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Static Quadrupole Moment of the First 2^+ State of Vibrational Nuclei*

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It is shown that reasonable models of vibrational nuclei can be constructed that explain the large magnitude of the static quadrupole moment Q_2 of the first excited 2^+ state of ^{114}Cd, recently derived from the analyses of the Coulomb-excitation experiments. Since the models considered are kept in very simplified forms, in order to make their qualitative implications as transparent as possible, they do not necessarily explain all the other known properties in this nculeus. Nevertheless, it seems that there is room for improving these models while still predicting large Q_2, and thus it is concluded that one can continue to consider ^{114}Cd as a vibrational nucleus, and not a rotational nucleus, in spite of the large magnitude of Q_2.

I. INTRODUCTION

CADMIUM–114 has long been known as one of the most typical quadrupole-type vibrational nuclei. The first excited state is 2^+ and there is a strong $B(E2)$ value between it and the ground state. The next higher states are 0^+, 2^+, and 4^+ triad states whose energy, and the $E2$ (and $M1$) transitions from them to the first 2^+ and the ground states, have every characteristic of a vibrational nucleus.[1]

Some time ago Breit *et al.*[2] suggested that it is possible to measure the static quadrupole moment of an excited state of a nucleus by utilizing the reorientation effect in the course of the Coulomb excitation of each state of interest. In the past year or two, such experiments were performed at at least three laboratories: Caltech,[3] Oak Ridge,[4] and Aldermaston,[5] and the static quadrupole moment Q_2 of the first 2^+ state of ^{114}Cd has been derived. The reported values differ somewhat from each other, partly because of different experimental setups and partly because of the difference in the theories used in deriving Q_2 from the observed excitation cross sections. Nevertheless we may safely say that the experimental value of Q_2 can be summarized as

$$Q_2^{(\mathrm{exp})} = -(0.50 \pm 0.25)\ \mathrm{b}. \qquad (1)$$

The value of $Q_2^{(\mathrm{exp})}$ as given in (1) is of surprisingly large magnitude, because if ^{114}Cd is assumed to be an ideal vibrational nucleus the expected value of Q_2 is zero (see Sec. IIA for a more detailed argument). The magnitude of $Q_2^{(\mathrm{exp})}$ is close to $Q_2 = -0.7$ b, a value which is obtained if ^{114}Cd is assumed to be a rotational nucleus (see Sec. IIA), and is thus the maximum possible theoretical magnitude that one expects to have, at least within the framework of simple-minded phenomenological models. Note, however, that the spectrum and electromagnetic transition probabilities known for ^{114}Cd are very much of the nature expected for a vibrational nucleus, and not for a rotational nucleus.

[*Note added in proof.* Recently P. Stoler, M. Slagowitz, W. Makofske, and T. Kruse (Phys. Rev., to be published) observed the (p,p_0) and (p,p') cross sections from various Sm isotopes ranging from $A=144$ to $A=154$. Of these the lightest elements are spherical while the heaviest ones are rotational nuclei. Stoler *et al.* found

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[1] For the experimentally known spectrum and the electromagnetic transition probabilities in ^{114}Cd see F. K. McGowan, R. L. Robinson, P. H. Stelson, and J. L. C. Ford, Jr., Nucl. Phys. **66**, 97 (1965); D. Eccleschall, P. M. Hinds, M. J. L. Yates, and N. MacDonald, *ibid.* **37**, 377 (1962).

[2] G. Breit, R. L. Gluckstern, and J. E. Russell, Phys. Rev. **103**, 727 (1956).

[3] J. de Boer, R. G. Stokstad, G. D. Symons, and A. Winther, Phys. Rev. Letters **14**, 564 (1964).

[4] P. H. Stelson, W. T. Milner, J. L. C. Ford, Jr., F. K. McGowan, and R. L. Robinson, Bull. Am. Phys. Soc. **10**, 427 (1965).

[5] J. J. Simpson, D. Eccleshall, and M. J. K. Yates (to be published).

that the observed cross sections from these two groups differ markedly from each other, the difference being in the way predicted by the coupled-channel calculations: [T. Tamura, Rev. Mod. Phys. **37**, 679 (1965)]. This result indicates that the analysis of (p,p') data gives a powerful tool in telling whether a given nucleus is spherical or rotational. We then recall that we already performed a (p,p') analysis from ^{114}Cd, assuming that it is a spherical nucleus and obtained good agreement with experiment,[10] which will thus supply us with another support to assume that ^{114}Cd is basically a spherical nucleus.]

Nevertheless ^{114}Cd is not an ideal (i.e., purely harmonic) vibrational nucleus, and many models have been proposed in the past to explain the observed deviations of various quantities from what are expected for an ideal vibrational nucleus. Therefore what one naturally asks is whether the models that explained such deviations successfully may also successfully explain the large magnitude of Q_2. As is seen in Sec. II, however, none of these models give sufficiently large Q_2.

It is the purpose of the present paper to show, in spite of this situation, that it is still possible to construct models within the framework of the vibrational nucleus that give sufficiently large magnitude of Q_2. In order to make such an argument conclusive and transparent, it is desirable to make the models as simple as possible. Such simplified models are considered in Sec. III, and it is shown there that a Q_2 of large magnitude can in fact be derived.

Because of their simplified nature, it is expected that the models considered in Sec. III will not necessarily explain all the known properties of ^{114}Cd. Nevertheless the results obtained in Sec. III are encouraging, and they indicate how the more detailed models are to be developed in the future. This is discussed in Sec. IV. The Q_2 values in some other vibrational nuclei are also discussed there.[6]

II. EVALUATION OF Q_2 IN TERMS OF KNOWN MODELS

The present section is devoted to the evaluation of Q_2 in terms of models which have been proposed by various authors for the description of the known properties of ^{114}Cd. In Sec. II.A phenomenological models are treated; in Sec. II.B, microscopic models.

A. Phenomenological Models

(1) *Harmonic Vibrational Model*

In this model, Q_2 is zero.[7] This is a trivial consequence of the fact that the quadrupole-moment operator is a linear combination of a creation and an annihilation operator of a phonon, and thus its diagonal matrix elements vanish with respect to states that have definite numbers of phonons.

(2) *Shell Model*

The simplest interpretation in the shell model[8] of the first excited 2^+ state in ^{114}Cd is that it is the 2^+ state that belongs to the proton configuration $(g_{9/2})^{-2}$. In general, the quadrupole moment of a state with spin I that belongs to a configuration $(j)^{\pm 2}$ (i.e., a two-particle or two-hole configuration) is given by

$$Q_I=\pm 2e(-)^{I-2}\left[\frac{16\pi}{5}\frac{(2I-1)I(2I+1)}{(I+1)(2I+3)}\right]^{1/2}\times\langle j\|r^2Y_2\|j\rangle W(jIjI;j2).$$

Putting $j=\frac{9}{2}$ and $I=2$ we obtain

$$Q_2=-0.4848\langle r^2\rangle.$$

By using $\langle r^2\rangle=\frac{3}{5}R_0^2$ with $R_0=1.2A^{1/3}$ F, we obtain for ^{114}Cd

$$Q_2=-0.10\text{ b}. \tag{2}$$

Consideration of the effective charge may increase the magnitude of Q_2 slightly over that of Eq. (2). If more complicated configurations are considered, the magnitude will become smaller than that of (2), as is exemplified by the random-phase-approximation (RPA) calculation which will be presented in Sec. IIB.

(3) *Rotational Model*

Assuming that the 2^+ state concerned is the 2^+ member of the rotational[7,9] band bulit upon the ground state, the Q_2 is obtained from the well-known relation:

$$Q_2=-(2/7)Q_0, \tag{3a}$$

where Q_0, the intrinsic quadrupole moment, is defined by

$$Q_0=3(5\pi)^{-1/2}ZeR_0^2\beta[1+0.16\beta+O(\beta^2)]. \tag{3b}$$

From analyses of the Coulomb excitation[1] and (p,p') experiments,[10] the value of β is known to be $\beta=0.2$. Thus, we get

$$Q_2=-0.70\text{ b}. \tag{4}$$

(4) *Wilets-Jean Model (γ-Unstable Model)*

In order to give a clue to explain the anharmonicity in the experimental spectrum of ^{114}Cd and other vibra-

[6] Preliminary accounts of the present paper have been reported; T. Tamura and T. Udagawa, Phys. Rev. Letters **15**, 765 (1965); in Proceedings of the Topical Conference on Spin and Parity Assignment, Gatlinburg, 1965, p. 453 (unpublished).

[7] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, No. 14 (1952); A. Bohr and B. R. Mottelson, *ibid.* **27**, No. 16 (1953).

[8] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).

[9] We include the rotational model here, in spite of the fact that it is quite different from the vibrational model, since it is the only known model, except those considered in Sec. III, that predicts large enough Q_2.

[10] M. Sakai *et al.*, Phys. Letters **8**, 197 (1964); M. Sakai and T. Tamura, *ibid.* **10**, 323 (1964).

tional nuclei, Wilets and Jean[11] modified the simple-harmonic vibrational model by replacing the potential-energy term[7] $V=\frac{1}{2}C\beta^2$ by

$$V=\tfrac{1}{2}C(\beta-\beta_0^2). \tag{5}$$

This model succeeded in lifting partially the degeneracy of the two-phonon triads and in giving the correct ratio of the energies of the lowest two 2^+ states, but as is easily seen from the fact that the potential (5) is still independent of the shape parameter γ, the Q_2 predicted by this model is zero.

(5) *Tamura-Komai Model*

In this model,[12] in addition to the Wilets-Jean potential (5), anharmonic potentials that depend on γ were also considered. A potential which was found to be particularly suited to the explanation of the spectrum and $B(E2)$ ratios of ^{114}Cd was

$$V'=-\tfrac{1}{2}\bar{k}(\beta-\bar{\beta})\cos 3\gamma. \tag{6}$$

Treating this potential as a perturbation, Q_2 will be given as

$$Q_2=(2/35)^{1/2}2f\langle 2_2^+\|\hat{Q}\|2_1^+\rangle, \tag{7}$$

where $|2_1^+\rangle$ and $|2_2^+\rangle$ denote the one- and two-phonon states (in the sense of the Wilets-Jean model), while f is the amplitude of mixing of the two-phonon state into the one-phonon state. $\hat{Q}$ in Eq. (5) is the quadrupole-moment operator. Choosing the parameters in (6) so that the experimental spectrum and $B(E2)$ values are best fitted, we obtain

$$Q_2=0.003\text{ b}. \tag{8}$$

The magnitude is very small and the sign is wrong.

(6) *Davidov-Filippov Model*

The expression for Q_2 in this model is found on p. 242 of Ref. 13;

$$Q_2=-Q_0\frac{6\cos 3\gamma_0}{7[9-8\sin^2 3\gamma_0]^{1/2}}. \tag{9}$$

Here Q_0 is the same as that given in Eq. (3b). The parameter γ_0 may be determined from the known level structure of ^{114}Cd. Using $\gamma_0=26.75°$ determined this way, we get

$$Q_2=-0.32\text{ b}. \tag{10}$$

(7) *Goldhaber-Weneser Model*

Based on the idea proposed by Goldhaber and Weneser,[14] MacDonald[15] made detailed numerical calculations of spectra and transition probabilities of vibrational nuclei, and found that (after an error made by Raz[16] in a previous calculation of similar content is corrected) this model predicts various known properties of ^{114}Cd fairly well. MacDonald also evaluated Q_2 and found that

$$Q_2=-0.30\text{ b}. \tag{11}$$

(8) *Summary*

Summarizing this subsection, we see that the predicted values of Q_2 are too small in magnitude, although the values obtained with the models (6) and (7) are barely within the lower boundary of the experimental value.[17]

B. Microscopic Models

(1) *Hamiltonian*

The calculations in this subsection are based on the pairing-plus-quadrupole force model,[18] where the pairing force is treated in the BCS approximation.[19] After performing the Bogoliubov-Valatin transformation,[20] the Hamiltonian may be written as[21,22]

$$H=H_0+H_Q, \tag{12}$$

$$H_0=\sum_{jm}E_j d_{jm}^{\dagger}d_{jm}, \tag{13a}$$

$$H_Q=-(\tfrac{1}{2}G_0)\textstyle\sum_\mu Q_{2\mu}^{\dagger}Q_{2\mu}, \tag{13b}$$

where H_0 is the quasiparticle Hamiltonian, while H_Q is the transformed quadrupole force, in which we have assumed that the coupling strengths of the proton-proton, neutron-neutron, and proton-neutron interactions are all equal to G_0. In (13a), $d_{jm}^{\dagger}$ and d_{jm} are the quasiparticle creation and annihilation operators, respectively, of a particle in a single-particle state labeled by (jm), while E_j is the corresponding quasiparticle energy. In (13b), $\hat{Q}_{2\mu}$ is the mass quadrupole-moment operator, which can be expressed as a sum of two terms, one involving only the pair creation and annihilation operators $A_{2\mu}{}^{\alpha\dagger}$ and $A_{2\mu}{}^{\alpha}$ (which are referred to as "pair operators" in the following), and the other involv-

[11] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).
[12] T. Tamura and L. G. Komai, Phys. Rev. Letters **3**, 344 (1959).
[13] A. S. Davidov and G. T. Filippov, Nucl. Phys. **8**, 237 (1958).
[14] G. S. Goldhaber and J. Weneser, Phys. Rev. **98**, 212 (1955).
[15] N. MacDonald, Nucl. Phys. **48**, 500 (1963).
[16] B. J. Raz, Phys. Rev. **114**, 1116 (1959); **128**, 2622 (1963).
[17] It may be that the lower limit for Q_2 given in Eq. (1) is too small. The experimental value reported in Ref. 3, e.g., is $Q_2=-(0.70\pm0.21)$ b, and if this value is taken, the predictions of the models (6) and (7) clearly disagree with experiment. A source of the large uncertainty in experimental Q_2 in (1) is the theoretical uncertainty in evaluating the $E1$ contribution to the Coulomb excitation cross section via giant dipole resonance, as pointed out by J. Eichler [Phys. Rev. **133**, B1162 (1964)]. According to N. MacDonald [Phys. Letters **10**, 334 (1964)], however, this contribution is likely to be small so long as we understand that the ^{114}Cd is a vibrational nucleus.
[18] See, e.g., L. S. Kisslinger and R. A. Sorensen, Rev. Mod. Phys. **35**, 853 (1963).
[19] T. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[20] N. N. Bogoliubov, Nuovo Cimento **7**, 794 (1958); J. G. Valatin, *ibid.* **7**, 843 (1958).
[21] T. Tamura and T. Udagawa, Progr. Theoret. Phys. (Kyoto) **26**, 947 (1961).
[22] T. Tamura and T. Udagawa, Nucl. Phys. **53**, 33 (1964).

ing the scattering operators $\mathfrak{A}_{2\mu}{}^{\alpha}$, which are defined shortly. Thus

$$\hat{Q}_{2\mu}=\hat{Q}_{2\mu}{}^{(c)}+\hat{Q}_{2\mu}{}^{(s)}, \tag{14}$$

$$\hat{Q}_{2\mu}{}^{(c)}=-\sum_{\alpha} g_c(\alpha)(A_{2\mu}{}^{\alpha\dagger}+(-)^{\mu}A_{2-\mu}{}^{\alpha}), \tag{15a}$$

$$\hat{Q}_{2\mu}{}^{(s)}=\sum_{\alpha} g_s(\alpha)\mathfrak{A}_{2\mu}{}^{\alpha}, \tag{15b}$$

with

$$A_{2\mu}{}^{\alpha}\equiv A_{2\mu}{}^{(12)\dagger}=D_{12}{}^{-1}\sum_{m_1m_2}\langle j_1j_2m_1m_2|2\mu\rangle \times d_{j_2m_2}{}^{\dagger}d_{j_1m_1}{}^{\dagger}, \tag{16a}$$

$$\mathfrak{A}_{2\mu}{}^{\alpha}\equiv\mathfrak{A}_{2\mu}{}^{(12)}=D_{12}{}^{-1}\sum_{m_1m_2}(-)^{j_2-m_2}\langle j_1j_2m_1-m_2|2\mu\rangle \times d_{j_1m_1}{}^{\dagger}d_{j_2-m_2}, \tag{16b}$$

$$D_{12}=[1+\delta_{j_1j_2}]^{1/2}, \tag{16c}$$

$$g_c(\alpha)\equiv g_c(12)=5^{-1/2}D_{12}{}^{-1}\langle j_1\|r^2Y_2\|j_2\rangle \times(U_{j_1}V_{j_2}+U_{j_2}V_{j_1}), \tag{17a}$$

$$g_s(\alpha)\equiv g_s(12)=5^{-1/2}D_{12}{}^{-1}\langle j_1\|r^2Y_2\|j_2\rangle \times(U_{j_1}U_{j_2}-V_{j_1}V_{j_2}). \tag{17b}$$

If (14) is inserted into (13b), H_Q is rewritten as

$$H_Q=H_Q{}^{(c)}+H_Q{}^{(31)}+H_Q{}^{(s)}, \tag{18}$$

$$H_Q{}^{(c)}=-\tfrac{1}{2}G_0\textstyle\sum_{\mu} Q_{2\mu}{}^{(c)\dagger}Q_{2\mu}{}^{(c)}, \tag{19a}$$

$$H_Q{}^{(31)}=\tfrac{1}{2}G_0\textstyle\sum_{\mu}(Q_{2\mu}{}^{(c)\dagger}Q_{2\mu}{}^{(s)}+\text{c.c.}), \tag{19b}$$

$$H_Q{}^{(s)}=-\tfrac{1}{2}G_0\textstyle\sum_{\mu} Q_{2\mu}{}^{(s)\dagger}Q_{2\mu}{}^{(s)}. \tag{19c}$$

(2) *The Static Quadrupole Moment Q_2*

If the wave function of the 2^+ state is denoted as $|\Psi_{2M}\rangle$, the quadrupole moment Q_2 is expressed as

$$Q_2=[16\pi/5]^{1/2}\langle\Psi_{22}|\mathfrak{M}(E2,0)|\Psi_{22}\rangle. \tag{20}$$

In (20) $\mathfrak{M}(E2,\mu)$ is the $E2$ transition operator, which can be written, if the mass quadrupole-moment operator $Q_{2\mu}$ is decomposed into the proton and neutron parts, $\hat{Q}_{2\mu}(p)$ and $Q_{2\mu}(n)$, respectively, as

$$\mathfrak{M}(E2,\mu)=e_pQ_{2\mu}(p)+e_nQ_{2\mu}(n), \tag{21}$$

and

$$e_p=e+e_{\text{eff}} \qquad e_n=e_{\text{eff}}, \tag{22}$$

where e_{eff} is the effective charge.[18] As in Eq. (14), each of $\hat{Q}_{2\mu}(p)$ and $\hat{Q}_{2\mu}(n)$ can be decomposed into two terms as

$$\hat{Q}_{2\mu}(p)=\hat{Q}_{2\mu}{}^{(c)}(p)+\hat{Q}_{2\mu}{}^{(s)}(p), \tag{23a}$$

$$\hat{Q}_{2\mu}(n)=\hat{Q}_{2\mu}{}^{(c)}(n)+\hat{Q}_{2\mu}{}^{(s)}(n). \tag{23b}$$

Inserting (23a) and (23b) into (21), $\mathfrak{M}(E2,\mu)$ is now re-expressed as

$$\mathfrak{M}(E2,\mu)=\mathfrak{M}^{(c)}(E2,\mu)+\mathfrak{M}^{(s)}(E2,\mu), \tag{24}$$

$$\mathfrak{M}^{(c)}(E2,\mu)=e_p\hat{Q}_{2\mu}{}^{(c)}(p)+e_n\hat{Q}_{2\mu}{}^{(c)}(n), \tag{25a}$$

$$\mathfrak{M}^{(s)}(E2,\mu)=e_p\hat{Q}_{2\mu}{}^{(s)}(p)+e_n\hat{Q}_{2\mu}{}^{(s)}(n). \tag{25b}$$

(3) *Evaluation of Q_2 in Random-Phase Approximation (RPA) and Higher Random-Phase Approximation (HRPA)*

Since HRPA[22] includes RPA[21] as a limit, it is convenient to write down the expression of Q_2 for HRPA first. In HRPA[22] the wave function $|\Psi_{2M}\rangle$ is assumed to have the form

$$|\Psi_{2M}\rangle=B_{2M}{}^{\dagger}|\Psi_0\rangle, \tag{26a}$$

$$B_{2M}{}^{\dagger}=\sum_{\alpha}\{a(\alpha)A_{2M}{}^{\alpha\dagger}+b(\alpha)(-)^{M}A_{2-M}{}^{\alpha}\} +\sum_{\alpha\le\beta}\{a(\alpha,\beta)\sum_{\nu\nu'}\langle 22\nu\nu'|2M\rangle A_{2\nu}{}^{\alpha\dagger}A_{2\nu'}{}^{\beta\dagger} +b(\alpha,\beta)\sum_{\nu\nu'}(-)^{M}\langle 22\nu\nu'|2-M\rangle A_{2\nu}{}^{\alpha}A_{2\nu'}{}^{\beta}\}, \tag{26b}$$

where $|\Psi_0\rangle$ denotes the ground state while α and β denote a pair of single-particle states.

Inserting (24) and (26a) into (20), Q_2 is rewritten as

$$Q_2=[16\pi/5]^{1/2}\langle\Psi_0|B_{22}\mathfrak{M}^{(s)}(E2,0)B_{22}{}^{\dagger}|\Psi_0\rangle +[16\pi/5]^{1/2}\langle\Psi_0|B_{22}\mathfrak{M}^{(c)}(E2,0)B_{22}{}^{\dagger}|\Psi_0\rangle \equiv Q_2{}^{(1)}+Q_2{}^{(2)}. \tag{27}$$

In evaluating the matrix elements that appear in (27), we shall retain only the dominant contributions. Thus, in calculating $Q_2{}^{(1)}$ we consider only the contributions from the one-pair modes in $B_{22}{}^{\dagger}$ and B_{22} [i.e., we consider only the first term of (26b)], while in calculating $Q_2{}^{(2)}$ we retain only terms that are nonvanishing when the boson commutation relations are assumed for the pair operators $A^{\dagger}$ and A. The resulting expression is given as

$$Q_2=Q_2{}^{(1)}+Q_2{}^{(2)}, \tag{28}$$

$$Q_2{}^{(1)}=4(10\pi/7)^{1/2}\sum_{\tau=p,n}e_{\tau}\sum_{123}{}^{(\tau)}D_{12}D_{13}D_{23}g_s(12) \times\{a(13)a(23)+b(13)b(23)\}W(j_1j_322;2j_2), \tag{28a}$$

$$Q_2{}^{(2)}=8(2\pi/7)^{1/2}\sum_{\tau=p,n}e_{\tau}\sum_{\alpha\le\beta}{}^{(\tau)} \times\{a(\alpha,\beta)[a(\alpha)g_c(\beta)+a(\beta)g_c(\alpha)] +b(\alpha,\beta)[b(\alpha)g_c(\beta)+b(\beta)g_c(\alpha)]\}. \tag{28b}$$

Because of the approximations made in evaluating $Q_2{}^{(1)}$, (28a) is exactly the same expression as one gets for Q_2 with RPA, although the values of the expansion coefficients $a(\alpha)$ obtained in RPA differ from those obtained in HRPA.

(4) *Numerical Calculation in RPA*

In RPA we neglect the last two terms of (18), and the second terms of (26b) and (28). Under these simplifications the energies of the 2^+ states ω_n are obtained as

solutions of the following dispersion equation:

$$G_0 \sum_\alpha \frac{2E_\alpha g_c^2(\alpha)}{E_\alpha^2-\omega_n^2}=1, \quad (E_\alpha=E_{j_1}+E_{j_2}). \tag{29}$$

The energy of the first excited 2^+ state is of course the smallest solution ω_1 of (29). With this ω_1, the expansion coefficients $a(\alpha)$ and $b(\alpha)$ in the first term of (26b) are given analytically as

$$a(\alpha)=-\frac{1}{N}\frac{g_c(\alpha)}{E_\alpha-\omega_1}, \quad b(\alpha)=-\frac{1}{N}\frac{g_c(\alpha)}{E_\alpha+\omega_1},$$

$$N^2=\sum_\alpha \frac{4E_\alpha\omega_1 g_c^2(\alpha)}{(E_\alpha^2-\omega_1^2)^2}. \tag{30}$$

By inserting (30) into (28) the value of Q_2 is computed, and turns out to be $Q_2=-0.077$ b.

(5) *Numerical Calculation in HRPA*

In Ref. 22 the result of a detailed numerical calculation based on HRPA was given, which showed a fairly good agreement with experiment of the energies of the one- and two-phonon states and also of a few low-lying quasiparticle states in ^{114}Cd. The $B(E2)$ values were also in fairly good agreement (although a critical argument will be given on this point in the next section). In the course of this calculation the expansion coefficients $a(\alpha)$, $b(\alpha)$, $a(\alpha,\beta)$, and $b(\alpha,\beta)$ that appeared in (26) have been calculated, and thus we can use them in (28) to evaluate Q_2. The resulting value of Q_2 is found to be $Q_2=-0.082$ b.

(6) *Summary*

Summarizing the arguments of this subsection, we conclude that both RPA and HRPA give the correct sign to Q_2, but with too small magnitudes.

III. MODELS THAT PREDICT LARGE Q_2

As we have seen in Sec. II, all the models so far proposed for the vibrational nuclei have achieved varying success in explaining known properties (other than Q_2) of ^{114}Cd but have failed to predict sufficiently large values for Q_2. The rotational model predicts a sufficiently large value, but as was noted in Sec. I the known spectrum and various transition probabilities of ^{114}Cd have all the characteristics of a vibrational nucleus rather than a rotational nucleus. Thus, to be consistent, we believe that there must exist one or more models that predict a sufficiently large theoretical value for Q_2 without going outside the framework of the vibrational nucleus.

^{114}Cd is a rather typical vibrational nucleus, as we have emphasized in Sec. I; but it is nonetheless not an ideal vibrational nucleus. The energy spectrum shows deviation from harmonicity, and the relative magnitudes of various $B(E2)$ transition probabilities are not exactly those predicted by the ideal vibrational model. It would then be a natural guess that the large magnitude of Q_2 is somehow correlated with the deviations from harmonicity of some of the above-mentioned quantities.

The quantity whose value deviates considerably from the ideal value, and thus is of particular concern to us, is the ratio R_1 which is defined as

$$R_1=B(E2;\ 2_2\rightarrow 2_1)/B(E2;\ 2_1\rightarrow \text{ground}), \tag{31}$$

where 2_1 and 2_2 mean, respectively, the first and the second excited 2^+ states. Experimentally it is known that[1]

$$R_1^{(\text{exp})}=1.2\pm 0.2, \tag{32}$$

which is to be compared with the ideal value $R_1=2$.

In order to see the relation between the large magnitude of Q_2 and the large deviation of $R_1^{(\text{exp})}$ from 2, we shall consider a phenomenological model, which is very simple and thus may not necessarily succeed in explaining all the known properties of ^{114}Cd, but nonetheless can demonstrate the relation very clearly.

A. Phenomenological Model (Model A)

Let us assume that the wave functions $\psi(2_1)$ and $\psi(2_2)$ of the first and second excited 2^+ states are given as orthogonal linear combinations of those of the one- and two-phonon harmonic-vibrational 2^+ states, $|1\rangle$ and $|2\rangle$. Thus

$$\begin{aligned}\psi(2_1)&=a_1|1\rangle+a_2|2\rangle,\\ \psi(2_2)&=-a_2|1\rangle+a_1|2\rangle,\end{aligned} \tag{33a}$$

with

$$a_1^2+a_2^2=1. \tag{33b}$$

In this model R_1 and Q_2 can be expressed as

$$R_1=2(2a_1^2-1)^2/a_1^2, \tag{34}$$

$$Q_2=(12/5)(7\pi)^{-1/2}a_1a_2ZR_0^2\beta, \tag{35}$$

where β is just the zero-point amplitude of the quadrupole-type surface vibration and is to be put equal to 0.2.[1,10]

On putting R_1 of (34) equal to 1.2, the mixing parameter a_1^2 (and a_2^2) is determined, and the result is

$$a_1^2=0.86 \quad (\text{and} \quad a_2^2=0.14) \tag{36}$$

which gives, by (35),

$$|Q_2|=0.58 \text{ b}. \tag{37}$$

This Q_2 has sufficiently large magnitude and thus indicates that the deviation of R_1 from 2 is indeed consistent with large Q_2.

Since the relative sign of the amplitudes a_1 and a_2 depends on the details of the interaction that causes the mixing of the one- and two-phonon states, the sign of the Q_2 is undetermined to the extent of this simple consideration. It does not seem difficult, however, to

think of an interaction of a reasonable form that predicts the negative sign to Q_2, and thus we can conclude that we have succeeded in constructing a model that explains the observed value of Q_2.

B. Microscopic Model (Model B)

As we have seen in Sec. II.B, neither RPA nor HRPA gave sufficiently large magnitude for Q_2, though the sign was predicted correctly. On the other hand we saw in the preceding subsection that we can conceive of a phenomenological model that can explain the large magnitude of Q_2 in a very clear and natural way.

The reason for the success of the model A was that we considered a mixture of the one- and two-phonon states in order to describe the wave function of the 2_1^+ state. If we note this fact, then it is by no means surprising that RPA failed to give a large magnitude for Q_2, since the wave function of the 2_1^+ state in RPA is just that of the one-phonon state. By the same reasoning, however, it is quite embarrassing that HRPA has also failed, since in this model the 2_1^+-state wave function has been described from the beginning as a linear combination of those of the one- and two-phonon states; see Eq. (26b).

Faced with this difficulty, we recall that, though our previous HRPA calculation[22] predicted the energy spectrum and various transition probabilities fairly successfully, it gave nevertheless a value 0.85 for R_1, which is too small compared with experiment by about a factor of 0.7. In Ref. 22 a possible origin of this small R_1 was discussed, and it was argued that perhaps it lay in the fact that the wave function assumed in (26b) was too restrictive.

Indeed, if we take the first term of (26b) as the operator (to be operated on the ground state) that creates the one-phonon state, then the operator to create the two-phonon state must be a bilinear product of these one-phonon creation operators (with appropriate Clebsch-Gordan coefficient and normalization factor). In other words the term which corresponds to the second term of (26b) is to be of the form (writing somewhat symbolically) $(A+A^\dagger)(A+A^\dagger)=AA+A^\dagger A^\dagger+AA^\dagger+A^\dagger A$. Compared with this, it is seen that the terms of the form $AA^\dagger+A^\dagger A$ are missing in (26b), and this fact might have been the origin of the too small R_1. There was a mathematical difficulty which forced us to neglect this type of term in the HRPA,[22] but it is by no means impossible to include them, if we are going to treat the mixing of the one- and two-phonon states in a simple perturbation calculation. We shall show that this can be done very easily, and that the resulting Q_2 has a sufficiently large magnitude. (For R_1 see the discussion in Sec. IV.)

The wave function $|\Psi_{2M}\rangle$ is now described, introducing a mixing amplitude f, as

$$|\Psi_{2M}\rangle=|\Phi_{2M}{}^{(1)}\rangle+f|\Phi_{2M}{}^{(2)}\rangle. \tag{38}$$

Here $|\Phi_{2M}{}^{(1)}\rangle$ and $|\Phi_{2M}{}^{(2)}\rangle$ are the wave functions of the one- and two-phonon states obtained in RPA, and thus are expressed—by using the one-phonon creation operator

$$B_{2M}{}^\dagger=\sum_\alpha\{a(\alpha)A_{2M}{}^{\alpha\dagger}+b(\alpha)(-)^M A_{2-M}{}^\alpha\}, \tag{39}$$

which is nothing but the first term of (26b)—as

$$|\Phi_{2M}{}^{(1)}\rangle=B_{2M}{}^\dagger|\Psi_0\rangle, \tag{40a}$$

$$|\Phi_{2M}{}^{(2)}\rangle=\frac{1}{\sqrt{2}}\sum_{\nu\nu'}\langle 22\nu\nu'|2M\rangle B_{2\nu}{}^\dagger B_{2\nu'}{}^\dagger|\Psi_0\rangle. \tag{40b}$$

In calculating the mixing amplitude f in (38) we first recall that the term $H_Q{}^{(31)}$ in (18) was left out and only $H_Q{}^{(c)}$ was used in RPA. We also note that the strength of $H_Q{}^{(c)}$ was fixed in order to give a correct value of ω_1 [in (29)], and this procedure also fixed the strength of $H_Q{}^{(31)}$. Therefore the perturbation calculation to mix $|\Phi_{2M}{}^{(2)}\rangle$ into $|\Phi_{2M}{}^{(1)}\rangle$ through $H_Q{}^{(31)}$ has no adjustable parameter. In the first-order perturbation calculation f is easily found to be

$$f=-(1/\omega_1)[\tfrac{2}{5}]^{1/2}G_0Q_2{}^{(s)}Q_2{}^{(c)}, \tag{41}$$

where $Q_2{}^{(s)}$ and $Q_2{}^{(c)}$ are defined by

$$Q_2{}^{(s)}=5(5)^{1/2}\sum_{123}D_{12}D_{13}D_{23}g_s(12)\times\{a(13)a(23)+b(13)b(23)\}\times W(j_1j_322;2j_2), \tag{42a}$$

$$Q_2{}^{(c)}=\sum_\alpha g_c(\alpha)\{a(\alpha)-b(\alpha)\}. \tag{42b}$$

The meaning of the notations that appear in (42) can be found in Sec. II.B.

Using (38) through (42) the evaluation of Q_2 is straightforward, in particular since we use the same approximations as those introduced in evaluating the matrix elements of (27). Thus, we obtain a quite similar result to that of HRPA of Sec. II.B, Q_2 being expressed as a sum of two contributions, one coming solely from the one-phonon state, and the other coming from the admixture of the two-phonon state to the one-phonon state. Furthermore, the first contribution is exactly the same as that in (28a). Thus the final result is

$$Q_2=Q_2{}^{(1)}+Q_2{}^{(2)\prime}, \tag{43}$$

where

$$Q_2{}^{(1)}=\text{right-hand side of Eq. (28a)} \tag{44a}$$

$$Q_2{}^{(2)\prime}=16[\pi/35]^{1/2}f\sum_{\tau=p,n}e_\tau\sum_\alpha{}^{(\tau)}g_c(\alpha)\times\{a(\alpha)-b(\alpha)\}. \tag{44b}$$

Using (41) and (44) and the values of various quantities that have already been fixed in Sec. II.B (1) (or

rather in Ref. 21), the evaluation of Q_2 in (43) is straightforward. We have evaluated it for three values of the effective charge e_{eff} and the result is summarized in Table I under the heading of model B, where the result in terms of HRPA is also given for comparison. (The values $e_{\text{eff}}=0.83$ and 1.2 are the ones that were required in getting the correct value for $B(E2; 2_1^+ \rightarrow \text{ground})$ in RPA[21] and HRPA.[22]) With $e_{\text{eff}}=0.83$, which must be taken for consistency here, we get $Q_2=-0.44$ b, which is somewhat smaller than that obtained in Sec. III.A, but still is sufficiently large to explain the experimental results. The sign is also given correctly. We thus find a second model that predicts Q_2 in agreement with experiment. It is to be noted that this model is not very different in its essence from the first model (model A). In both models the most important point is that the 2_1^+-state wave function is written as a linear combination of those of the one- and two-phonon states.

TABLE I. Calculated and experimental values of the quadrupole moment of the first excited 2^+ state of ^{114}Cd, in barns.

	HRPA			Model B			
e_{eff}	$Q_2^{(1)}$	$Q_2^{(2)}$	Q_2	$Q_2^{(1)}$	$Q_2^{(2)\prime}$	Q_2	$Q_2^{(\text{exp})}$
0.83	−0.054	−0.028	−0.082	−0.077	−0.362	−0.439	
1.0	−0.060	−0.031	−0.091	−0.079	−0.432	−0.511	0.50±0.25
1.2	−0.068	−0.035	−0.101	−0.082	−0.520	−0.602	

IV. DISCUSSION

As we have seen in Sec. III, there exist models that can give rise to sufficiently large theoretical magnitudes of Q_2, and this convinces us that we can treat ^{114}Cd within the framework of the vibrational model in spite of the large Q_2, at least so far as we are interested in the properties of the low-lying states.

The models considered in Sec. III are very crude, and not much argument about the properties of ^{114}Cd other than Q_2 has been made there. For example, we have made very little discussion about the energies or the transition probabilities in connection with the two-phonon states, and to be consistent such investigations certainly have to be made. Such a detailed investigation cannot be made with the simple models of Sec. III and have to wait for a more sophisticated extension of these models.

Still, within the framework of the models of Sec. III, it is possible to make semiquantitative, or at least qualitative arguments about the extent to which our models are consistent with the known data, and this we will now do.

In evaluating Q_2 in model A in Sec. III.A, we utilized the experimental knowledge of R_1 defined in (31). Another quantity whose value is characteristic of vibrational nuclei, and thus is important in identifying a nucleus as a good vibrational nucleus, is another ratio R_2 which is defined as

$$R_2=B(E2; 2_2 \rightarrow \text{ground})/B(E2; 2_2 \rightarrow 2_1); \quad (45)$$

that is, the (reduced) branching ratio of the $E2$ transitions from the 2_2^+ state to the ground and 2_1^+ states. For an ideal vibrational nucleus $R_2=0$, and in many actual vibrational nuclei R_2 differs from zero but still is very small. In particular in ^{114}Cd we have[1]

$$R_2^{(\text{exp})}=0.015\pm0.005. \quad (46)$$

We can evaluate R_2 in model A, since we know the wave functions of the 2_1^+ and 2_2^+ states. The result is

$$R_2^{(\text{A})}=a_2^2/2(2a_1^2-1)^2=0.14. \quad (47)$$

This value is about one order of magnitude too large compared with (46), and this constitutes one of the difficulties of model A as it stands. It should, however, be noted that model A is a one-parameter (a_1) theory in that only the mixture of one- and two-phonon states were considered, and further that there exists no value of a_1 that gives simultaneous agreement of R_1 and R_2 of Eqs. (34) and (47) with their respective experimental values. In a more refined calculation, which has to be made in the future, clearly a more complicated mixing must be considered, and if this were done we might get small values for R_2. Incidentally, it may be worthwhile to remark here that several phenomenological models[11,13,15] and a microscopic model[22] gave values for R_2 in good agreement with experiment; see Eq. (46).

As for the energies of the 2_2^+ and 2_1^+ states, we get $E(2_2)/E(2_1)=2.72$ in model A, which is large compared with the experimental value, 2.17. This is not a difficulty, however, since no shift of the ground-state energy has been considered yet. With the introduction of reasonable interactions between the ground and the excited 0^+ states, it is expected that the ground-state energy is depressed from what it was before the interaction was considered, and this certainly decreases the ratio $E(2_2)/E(2_1)$ from the above value. This interaction may also push up the first excited 0^+ state to slightly below the 2_2^+ state, in agreement with experiment. All these arguments, including further that of the energy of the first excited 4^+ state, can, however, be made more definitely only after more complicated calculations are made.

As for model B, the quantities discussed above are calculated as follows:

$$\begin{aligned} &E(2_2)/E(2_1)=2.15\ (2.17), \\ &R_1^{(\text{B})}=1.65\ (1.25\pm0.25), \\ &R_2^{(\text{B})}=0.047\ (0.015\pm0.005), \end{aligned} \quad (48)$$

where for ease of comparison the corresponding experimental values are given in parentheses. $R_2^{(\text{B})}$ is still too large compared with experiment, but is much smaller than $R_2^{(\text{A})}$. Indeed $R_2^{(\text{B})}$ is only a little larger than twice the upper limit of $R_2^{(\text{exp})}$, and thus embodies the characteristic smallness of R_2 in vibrational nuclei. $R_1^{(\text{B})}$ is also larger, but only very slightly so, than the upper limit of the experimental $R_1^{(\text{exp})}$.

The value of $Q_2=-0.44$ b given for model B in Sec. III.B may seem slightly too small to make this model satisfactory. It should be noted, however, that Q_2 in this model is a rather sensitive function of the position of the single-neutron state $1h_{11/2}$ in the shell model. In the calculation of Ref. 21, we assumed that $\epsilon_{1h_{11/2}}-\epsilon_{2d_{5/2}}=2.4$ MeV. If this energy difference is increased to 2.8 MeV and the calculation of Sec. III.B is repeated [which now requires $e_{\text{eff}}=0.904$ in order to fit the experimental value of $B(E2; 2_1^+ \rightarrow \text{ground})$] we get

$$Q_2=-0.586 \text{ b}, \tag{49}$$

which is of quite large magnitude.

As far as the above arguments are concerned, model B seems somewhat preferred to model A. The situation is different, however, if we consider the Q_2 values of other Cd isotopes. Recently in Oak Ridge experiments[4] similar to those for ^{114}Cd have been made for ^{112}Cd and ^{116}Cd, and the results indicate that[23]

$$|Q_2(^{112}\text{Cd})|<|Q_2(^{114}\text{Cd})|<|Q_2(^{116}\text{Cd})|, \tag{50}$$

the sign being always negative. Since it is known experimentally[1] that

$$R_1(^{112}\text{Cd})>R_1(^{114}\text{Cd})>R_1(^{116}\text{Cd}), \tag{51}$$

and the larger R_1 gives rise to the smaller Q_2 in model A, as is easily seen by comparing Eqs. (34) and (35) [for a fixed value of $B(E2; 2_1^+ \rightarrow \text{ground})$, which is known to be approximately the case[1]], we can conclude that model A is in accord with the experimental result (50).

On the other hand model B predicts that

$$|Q_2(^{112}\text{Cd}|\rangle|Q_2(^{114}\text{Cd})|\rangle|Q_2(^{116}\text{Cd})|, \tag{52}$$

which is just opposite to (50). The reason we get (52) is that in going from ^{112}Cd to ^{116}Cd, the neutron orbit $1h_{11/2}$ is more and more filled, and its large positive contribution to Q_2 increases accordingly, making Q_2 a smaller negative quantlty.

[23] P. H. Stelson (private communication).

In this connection it may be of interest to measure Q_2 in vibrational nuclei that are heavier than ^{114}Cd, for example, the Te isotopes. In these nuclei the above $1h_{11/2}$ orbit is more and more filled (in the sense of the BCS theory), and contributes a large positive value to Q_2, making Q_2 itself a positive quantity. In particular, model B indicates that the Te isotopes may have positive Q_2. Since (52) contradicts the experimental result (50), this argument should not be taken too seriously. Nevertheless such experiments still seem of great interest in clarifying the structure of the vibrational nuclei.

We conclude this section by considering the magnetic moment, μ_2, of the first 2^+ state of ^{114}Cd. In model A (plus the hydrodynamic model), we would get

$$\mu_2^{(\text{A})}=2Z/A=0.86 \text{ nm}, \tag{53}$$

while in the model B we expect to get

$$\mu_2^{(\text{B})}=2.38 \text{ nm}. \tag{54}$$

Since the difference between (53) and (54) is so large, the observation of μ_2 certainly would be quite useful in discriminating between models A and B [though it should be noted that we would not get the result (53) for model A, if the assumption of the hydrodynamic model were not used]. We have been informed that an experiment to observe μ_2 is under way.[24]

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[24] S. K. Bhattacherjee (private communication).