Since one is studying an integral equation, this necessarily means extending k as well as p into the complex planes. Since for scattering, the kernel is complex even for k and p real, their extension requires no additional computer storage for the matrix representing the kernel. In fact, it leads to great saving of computer space as one can choose the integration path as far as possible from singularities and therefore the kernel is very smooth and relatively few points are needed to represent the integral. However, the resulting solutions are for k complex and experimentally k is real. To fix this, one takes Eq. (1) again, using the amplitude defined on the deformed contour for $(p,b|T_S(E)|k',d)$ with p complex and one now does the integral with k real. These contour deformations are justified as long as no singularities are crossed. The location of the singularities of the Born terms with one or both momenta complex and of the propagators, is straightforward. The problem of the location of the singularities of the scattering amplitude is much more difficult, and remains, so far as we know, unresolved. Hetherington and Schick assumed that attention to the singularities of the Born terms and of the propagators is sufficient. This is certainly not true in general, but seems to have worked in their case. We shall also make this assumption (for lack of a better one) but will return to the problem at the end of this section.

We begin by studying the singularities of the Born terms. This is most easily done for S waves. The other partial waves have singularities at the same places. The S-wave projection of (2) is

$$(k,a|B_0(E)|k',b) = -\frac{1}{2kk'}\left[\frac{1}{(B-A)(C-A)}\ln\left(\frac{A+kk'}{A-kk'}\right) + \frac{1}{(A-B)(C-B)}\ln\left(\frac{B+kk'}{B-kk'}\right) + \frac{1}{(A-C)(B-C)}\ln\left(\frac{C+kk'}{C-kk'}\right)\right], \quad (17)$$

where

$$A=k'^2+\tfrac{1}{4}k^2+\beta_a^2, \quad B=k^2+\tfrac{1}{4}k'^2+\beta_b^2, \quad C=k^2+k'^2-\tfrac{1}{2}E. \quad (18)$$

There are no singularities at $k=k'=0$ at $A=B$, $B=C$, or $A=C$. These apparent singularities come from the partial fractions used to do the $\cos\theta$ integral. The only singularities therefore come from the vanishing of the arguments of the logarithm. This leads to branch points in k at

$$k=\pm 2(k'\pm i\beta_a) \quad \text{and} \quad k=\pm(\tfrac{1}{2}k'\pm i\beta_b) \quad (19)$$

or in k' at

$$k'=\pm(\tfrac{1}{2}k\pm i\beta_a) \quad \text{and} \quad k'=\pm 2(k\pm i\beta_b) \quad (20)$$

from the first two terms, and at

$$k=\tfrac{1}{2}[\pm k'\pm(2E-3k'^2)^{1/2}] \quad \text{or} \quad k'=\tfrac{1}{2}[\pm k\pm(2E-3k^2)^{1/2}] \quad (21)$$

from the third. It is this last logarithmic singularity which causes trouble for real k and k'. The propagator $P(E,p^2)$ has branch points at $p=\pm(2E/3)^{1/2}$. It should be recalled in all this that E has a small positive imaginary part. In addition to this cut present in P_d and P_ϕ, P_d has a pole at $p=[2(E+\epsilon)/3]^{1/2}$.

In deforming the contour we must ask not only what singularities will be produced by the kernel but also by the inhomogeneous Born term. In this term we put k' on the energy shell. This makes

$$k'=[2(E+\epsilon)/3]^{1/2}, \quad (22)$$

and since we are studying breakup we have $E>0$. This substituted in (21) gives branch points in the complex k plane at

$$k=\tfrac{1}{2}(\pm k'\pm i\epsilon). \quad (23)$$

The other singularities will come from (19) and (20) with k' taken from (22).

The most convenient contour deformation is the one used by Hetherington and Schick, namely a rotation; $k\to ke^{-i\Phi}$, $\Phi>0$. We rotate into the lower half-plane to avoid the propagator singularities. The fixed singularities from the inhomogeneous Born term place an

[10] J. H. Hetherington and L. H. Schick, Phys. Rev. 135, B935 (1965).

upper limit on Φ. We see that Φ must be smaller than the least of

$$\arctan(\epsilon/k'),\quad \arctan(\beta_a/k'). \tag{24}$$

In the kernel, if we put $k\rightarrow ke^{-i\Phi}$ and $p\rightarrow pe^{-i\Phi}$, the singularities due to the Born terms will come at

$$k=\pm 2(p\pm i\beta_a e^{i\Phi}),\quad k=\pm(\tfrac{1}{2}p\pm i\beta_b e^{i\Phi}), \tag{25}$$

$$p=\pm(\tfrac{1}{2}k\pm i\beta_a e^{i\Phi}),\quad p=\pm(k\pm i\beta_b e^{i\Phi}), \tag{26}$$

and

$$k=\tfrac{1}{2}[\pm p\pm(2Ee^{2i\Phi}-3p^2)^{1/2}]\quad\text{or}\quad p=\tfrac{1}{2}[\pm k\pm(2Ee^{2i\Phi}-3k^2)^{1/2}], \tag{27}$$

where we now understand p and k to be real. Clearly, so long as $\Phi<\frac{1}{4}\pi$, the effect will be to move the offending energy-dependent logarithm away from the integration path without bringing the β-dependent ones closer. Hence the prescription is to choose Φ between 0 and $\frac{1}{4}\pi$ and sufficiently far from the limits imposed by the inhomogeneous term, and solve the equations.

The remaining problem is to obtain the amplitude for real k from the amplitude along the contours by doing the p integral in (1) once more. Whether or not we can do this depends on the singularities of the Born term one momentum real. For k real and on the elastic n-d energy shell, we have already studied the equivalent problem in the case of the inhomogeneous term, and there are no singularities in the Born term which prevent the rotation of the p contours by $-\Phi$. This is the reason for the success of Hetherington and Schick in the elastic scattering case. However, since we are interested in breakup, we want k real and between zero and $(\frac{2}{3}E)^{1/2}$. From (21) we see that for $(\frac{1}{2}E)^{1/2}<k<(\frac{2}{3}E)^{1/2}$, $(k,c|B_0(E)|p,b)$ has a branch point in the p plane just below the positive real p axis. For $k<(\frac{1}{2}E)^{1/2}$ this branch point has negative real part and does not trouble us. Hence, for $k<(\frac{1}{2}E)^{1/2}$ we may rotate the contours to $-\Phi$ but for $(\frac{1}{2}E)^{1/2}<k<(\frac{2}{3}E)^{1/2}$ we must take the contours shown in Fig. 4. We then have

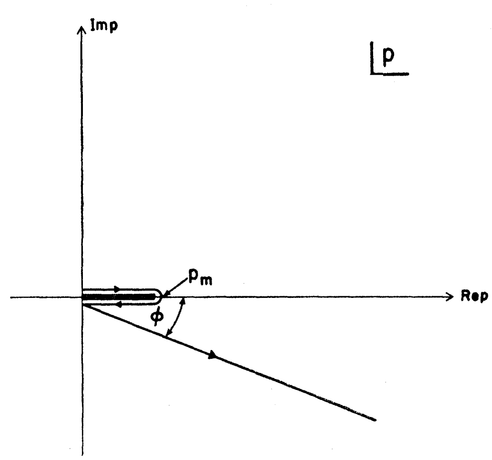

FIG. 4. Contour in the complex p plane for evaluation of Eq. (28).

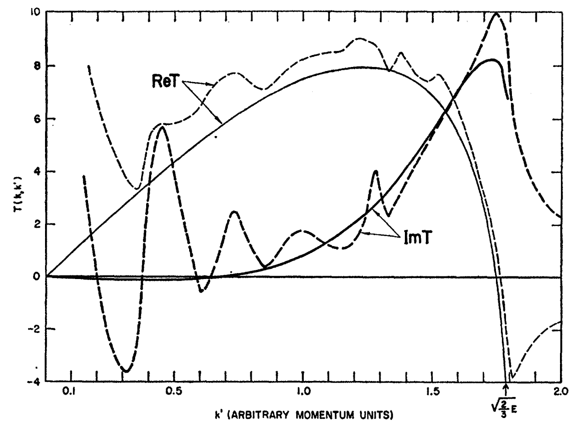

FIG. 5. Comparison of amplitudes calculated along the deformed contour, solid line, and along the real axis, dashed line. The amplitude shown is the doublet p-wave $n+d\rightarrow n+d$ with one leg on the elastic energy shell at incident laboratory neutron energy of 14.1 MeV. $T(k,k')$ has units of inverse momentum.

$$\int_0^\infty (k,c|B_0(E)|p,b)P_b(E,p^2)(p,b|T_0(E)|k',d)p^2dp$$
$$=\int_0^\infty (k,c|B_0(E)|pe^{-i\Phi},b)P_b(E,p^2e^{-2i\Phi})\times(pe^{-i\Phi},b|T_0(E)|k',d)p^2dpe^{-3i\Phi}$$
$$+\int_0^{p_m}[(k,c|B_0(E)|p^+,b)-(k,c|B_0(E)|p^-,b))\times P_b(E,p^2)(p,b|T_0(E)|k',d)p^2dp]. \tag{28}$$

p^+ and p^- refer to p above and below the cut and p_m is the maximum extent of the cut along the real p axis. We have assumed the amplitude does not have such a cut. We can easily compute the discontinuity in the Born term across the cut, but we do not know the amplitude on the real axis, in general. Examination of (21), however, shows that as k goes from $(\frac{1}{2}E)^{1/2}$ to $(\frac{2}{3}E)^{1/2}$, p_m goes from zero to $(E/6)^{1/2}$. Since $(E/6)^{1/2}<(\frac{1}{2}E)^{1/2}$ and for $k<(\frac{1}{2}E)^{1/2}$ we know the amplitude along the real axis from the rotated contour integral only, the problem is solved. To calculate the amplitude for $k<(\frac{1}{2}E)^{1/2}$ we calculate the integral along the rotated contour. For $k>(\frac{1}{2}E)^{1/2}$ we add the part from the Born cut, but we only need the amplitude for $k<(\frac{1}{2}E)^{1/2}$ to do this, and we have that without the extra cut. This is the prescription we have used in the paper.

In Fig. 5 we show an amplitude calculated in this way as well as the erratic amplitude obtained from integrating along the real axis and averaging over the logarithmic singularities. The oscillatory behavior shown in the figure depends on the integration mesh. For the amplitude on the energy shell there is also considerable gain in phase-shift accuracy in a few partial waves from going to deformed contours, although there is no significant effect on the cross sections.

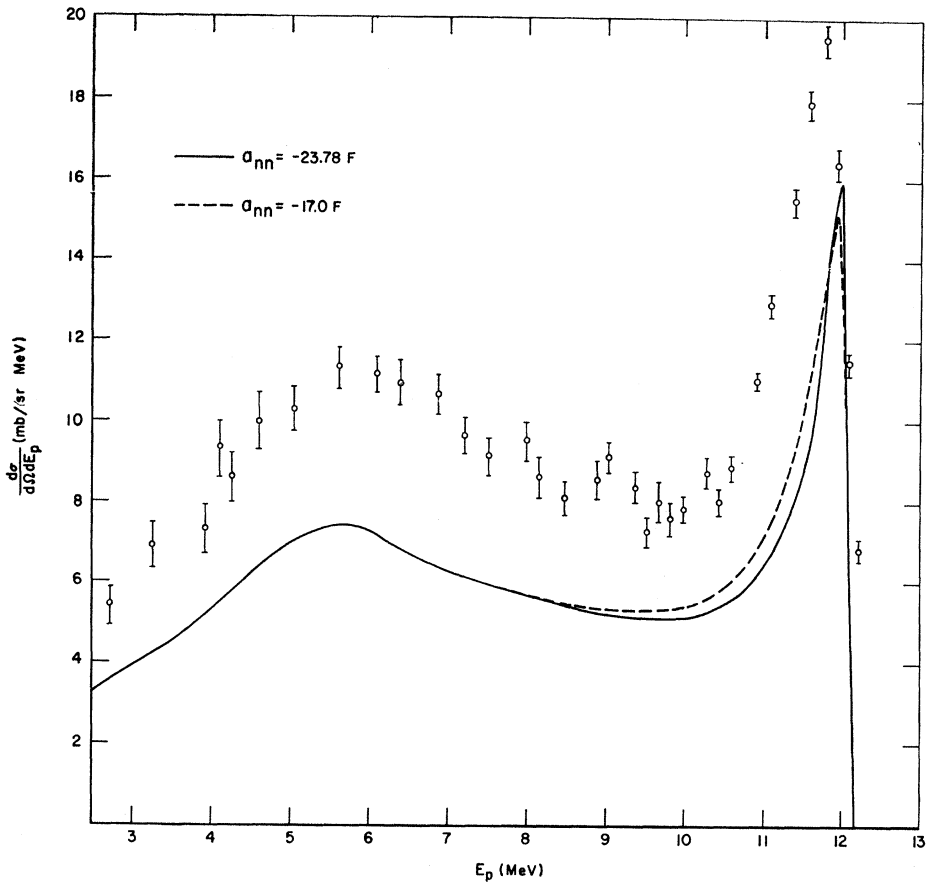

FIG. 6. Proton energy spectrum in the breakup at proton laboratory angle of 4.8° for two choices of neutron-neutron scattering length. The experimental data are from Ref. 3. All quantities are in the laboratory frame.

In all this analysis we have assumed the scattering amplitude itself has no singularities that hamper our contour deformation. In principle we should verify this. A method for locating the singularities of the amplitude is to assume they all come from the Born term, put their location in under the integral in (1) and study by contour pinching what new singularities are generated in the amplitudes; put these in in turn, and so on until no new singularities appear. This is essentially equivalent to finding all the singularities in the Neumann series. Since it has been shown that this series exists for sufficiently large energy,[11] it would be surprising if the number and location of singularities depended drastically on the energy. Unfortunately, we have not been able to carry out this program in full largely because the number of singularities generated in each iteration increases. It does seem that they get farther away from the real axis in each iteration so that the contour rotation procedure is valid, but we cannot prove it. In the special case of no vertex function, that is, replacing all the factors of $1/(k^2+\beta^2)$ by 1, the number of singularities is small and does not increase and we can show that the rotation procedures are valid. Unfortunately, this is a singular case for which Fredholm theory is not valid.[12] Therefore the problem remains open.

[11] Y. Y. Yam, doctoral dissertation, University of Pennsylvania, 1965 (unpublished).

IV. RESULTS

We have calculated by the methods outlined above, the differential proton energy spectrum at fixed angle $d\sigma/d\Omega_p dE_p$ for the reaction $n+d \rightarrow 2n+p$ at several laboratory angles for incident laboratory neutron energy of 14.4 MeV. Our theoretical results are shown in Figs. 6 and 7 where they are compared with the available experimental data.[3,13,14] The experimental proton energy resolution is 0.75 MeV. Our theoretical curves do not include any energy smearing. At 4.8° where the neutron-neutron low-energy interaction is most important, we show the theory with both choices of neutron-neutron scattering length in the final propagator. The

[12] R. D. Amado, Phys. Rev. **132**, 485 (1963).

[13] K. Ilakovac, L. G. Kuo, M. Petravić, I. Šlaus, and P. Tomaš, Nucl. Phys. **43**, 254 (1963).

[14] The results of an apparently similar but less extensive calculation have been reported by Phillips [A. C. Phillips, Phys. Letters **20**, 50 (1966)]. His results are quite different from ours and shed no light on the scattering-length problem. The reasons for the discrepancy of the two calculations are not clear. We should note, however, that Phillips obtains his two-body off-energy-shell amplitudes by straightforward solution of the integral equations along the real axis, and as we noted above, this can be a dangerous procedure.

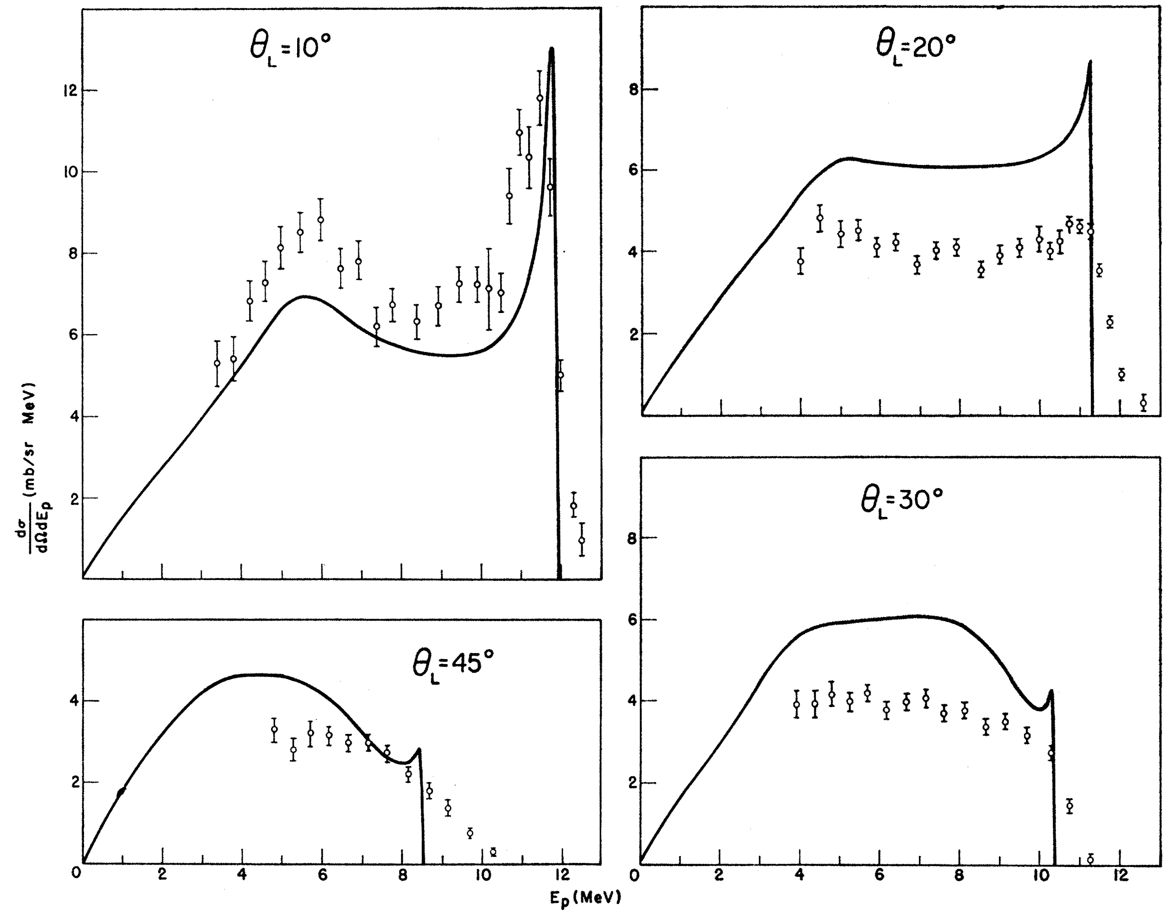

FIG. 7. Proton energy spectra at proton laboratory angles of 10°, 20°, 30°, 45°. The experimental data is from Ref. 13.

general features and trends of the experimental data are reproduced by the theory. The magnitudes are not so well reproduced. The fact that we are low at the forward angles and high at the backward angles is consistent with our having the correct total cross section.[1] The disagreement, particularly at the forward angles, presumably arises from our neglect of higher partial waves in the nucleon-nucleon force. To send a proton nearly straight ahead with most of the neutron incident energy, we require a fairly energetic "head-on" collision involving many forces we have left out. In spite of these inadequacies we find the ability of our simple zero-parameter three-body theory to fit the data as well as it does gratifying. In fact the experimental data at the smallest angle is actually an average over 0° to 8° and the cross section is rapidly varying with angle in this region. Averaging our theory over such angles might improve agreement with experiment. In Fig. 8 we plot separately the doublet and quartet contribution to the differential cross section at a proton lab angle of 4.8°. The pronounced peak near maximum proton energy in the doublet is a reflection of the large neutron-neutron scattering length. At all angles the predominant structure of the differential cross sections comes from the doublet contribution, the shape of the quartet contribution being essentially featureless. We demonstrate this point in Fig. 9 where the doublet and quartet contributions at proton angles of 30° and 45° are plotted separately and compared with phase space at 45°. Note that the peak near maximum proton energy persists in the doublet (although greatly diminished) even at these angles. In the cross section it would be difficult to see with present energy resolution.

V. WATSON THEORY AND THE NEUTRON-NEUTRON SCATTERING LENGTH

We have seen that our theory does not fit experiment sufficiently well to allow the question of the neutron-neutron scattering lengths to be settled. Since our theory is exact (within the limitations imposed by the restrictive interactions we take) it may be used, not just to compare with experiment, but also to compare with the approximate theories used to study breakup. The most popular of these is the final-state-interaction theory of Watson.[2] The strong peak at the upper end of the proton

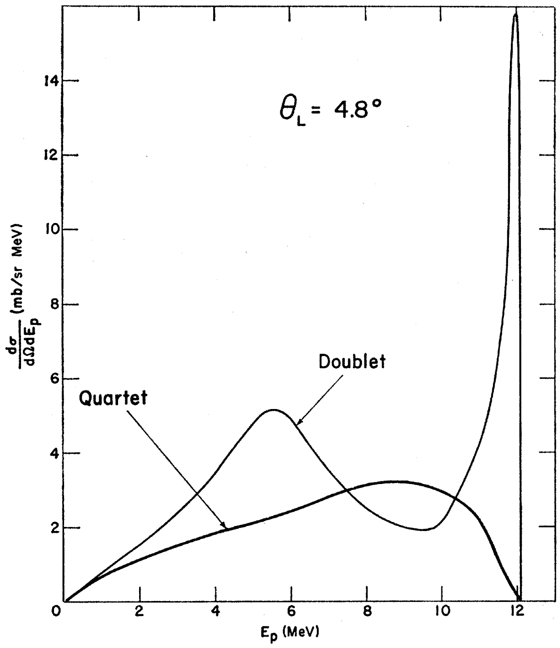

FIG. 8. Doublet and quartet contributions to proton energy spectrum at 4.8° with neutron-neutron scattering length of −23.78 F.

energy spectrum for forward proton angles in the experimental data and in our theory seems to indicate considerable enhancement due to final-state interactions between the two neutrons at very small neutron-neutron relative energy. Watson theory would describe this enhancement by saying that in this region the breakup amplitude has the phase of neutron-neutron scattering and therefore has the rapid dependence of that amplitude. Thus, near maximum proton energy in the forward direction, we can write the break-up cross section in the three-body center-of-mass system as

$$d\sigma/dE_p d\Omega = C^2 k / |ik + \tfrac{1}{2} r_0 k^2 - 1/a|^2 , \qquad (29)$$

where a and r_0 are the neutron-neutron scattering length and effective range and k is the magnitude of the relative neutron-neutron momentum. It is related to the proton momentum p and the total center-of-mass energy E by

$$k = (\tfrac{1}{2}E - \tfrac{3}{4}p^2)^{1/2} . \qquad (30)$$

The factor of k in front of the right-hand side of (29) is a phase-space factor and C is a positive constant not given by the theory. It represents the part of the breakup that occurs before the final neutron-neutron scattering and is assumed to depend weakly on k or p in this region. It is used to normalize to the data. Equation (29) for the breakup is plotted for the two choices of scattering length in Fig. 10, along with our exact results for the doublet part of the breakup only. The Watson result has been normalized to the exact one at the peaks, and only the upper end of the spectrum is shown, since only it is relevant. We see that for the two choices of a, both the exact and the Watson result peak at about the same proton energy, but the Watson peak is wider in both cases than the exact result. Since in comparing the Watson theory with the experiment one would normalize it and smear it over the experimental resolution, which has always been much wider than the 0.05-MeV difference in the position of the peaks, it is the width of the peak that is relevant to extracting the scattering length. Since, furthermore, the larger scattering length gives the narrower peak, an analysis by Watson theory of our data smeared to simulate the experiment, would give too large a scattering length. Addition of the breakup cross section from the quartet channel (which is incoherent with the doublet and does not depend at all on the neutron-neutron scattering length) would complicate the issue further since it turns out to be small, but rapidly varying in this region.

One further point about the relation of our theory to Watson theory deserves mention. One might guess that the statement of the validity of Watson theory translated into our model is that near maximum proton energy only the term in which the last interaction is between the neutrons is important. That is, of all the terms only the one in which the correlated pair in Fig. 1 is a neutron pair is important. Since the entire rapid Watson-type dependence of this term is in the neutron-neutron propagator, a further criterion for the validity

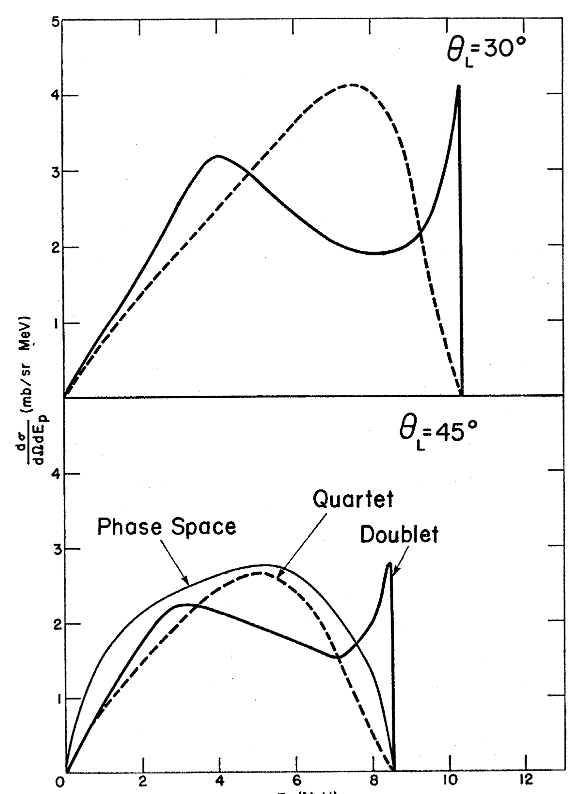

FIG. 9. Doublet and quartet contributions to the proton energy spectrum at 30° and 45°. At 45° we also show phase space.

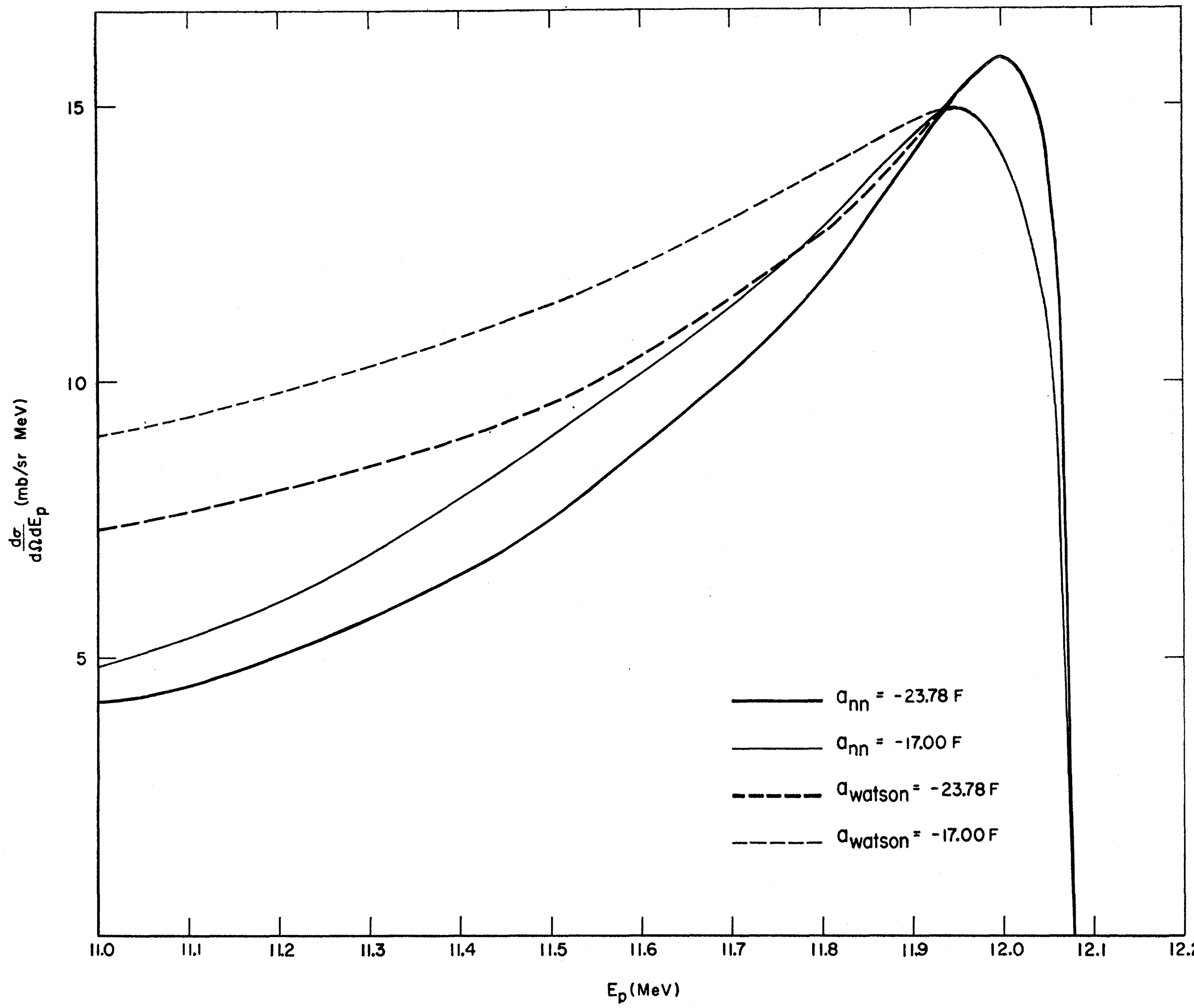

FIG. 10. Comparison of the exact doublet breakup cross section and the Watson theory (normalized) for the two choices of *n-n* scattering length.

of Watson theory is that the off-energy-shell $nd \to p\varphi$ amplitude depend weakly on the proton momentum. In fact this does not happen. The propagator does give a factor just like the Watson factor in (29), but the $nd \to p\varphi$ amplitude is rapidly varying in this region and the term taken as a whole does not have the sharp peak. In Fig. 11 the contribution to the cross section from the term corresponding to Fig. 1 with final neutron-neutron term is plotted and compared to the full theoretical cross section. The neutron-neutron term is of the same order of magnitude as the full cross section but because of the rapid dependence of the off-shell amplitude, much less strongly peaked. When the other terms contributing to the breakup are taken, however the peak reappears. This occurs through a complex interplay of amplitudes and phases and it is not clear whether it is an accident or not. The only clue we have so far is that the off-shell amplitude seems to have a square-root singularity at maximum proton energy, which would account for its rapid variation. However, the general question of whether the resemblance we find between Watson theory and the exact theory is an accident remains. If the confusion of Watson theory in this problem is due to the square-root singularity in the off-shell amplitude, it is a threshold effect and would not apply to problems in which there is a resonant two-particle interaction above threshold. We are presently studying these questions.

VI. CONCLUSIONS

As a result of our calculations of the reaction $n+d \to n+n+p$ with separable *S*-wave spin-dependent interactions between pairs we conclude:

(A) The exact treatment of the three-body problem with abbreviated two-body forces accounts for the major features of the breakup reaction.

(B) Better agreement with experiment requires a better two-body force. Probably this would mean going to a considerably higher level of difficulty and would re-

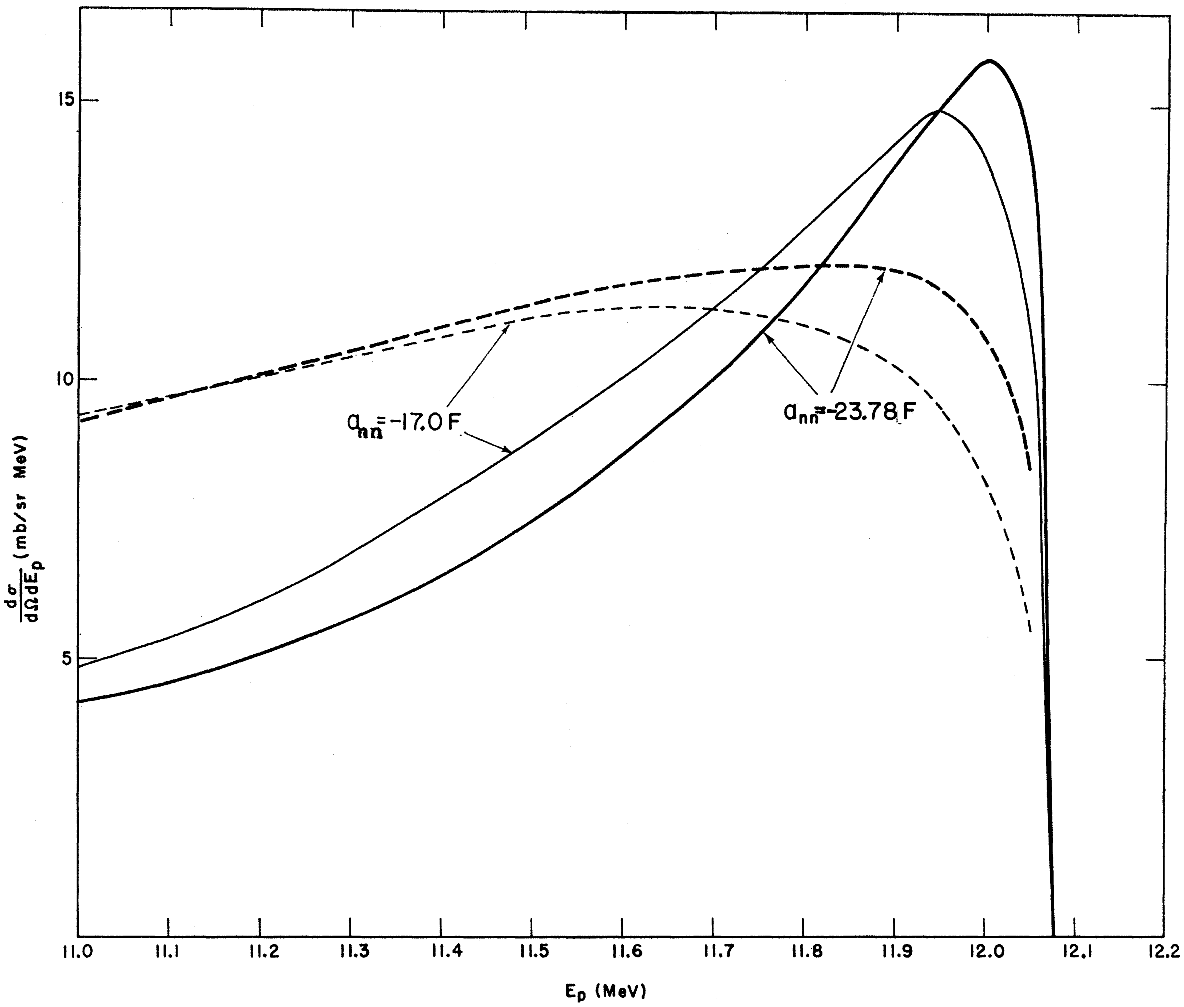

FIG. 11. Contribution to the cross section involving only final neutron rescattering, dashed line, compared with full doublet for the two scattering lengths, solid line.

quire the use of very large fast computer memories, which are now just becoming available. It would certainly require the contour-deformation methods we have outlined in Sec. III, because of the reduction in computer storage space it allows.

(C) Short of a really good theory that fits all the data, it does not seem hopeful that one will be able to extract the neutron-neutron scattering length from *n*-*d* breakup experiments with confidence. In particular, Watson theory seems to be misleading in this reaction.