By inspection, we can conclude that the spread of the spins around the direction of the plane perpendicular to the beam does not alter the angular distribution significantly, yet it makes it a little more isotropic, as expected.

Now the angular distribution is calculated, and the collective states are assumed to be in the statistical region. Equations (11) and (12) are used to evaluate $A_{M'}$. For a nucleus with 200 nucleons having rigid moments of inertia and a temperature of 1 MeV, we obtain for $J=30$:

$$P(\theta)\alpha 1+1.969\cos^2\theta-1.404\cos^4\theta. \quad (16)$$

Again this angular distribution has to be compared with Eq. (13c) when $J\rightarrow J-2$ and $J=30$.

$$P(\theta)\alpha 1+1.355\cos^2\theta-0.723\cos^4\theta. \quad (13c)$$

Comparison between Eq. (13c) and (16) shows that the angular distribution for the two cases differs considerably.

Finally, the angular distribution for levels in the statistical region based on the collective model has to be compared with results in the same region based on the single-particle model,[5] Eq. (17), and on the liquid-drop model,[6] Eq. (18). For convenience, again a comparison for $J=30$ is made:

$$P(\theta)\alpha 1-0.04\cos^2\theta, \quad (17)$$

$$P(\theta)\alpha 1+2.967\cos^2\theta-2.868\cos^4\theta, \quad (18)$$

$$P(\theta)\alpha 1+1.969\cos^2\theta-1.404\cos^4\theta. \quad (16)$$

Equations (16), (17), and (18) show that the angular distributions differ from one another significantly.

IV. SUMMARY

For a cascade of rotational states in which $J\rightarrow J-2\rightarrow J-4$, the angular distribution is weakly dependent on J and M and is not very far from the asymptotic form as shown in Eq. (13e). For collective transitions in the statistical region, on the other hand, the angular distribution is strongly model-dependent.

Therefore significant information about the nature of nuclear states can be obtained from measurement of the angular distribution of gamma rays from high-spin states with their spin close to the plane perpendicular to the heavy-ion beam.

PHYSICAL REVIEW VOLUME 150, NUMBER 3 21 OCTOBER 1966

Smooth Velocity-Dependent Potential and Nuclear Matter

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(Received 20 May 1966)

A velocity-dependent potential of the form

$$V(\mathbf{r},\mathbf{p})=-V_1(r)+p^2e^{-s^2p^2}V_2(r)+V_2(r)p^2e^{-s^2p^2},$$

where $V_1(r)$ and $V_2(r)$ are Gaussians, is used to fit the singlet (0–320-MeV) phase shifts for elastic nucleon-nucleon scattering. This exponential velocity dependence replaces the hard core with a short-range repulsion which is much softer than the case of quadratic ($s=0$) velocity dependence used by Green, Levinger, *et al.* The two-body scattering problem is solved in momentum space by numerical summation of the Born series; the $^1S_0$ scattering length is calculated separately by a rapid matrix-inversion method. The applicability of ordinary many-nucleon perturbation theory for this interaction is tested by calculation of the first-order (P_1) and second-order (P_2) potential energy per particle of nuclear matter. A rapid singlet-even convergence rate of $P_2/P_1=4.3\%$ at $k_F=1.5$ F follows as a result of the reduced off-energy-shell matrix elements of this two-body interaction. One also finds qualitative agreement with the singlet-state potential energy per particle obtained by Sprung *et al.* in their complete nuclear-matter calculation. The harmonic-oscillator matrix elements required for Hartree-Fock calculations of spherical nuclei are evaluated quite simply for this potential. Also, since this potential readily separates into relative x, y, and z coordinates, it is well suited for Hartree-Fock calculations of deformed nuclei.

I. INTRODUCTION

IN 1951, Jastrow[1] proposed a hard-core repulsion as a simple explanation for the observed isotropic proton-proton scattering at 350 MeV. He realized then that "Since there may be appreciable nonstatic contributions at 350 MeV, the quantitative implications of a static potential which has been fitted, in part, to a cross section at that energy must not be given undue weight."[1] Indeed, a quadratically velocity-dependent repulsion was later found to be consistent with high-energy (310-MeV) elastic nucleon-nucleon scattering by Levinger and co-workers.[2]

* Research supported in part by the National Science Foundation.
† Oak Ridge National Laboratory operated by the Union Carbide Corporation for the U. S. Atomic Energy Commission.
[1] R. Jastrow, Phys. Rev. **81**, 165 (1951).
[2] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962); O. Rojo and L. M. Simmons, *ibid.* **125**, 273 (1962).

The inapplicability of perturbation theory for calculating nuclear binding energies using such velocity-dependent potentials was demonstrated by Green and da Providencia[3] when they found a relatively slow convergence rate. It became necessary to invoke the methods developed especially for strong, short-range interactions by Brueckner, and others.[4] Using such methods, Bhaduri and Preston[5] concluded that a modified Levinger-Green potential produces saturation of nuclear matter provided the two-nucleon wave function has the appreciable distortion characteristic of a strong, short-range repulsion.

The numerical approximations usually made in nuclear matter calculations were recently examined by Brown, Schappert, and Wong,[6] who concluded that nuclear matter was underbound for potentials having a hard-core repulsion. Saturation was significantly improved by using a "soft" finite repulsion which was adjusted to fit two-nucleon scattering data.[7] One might hope again that perturbation theory is applicable; however, a nuclear-matter calculation by Sprung[8] clearly demonstrated that ordinary perturbation theory cannot be employed even for these static "soft" core repulsions.

These parallel developments indicate that for either static or nonstatic potentials, which are consistent with two-nucleon scattering, one must deal with very strong, if not infinite, repulsions to obtain the correct binding of nuclear matter. Consequently, *for the type of potentials considered*, one must face the extremely difficult task of applying Brueckner theory to finite nuclei.

Nevertheless, it has now been clearly shown that the Hartree-Fock method is a practicable scheme for determining nuclear properties; the harmonic-oscillator matrix elements of a *smooth* two-body interaction are the basic input information for these calculations. Muthukrishnan, Krieger, Davies, and Baranger[9] have emphasized that the interaction employed must yield the *correct* saturation with small second-order corrections for nuclear matter, whereas Svenne, Kerman, and Villars[10] have stressed the importance of fitting the two-nucleon scattering data.

We are motivated by the establishment of these Hartree-Fock procedures to re-examine the role of the short-range repulsion. Our goal is to simultaneously fit two-nucleon scattering experiments and the saturation curve for nuclear matter with a *smooth* interaction. This approach will perhaps indicate the extent to which strong, short-range correlation effects should or should not be included in finite-nuclei calculations.

In a previous study of this problem,[11] a procedure of reducing the off-energy-shell matrix elements of the two-nucleon interaction was used to generate a very smooth effective potential which still matched the scattering data up to 320 MeV. As a *preliminary* example, separable potentials were used to fit the two-nucleon phase shifts. Successful application of this very phenomenological interaction to nuclear Hartree-Fock calculations,[10,12] shell-model calculations,[13] proton-proton bremsstrahlung,[14] the photonuclear dipole sum rule,[15] and to excited states of the alpha particle,[16] indicate that the reduction of off-energy-shell matrix elements while fitting the two-nucleon phase shifts is a reasonable and useful procedure. *However, the establishment of a correct saturation of nuclear matter was not complete, particularly in the P-wave potential energy calculation and in the approximate calculation of the second-order energy per particle terms.* A valid calculation of these second-order terms is now possible.[17] In addition, we now introduce a velocity-dependent interaction, including a long-range local part, which we believe is more convenient and more realistic.

In Sec. II, we define this smooth velocity-dependent potential as a simple extension of the Levinger-Green interaction and in Sec. III show that a reasonable fit to the singlet two-nucleon phase shifts is possible. A nuclear-matter calculation is presented in Sec. IV along with the demonstration of a rapid convergence of ordinary perturbation theory. We compare our singlet state results with those obtained by Sprung and Reid[18] in their impressive calculation using the reference spectrum method of Bethe, Brandow, and Petschek.[19]

[3] A. M. Green, Nucl. Phys. **33**, 218 (1962); J. da Providencia, *ibid.* **40**, 321 (1963); J. S. Levinger, M. Razavy, O. Rojo, and N. Webre, Phys. Rev. **119**, 230 (1960); **121**, 1863 (1961).

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[6] G. E. Brown, G. T. Schappert, and C. W. Wong, Nucl. Phys. **56**, 191 (1964).

[7] C. W. Wong, Nucl. Phys. **56**, 213 (1964); **71**, 385 (1965); C. Bressel, A. Kerman, and E. Lomon, Bull. Am. Phys. Soc. **10**, 584 (1965).

[8] D. W. L. Sprung, MIT Report No. 2098-201, 1965 (unpublished).

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[10] J. P. Svenne, thesis, MIT, 1965 (unpublished); J. P. Svenne, A. K. Kerman, and F. Villars, Phys. Rev. **147**, 710 (1966).

[11] F. Tabakin, Ann. Phys. (N. Y.) **30**, 51 (1964).

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[19] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

Finally, the convenience of this velocity-dependent potential for Hartree-Fock calculations is illustrated by an explicit evaluation of the relevant harmonic-oscillator matrix elements (Sec. V).

II. THE VELOCITY-DEPENDENT INTERACTION

Let us define a velocity-dependent interaction in the Hermitian form

$$V(\mathbf{r},\mathbf{p})=-V_1(r)+C(p^2)V_2(r)+V_2(r)C(p^2), \quad (1)$$

where $\mathbf{r}$ and $\mathbf{p}$ are the relative coordinate and momentum, respectively. We seek a generalization of the Levinger-Green potential which should significantly reduce the off-energy-shell matrix elements of the potential and thereby generate the smoothness requisite for a rapid convergence of many-body perturbation theory. We have taken $V_1(r)$ and $V_2(r)$ to be Gaussian functions

$$\begin{aligned} V_1(r)&=V_1e^{-a^2r^2},\\ V_2(r)&=V_2e^{-b^2r^2}. \end{aligned} \quad (2)$$

A particularly convenient choice for $C(p^2)$, the cutoff operator, is

$$C(p^2)=D+(p^2/\hbar^2b^2)e^{-(s^2p^2/\hbar^2)}, \quad (3)$$

where $C(p^2)$ is made dimensionless by introduction of the inverse range b of the local function $V_2(r)$. The quantity s, the smoothness parameter, controls the magnitude of the off-energy-shell matrix elements of the interaction, and the energy at which this cutoff becomes effective is determined by $E_s=(2\hbar^2/m)s^{-2}$. Larger values of s should lead to smaller second-order corrections in nuclear matter. The dimensionless parameter D specifies the relative amount of static to nonstatic repulsion; the static force is associated with the strong, short-range correlations which, if possible, are to be minimized. Therefore, we have *tentatively* set $D=0$.

The smoothness parameter s can also be understood as the range of nonlocality of the operator $C(p^2)$,

$$\begin{aligned} (\mathbf{r}|C(p^2)|\mathbf{r}')&=\int\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{(2\pi)^3}d\mathbf{k}\frac{k^2}{b^2}e^{-s^2k^2}\\ &=(16\pi^{3/2}b^2s^5)^{-1}(3-\tfrac{1}{2}s^{-2}\xi^2)e^{-(\xi^2/4s^2)}. \end{aligned} \quad (4)$$

In the limit $s\to 0$ this operator becomes the second derivative of a delta function and we return to the Levinger-Green potential. For $s>0$, Eq. (4) is a function of the nonlocality $\xi=|\mathbf{r}-\mathbf{r}'|$ which peaks at $\xi=0$ with a smear of range s. The repulsive term in the Schrödinger equation is therefore the nonlocal kernel

$$\int\{(\mathbf{r}|C|\mathbf{r}')V_2(r')+V_2(r)(\mathbf{r}|C|\mathbf{r}')\}\Psi(\mathbf{r}')d\mathbf{r}', \quad (5)$$

where the two ranges b^{-1} and s both determine the dominant regions of the integration over $\mathbf{r}'$.

It is clear that the introduction of the operator $C(p^2)$ complicates the solution of the two-body scattering problem considerably; however, we are willing to pay this price if it is at all possible to use perturbation theory in many-nucleon calculations. We now discuss one way of determining the two-body phase shifts and wave function for such nonlocal interactions. In order to avoid the complications of the ${}^3S_1+{}^3D_1$ state, we have restricted ourselves to the singlet states in this study.

III. SOLUTION OF THE SCHRÖDINGER EQUATION

To solve the Schrödinger equation, we iterate and then sum the Born series. There is no bound state in the singlet states and one could expect convergence of this series for a sufficiently smooth operator $C(p^2)$. The ${}^1S_0$ state is almost bound and the Born series consequently converges very slowly at low energies; therefore, the scattering length has been calculated by a more rapid method.

The Lippmann-Schwinger equation for standing-wave boundary conditions is

$$\begin{aligned} R_l(k|k_0)&=V_l(k|k_0)-\frac{2}{\pi}P\int_0^\infty\frac{dk'k'^2V_l(k|k')R_l(k'|k_0)}{k'^2-k_0^2}\\ &=V_l(k|k_0)-\frac{2}{\pi}\int_0^\infty dk'\\ &\times\left\{\frac{k'^2V_l(k|k')R_l(k'|k_0)-k_0^2V_l(k|k_0)R_l(k_0|k_0)}{k'^2-k_0^2}\right\}, \end{aligned} \quad (6)$$

where a zero term is included to provide a nonsingular integrand. Here k, k', and k_0 refer to relative momenta, and the laboratory collision energy is $E_L=2\lambda k_0^2$ with $\lambda=\hbar^2/m=41.5$-MeV fermi^2. The potential is expanded as

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\frac{2}{\pi}\lambda\sum_{lm}V_l(k|k')Y_{lm}(\hat{k})Y_{lm}{}^*(\hat{k}'), \quad (7)$$

and the potential function $V_l(k|k')$ is

$$\begin{aligned} V_l(k|k')=&-(\pi^{1/2}V_1/4a^3\lambda)e^{-(k^2+k'^2)/4a^2}i_l(kk'/2a^2)\\ &+(\pi^{1/2}V_2/4b^3\lambda)e^{-(k^2+k'^2)/4b^2}i_l(kk'/2b^2)\\ &\times\{C(k)+C(k')\}, \end{aligned} \quad (8)$$

with $C(k)=(k^2/b^2)e^{-s^2k^2}$; $i_l(x)$ is a modified spherical Bessel function of the first kind.[20] The R matrix,

$$\langle\mathbf{k}'|V|\Psi_{k_0}\rangle=\frac{2}{\pi}\lambda\sum_{lm}R_l(k'|k_0)Y_{lm}(\hat{k}')Y_{lm}{}^*(\hat{k}_0) \quad (9)$$

[20] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1965), Chap. 10, p. 443.

yields the lth partial-wave phase shifts for diagonal matrix elements

$$\tan\delta_l(k_0) = -k_0 R_l(k_0|k_0), \tag{10}$$

whereas knowledge of the off-energy-shell elements ($k_0 \neq k'$) determines the complete two-body wave function

$$\psi_l(r) = j_l(k_0 r) - \frac{2}{\pi} P\int_0^\infty \frac{dk' k'^2 j_l(k'r) R_l(k'|k_0)}{k'^2 - k_0^2} = j_l(k_0 r) - \frac{2}{\pi}\int_0^\infty dk' \left\{ \frac{k'^2 j_l(k'r) R_l(k'|k_0) - k_0^2 j_l(k_0 r) R_l(k_0|k_0)}{k'^2 - k_0^2} \right\}. \tag{11}$$

The nonsingular integrands in Eqs. (6) and (11) evaluated at $k_0 = k'$ are easily expressed in terms of the derivative of the R matrix,

$$\frac{\partial}{\partial k} R_l(k|k_0) = \frac{\partial}{\partial k} V_l(k|k_0) - \frac{2}{\pi}\int_0^\infty dk' \left\{ \frac{k'^2 (\partial/\partial k) V_l(k|k') R_l(k'|k_0) - k_0^2 (\partial/\partial k) V_l(k|k_0) R_l(k_0/k_0)}{k'^2 - k_0^2} \right\}. \tag{12}$$

Equations (6) and (12) are simultaneously iterated beginning with the known values of $V_l(k|k')$ and $(\partial/\partial k)V_l(k|k')$. The mapping $k' = 2s^{1/2}/(1-s^{20})$ to the finite region $0 \leq s \leq 1$ gives convenient values of k, k', and k_0, for which the integrals are accurately evaluated with twenty-point Gaussian integrations.[20]

At low energies the Born series for the $^1S_0$ scattering converges very slowly; therefore, the singlet even scattering length a_s was evaluated using the relation ($l=0$),

$$a_s = \lim_{k\to 0}(-\tan\delta_0/k) = \lim_{k\to 0} R_0(k|0), \tag{13}$$

where $R_0(k|0)$ satisfies the integral equation

$$R_0(k|0) = V_0(k|0) - \frac{2}{\pi}\int_0^\infty dk' V_0(k|k') R_0(k'|0). \tag{14}$$

After the integral is replaced by a ten-point Hermite integration formula,[20] the function $R(k|0)$ is rapidly found to an accuracy of 1% by a matrix inversion. To calculate the effective range it is then only necessary to iterate Eqs. (6) and (12) twenty times. At higher energies the Born series converges rapidly provided the potential is smooth enough. For $s=0$, we find that the Born series diverges as a result of the dominance of the operator $C(p^2)$ with increasing energy.

Our parameter search procedure has been to adjust V_1 to give the scattering length a_s using Eqs. (13) and (14); the effective range determines another parameter a and we are left with three free parameters V_2, b, and s to fit the $^1S_0$ and $^1D_0$ phase shifts.[21,22] The results of this adjustment are presented in Figs. 1 and 2 and Tables I and II. The corresponding ($E_L = 1.14$ MeV) wave function is shown in Fig. 3 and is seen to be very smooth. At higher energies the wave function is, of course, pushed out by the repulsion; however, the wave

TABLE I. The singlet-even (SE1, SE2) and singlet-odd (SO) potential strengths V_1, V_2 given in MeV, the inverse ranges a and b given in (fermis)$^{-1}$; the scattering length a_s and effective range r are in fermis and the cutoff energy $E_s = 2\lambda s^{-2}$ is in MeV.

	V_1	V_2	a	b	E_s	a_s	r
SE1	96.07	5500	0.85	3.0	240	−24.17	2.42
SE2	94.44	4500	0.85	3.0	332	−24.08	2.34
SO	0	1000	···	1.14	83	···	···

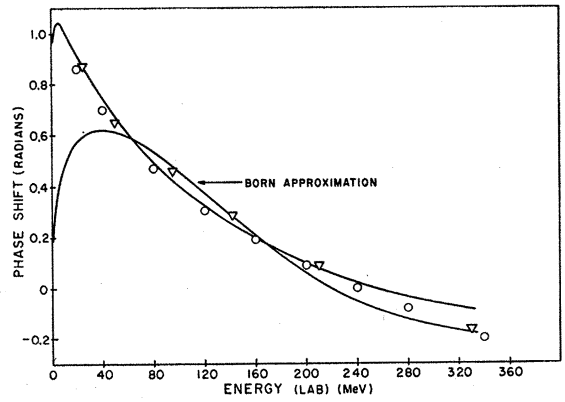

FIG. 1. The $^1S_0$ phase shifts for the SE1 potential parameters (see Table I) as a function of laboratory energy. Also shown are the first-Born-approximation phase shifts. The circles are the YLAM (Ref. 22) and the triangles are the Arndt-MacGregor (Ref. 21) phase shifts.

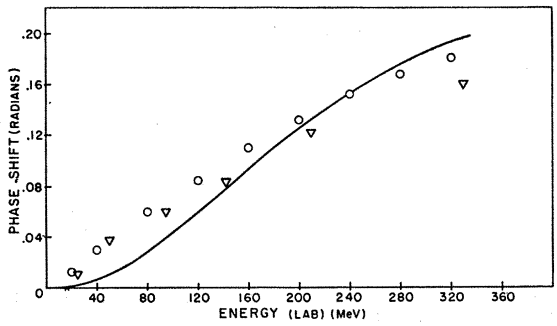

FIG. 2. The $^1D_2$ phase shifts for the SE1 potential parameters as a function of laboratory energy. The circles are the YLAM (Ref. 22) and the triangles are the Arndt-MacGregor (Ref. 21) phase shifts.

[21] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).

[22] G. Breit *et al.*, Phys. Rev. **120**, 2227 (1960); **122**, 1606 (1961).

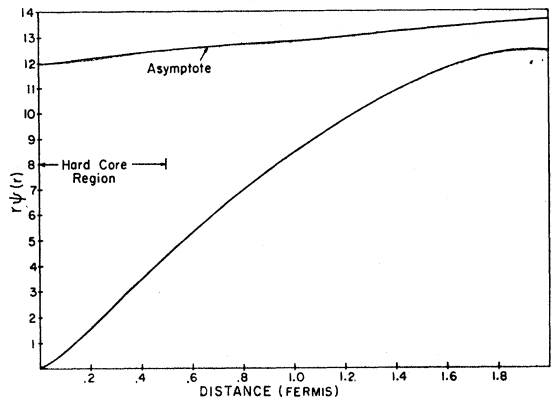

FIG. 3. The two-body wave function of relative motion in the $^1S_0$ (SE1); this case if for E(LAB)=1.14 MeV and δ_0=0.952 radians.

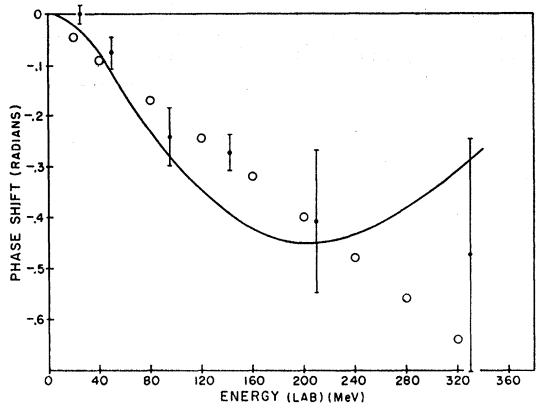

FIG. 4. The $^1P_1$ phase shifts for the singlet odd (SO) potential parameters. The circles are the YLAM phase shifts (Ref. 22) while the Arndt-MacGregor phase shifts are indicated by the error bars.

function is smooth at all collision energies. It was also found that fitting the singlet-even phase shifts requires small values of s and b^{-1}, thereby confining nonlocality effects to short distances. A singlet-odd ($^1P_1$) fit is presented in Fig. 4 and Tables I and II; the errors indicated for these phase shifts are quite large and are taken from Ref. 21. We have, of course, several alternative singlet phase-shift fits; the fits presented here have been selected on the basis of the following nuclear-matter calculation as well as the phase-shift fit.

The qualitative phase-shift fits obtained for this interaction indicate that nucleon-nucleon elastic scattering does not by itself demand a hard-core repulsion if nonlocal or velocity-dependent effects are permitted. Therefore, we now investigate the binding energy of nuclear matter as a further indication of the role of the short-range repulsion.

IV. NUCLEAR MATTER

Nuclear matter is an infinite system of interacting nucleons which is made amenable to theoretical studies by elimination of the nuclear-surface and Coulomb-force difficulties. This hypothetical system determines whether a given two-nucleon force would yield the correct binding and density when applied to finite nuclei. For example, the intimate relation between the saturation of energy and density for the infinite and finite systems has been demonstrated in recent calculations employing saturating forces for which the Hartree-Fock method should be applicable.[9]

Moreover, a potential[7] having a soft-core repulsion failed to give binding in both the infinite and finite systems when calculated by perturbation theory and the Hartree-Fock method.[8,10] This does *not* imply that such forces are unrealistic, but rather that the use of perturbation theory and of the Hartree-Fock method is invalid for such forces. These calculations illustrate another service provided by nuclear-matter studies, namely, the validation of the calculational method, which is the aspect of nuclear matter emphasized in this paper. To validate the use of the Hartree-Fock method for finite nuclei, a perturbation-theory calculation of the binding energy of nuclear matter should converge rapidly.

For this reason, the energy per particle E/A of infinite nuclear matter has been calculated using

$$E/A=\tfrac{3}{5}\epsilon_F+P_1+P_2\cdots, \qquad (15)$$

where the Fermi energy is $\epsilon_F=\frac{1}{2}\lambda k_F^2$. ($\lambda=\hbar^2/m$) and

TABLE II. The calculated phase shifts (radians) as a function of laboratory energy for the $^1S_0(\delta_0)$, $^1P_1(\delta_1)$, and $^1D_2(\delta_2)$ singlet states. Also shown are the corresponding first-Born-approximation phase shifts δ_{0B}, δ_{1B}, and δ_{2B}.

	$l=0$				$l=2$				$l=1$	
	δ_0		δ_{0B}		δ_2		δ_{2B}		δ_1	δ_{1B}
E	SE1	SE2	SE1	SE2	SE1	SE2	SE1	SE2	SO	SO
5.98	1.05	1.06	0.396	0.391	0.75×10^{-4}	0.74×10^{-4}	0.71×10^{-4}	0.70×10^{-4}	-0.12×10^{-2}	-0.18×10^{-2}
26.71	0.854	0.871	0.602	0.602	0.27×10^{-2}	0.27×10^{-2}	0.25×10^{-2}	0.25×10^{-2}	-0.38×10^{-1}	-0.54×10^{-1}
60.42	0.610	0.632	0.593	0.602	0.016	0.016	0.015	0.015	−0.155	−0.232
103.96	0.394	0.414	0.446	0.459	0.047	0.046	0.041	0.041	−0.303	−0.424
153.3	0.221	0.233	0.236	0.235	0.088	0.086	0.077	0.075	−0.413	−0.493
203.8	0.094	0.092	0.048	0.011	0.128	0.125	0.111	0.110	−0.449	−0.455
252.7	0.45×10^{-2}	−0.015	−0.078	−0.161	0.160	0.157	0.140	0.138	−0.419	−0.371
332.6	−0.087	−0.147	−0.180	−0.337	0.198	0.193	0.176	0.172	−0.290	−0.220
462.0	−0.130	−0.276	−0.177	−0.432	0.229	0.222	0.209	0.204	−0.109	−0.072

TABLE III. The potential energy per particle of nuclear matter (MeV/A) as a function of k_F (fermis) for the various singlet-star potentials. Here P_1 and P_2 are the first- and second-order potential energy per particle, respectively, and $P_1[^1S_0]$ is the first-order potential energy per particle from the $^1S_0$ state alone.

	$P_1[^1S_0]$		P_1			P_2		
k_F	SE1	SE2	SE1	SE2	SO	SE1	SE2	SO
1.2	−12.78	−12.78	−13.27	−13.27	+ 1.66	−1.01	−1.06	−0.43
1.4	−17.74	−17.81	−18.99	−19.04	+ 3.58	−0.94	−1.06	−0.74
1.5	−20.29	−20.40	−22.16	−22.24	+ 4.91	−0.96	−1.15	−1.88
1.6	−22.81	−22.97	−25.51	−25.61	+ 6.50	−1.04	−1.32	−1.01
1.8	−27.63	−27.81	−32.74	−32.83	+10.37	−1.38	−1.88	−1.18

the density of nuclear matter is

$$\rho=(2/3\pi^2)k_F^3. \tag{16}$$

The first-order potential energy P_1 is calculated from the potential function $V_l(k|k)$ [Eq. (8)] using

$$P_1=\frac{4}{\pi}\lambda\sum_{JTSl}(2J+1)(2T+1)\int_0^{k_F}dk \times k^2\left(1-\frac{3}{2}\frac{k}{k_F}+\frac{1}{2}\frac{k^3}{k_F^3}\right)V_l(k|k), \tag{17}$$

whereas the second-order potential energy per particle P_2 is given by

$$P_2=\frac{1}{A\lambda}\sum_{\substack{\mu\nu<F\\ \sigma\tau>F}}\frac{\langle\mu\nu|V|\sigma\tau\rangle\langle\sigma\tau|V|\mu\nu-\nu\mu\rangle}{k_\mu^2+k_\nu^2-k_\sigma^2-k_\tau^2} =\sum_{JTSll'}\int_0^{2k_F}d\mathbf{K}\int_{\mathfrak{D}}d\mathbf{k}d\mathbf{k}'F_{ll'}^{JTS}(\mathbf{K},\mathbf{k},\mathbf{k}'). \tag{18}$$

Here the quantum numbers JTS denote the various allowed two-body states and we have taken the effective mass $m^*=m$. Equation (18) involves an integration over the domain $\mathfrak{D}$ defined by the overlap of two Fermi spheres separated by K with $|\frac{1}{2}\mathbf{K}\pm\mathbf{k}|<k_F$ and $|\frac{1}{2}\mathbf{K}\pm\mathbf{k}'|>k_F$. Previous rough approximations[11] for the integration over $\mathfrak{D}$ have not been employed here; instead, the second-order potential energy P_2 has been accurately calculated using a complete method of integrating which is described in Ref. 17. The function $F_{ll'}^{JTS}$ is given in Refs. 11 and 17.

The first- and second-order potential energy per particle are presented in Table III and in Figs. 5 and 6. The convergence rate $P_2/P_1=4.3\%$ at $k_F=1.5$ F is seen to be very rapid in the singlet-even state when $s=0.585$ F. A smaller smoothness parameter gives much slower convergence which becomes $P_2/P_1=24\%$ ($k_F=1.5$ F) in the Levinger-Green $s=0$ limit. For the singlet-odd ($^1P_1$) state the convergence rate is only $P_2/P_1=18\%$ at $k_F=1.5$ F, which is sufficiently rapid for these smaller potential energy contributions. The significance of these results will be discussed in Sec. VI.

V. HARMONIC-OSCILLATOR MATRIX ELEMENTS

In this section, the velocity-dependent potential defined by Eqs. (1), (2), and (3) is shown to be particularly convenient for Hartree-Fock calculations which employ harmonic-oscillator wave functions $\varphi_\nu(\mathbf{r})$ as the initial basis for the single-particle orbitals. The essential two-body information is then contained in the matrix element

$$\langle\varphi_\mu|V|\varphi_\nu\rangle=-\int\varphi_\mu^*(\mathbf{r})V_1(r)\varphi_\nu(\mathbf{r})d\mathbf{r} +\int\varphi_\mu^*(\mathbf{r})V_2(r)\tilde{\varphi}_\nu(\mathbf{r})d\mathbf{r} +\int\tilde{\varphi}_\mu^*(\mathbf{r})V_2(r)\varphi_\nu(\mathbf{r})d\mathbf{r}, \tag{19}$$

where μ and ν denote the quantum numbers nlm for the *relative* motion of two nucleons in a common harmonic-oscillator potential. The function $\tilde{\varphi}_\nu(\mathbf{r})$ is related to the harmonic-oscillator wave function $\varphi_\nu(\mathbf{r})$ by

$$\tilde{\varphi}_\nu(\mathbf{r})=\langle\mathbf{r}|C(p^2)|\varphi_\nu\rangle. \tag{20}$$

We assume that the Moshinsky-Brody transformation to relative coordinates as well as proper treatment of spin and isospin have already been included. Exchange matrix elements present no difficulty; a factor of $1-(-1)^{S+T+l}$ is simply introduced.

The first term of Eq. (19) is easily evaluated using the normalized harmonic-oscillator wave function

$$\varphi_\nu(\mathbf{r})=N_{nl}(\beta r)^l e^{-(\beta^2r^2/2)}L_n^{l+1/2}(\beta^2r^2)Y_{lm}(\hat{r}), \tag{21}$$

where the oscillator length is $\beta^{-1}=(2\hbar/m\omega)^{1/2}$ for the standard definition of coordinates $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. The normalization is $N_{nl}^2=2\beta^3\Gamma(n+1)/[\Gamma(n+l+\frac{3}{2})]$. The required integral involves the Laguerre polynomials $L_n^{l+1/2}$ and is found in a table of Laplace transforms.[23] The result is given in the Appendix.

[23] Bateman Manuscript Project, *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., 1954), Vol. I and II.

For the nonlocal part of the matrix element, we must first evaluate $\tilde{\varphi}_\nu(\mathbf{r})$, which is expressed as

$$\tilde{\varphi}_\nu(\mathbf{r})=\int(\mathbf{r}|C(p^2)|\mathbf{k})d\mathbf{k}\varphi_\nu(\mathbf{k})$$

$$=-\frac{1}{b^2}\frac{\partial}{\partial s^2}\int\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}}e^{-s^2k^2}\varphi_\nu(\mathbf{k}). \quad (22)$$

The harmonic-oscillator wave function in momentum space is

$$\varphi_\nu(\mathbf{k})=N_{ln}\beta^{-3}i^{2n-l}(k/\beta)^l e^{-\frac{1}{2}(k/\beta)^2}\times L_n^{l+1/2}(k^2/\beta^2)Y_{lm}(\hat{k}). \quad (23)$$

Fortunately, the integration required in Eq. (22) is found in tabulated transforms[23]

$$\tilde{\varphi}_\nu(\mathbf{r})=-\frac{1}{b^2}\frac{\partial}{\partial s^2}I,$$

with

$$I=\frac{N_{nl}}{t^{l+3/2}}(2t^{-1}-1)^n(\beta r)^l e^{-(\beta^2r^2/2t)}L_n^{l+1/2}(U\beta^2r^2)Y_{lm}(\hat{k}),$$

and

$$t\equiv 1+2s^2\beta^2;\quad U\equiv t^{-1}(2-t)^{-1}.$$

Here $\tilde{\varphi}_\nu(\mathbf{r})$ is very much like a harmonic oscillator which has been scale distorted by the action of $C(p^2)$. The final integration required for the last two terms of Eq. (19) is also found in Ref. 23. The harmonic-oscillator matrix element is then the explicit, finite sum given in the Appendix; this sum is quickly evaluated on a computer.

Another great advantage of using this form of exponential velocity dependence, along with Gaussian forms for $V_1(r)$ and $V_2(r)$, is that it is also easy to find a simple expression for the matrix elements using an anisotropic harmonic-oscillator basis which separates into x, y, and z degrees of freedom. We expect therefore that this type of potential should be useful in Hartree-Fock calculations of deformed nuclei.

VI. CONCLUSION

The qualitative phase-shift fits found using this exponentially velocity-dependent potential (Figs. 1, 2, and 4) again demonstrate the possibility of replacing the hard core by a nonstatic, finite repulsion. Moreover, the off-energy-shell matrix elements of this interaction are significantly reduced by the cutoff operator $C(p^2)$ [24]; the resulting two-nucleon wave function (Fig. 3) is smooth as is required for the rapid convergence of many body perturbation theory (see Table III). The singlet-even convergence rate $P_2/P_1=4.3\%$ at $k_F=1.5$ F is rapid and would be reduced even more by introducing an effective mass $m^*<m$.

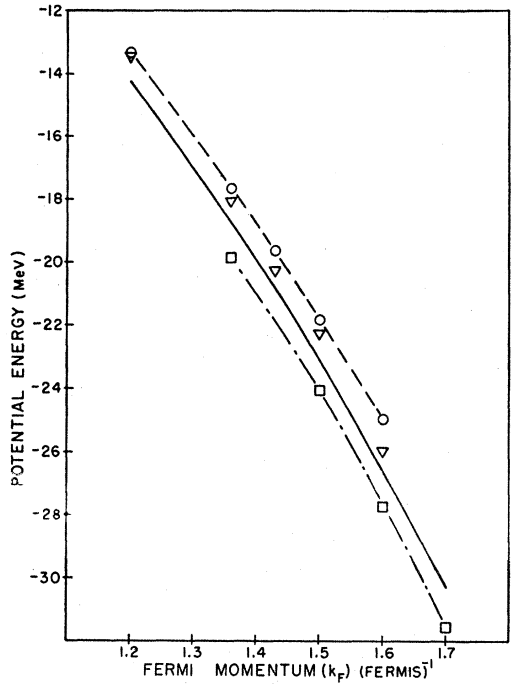

FIG. 5. The singlet even (SE1) potential energy per particle of nuclear matter including the complete second-order contributions (Table III) as a function of density measured by k_F. The results of Ref. 18 are indicated by circles for the Reid potential, by triangles for the Hamada-Johnston potential, and by squares for the Bressel-Kerman-Lomon potential.

Solution of the two-body Schrödinger equation is greatly complicated by this radical velocity dependence; nevertheless, it is possible to sum the Born series for scattering in all but the $^3S_1+^3D_1$ state. For this

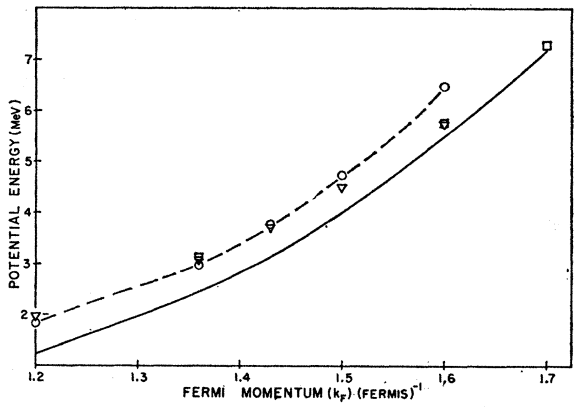

FIG. 6. The singlet-odd (SO) potential energy per particle of nuclear matter, including the complete second-order contributions (Table III), as a function of density measured by k_F. The results of Ref. 18 are indicated by circles for the Reid potential, by triangles for the Hamada-Johnston potential, and by squares for the Bressel-Kerman-Lomon potential.

[24] D. Y. Wong, Nucl. Phys. 58, 212 (1964); E. Predazzi, Ann. Phys. (N. Y.) 36, 250 (1966); A. E. S. Green and R. D. Sharma, Phys. Rev. Letters 14, 380 (1965). These authors have shown that strong velocity dependence exists in the two-nucleon interaction.

state, Weinberg's quasi-Born approximation should be useful.[25]

We wish to emphasize that the phase-shift fits presented in this paper were not chosen solely for the quality of fit. It is especially clear that the $^1D_2$ and $^1P_1$ fits can and should be improved. The potential parameters (Table I) were chosen by considering the corresponding energy per particle of nuclear matter as well as the phase shifts.

It is not yet possible to conclude that nuclear matter saturates with this force (for $D=0$) since the triplet state contributions are not known. Nevertheless, it is of interest to compare our singlet results, including the second-order terms, with the singlet potential energies per particle calculated by Reid, Sprung, Bhargava, and Dahlblom[18] which were kindly made available to us. In Figs. 5 and 6 we see that the agreement is very good. Again, this fit is not altogether accidental, since we have selected from many singlet fits those which give potential energies per particle closest to the Reid-Sprung values. This procedure makes our calculation *an adjustment rather than a prediction*. However, we view nuclear matter as a means of preparing a potential for use in Hartree-Fock calculations, and consider it significant that it is possible to simultaneously fit the phase shifts, have rapid convergence of perturbation theory, and agreement with the singlet nuclear-matter results of a more complete calculation which is known to give saturation.[18] In addition, the harmonic-oscillator matrix elements (Sec. V) are simple and should be useful in Hartree-Fock calculations for both spherical and deformed nuclei.

Our fits should be improved by addition of the one-pion-exchange potential, which is readily accommodated by the local part $V_1(r)$. The assumption of zero static repulsion $D=0$ is perhaps too strict and the introduction of some static repulsion could be helpful. The determination of the two-nucleon wave function at short distances and the magnitude of off-energy-shell matrix elements is, of course, an experimental problem; we believe that this smooth potential form may also be useful in analysis of such experiments.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge several helpful conversations with Professor M. Baranger and Professor E. Baranger, Dr. S. Krieger and Dr. K. Amos. The authors also wish to thank Professor H. A. Bethe, Dr. D. Sprung, and R. V. Reid for providing useful information about nuclear-matter calculations. Use of the University of Pittsburgh Computor under Grant G-11309 is also acknowledged.

APPENDIX

The harmonic-oscillator matrix elements of this velocity-dependent interaction [Eqs. (1), (2), and (3)] are given by the finite sum

$$\langle\varphi_{nlm}|V|\varphi_{n'l'm}\rangle=\delta_{ll'}\delta_{mm'}N_{nl}N_{n'l'}\times\sum_{k=0}^{n/n'}\frac{(-1)^k\Gamma(n+n'+l+\frac{3}{2}-k)\mathfrak{M}_k(n,n',l)}{(n-k)!(n'-k)!k!} \quad \text{(A1)}$$

for the normalization $N_{nl}^2=2\beta^3\Gamma(n+1)/[\Gamma(n+l+\frac{3}{2})]$. The index k takes on all integer values up to n or n', whichever is smaller. Following the steps outlined in Sec. V, one finds $\mathfrak{M}_k(n,n',l)$ as a function of the potential parameters V_1, V_2, a, b, and s and of the oscillator length $\beta^{-1}=(2\hbar/m\omega)^{1/2}$,

$$\begin{aligned}\mathfrak{M}_k(n,n',l)&=\mathfrak{M}_k(n',n,l)\\&=-\tfrac{1}{2}V_1\beta^{2l}a^{2(n+n')-4k}(a^2-\beta^2)^k\\&\quad\times(a^2+\beta^2)^{k-n-n'-l-\frac{3}{2}}+(V_2b^{-2}\beta^{-1})\\&\quad\times\{M_k(n,n',l)+M_k(n',n,l)\}.\end{aligned} \quad \text{(A2)}$$

The function $M_k(n,n',l)$, the nonlocal contribution to Eq. (19), is expressed in terms of the quantities t, U, and V as

$$M_k(n,n',l)=\frac{(V-U-1)^k(2t^{-1}-1)^n(V-U)^{n-k}(V-1)^{n'-k}}{V^{n+n'+l+\frac{3}{2}-k}t^{l+\frac{3}{2}}}\left\{\frac{kt^2U^2}{2(V-U-1)}-\frac{(n+n'+l+\frac{3}{2}-k)}{2Vt^2}+(l+\tfrac{3}{2})t^{-1}+2nU+\frac{(n-k)U^2t^2}{2(V-U)}+\frac{(n'-k)}{2t^2(V-1)}\right\}, \quad \text{(A3)}$$

where

$$\begin{aligned}t&=1+2s^2\beta^2,\\U&=t^{-1}(2-t)^{-1},\\V&=\tfrac{1}{2}(1+t^{-1})+\beta^{-2}b^2.\end{aligned} \quad \text{(A4)}$$

Computer evaluation of Eqs. (A1) to (A4) permits rapid calcultaion of these harmonic-oscillator matrix elements.

[25] M. Scadron and S. Weinberg, Phys. Rev. **133**, B1589 (1964).