Tomasch Effect as a Probe of the Dispersion Relation of Electrons in "Gapless" Superconductors*

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(Received 18 April 1966)

The recent theory of McMillan and Anderson is used to discuss the effect of nonmagnetic impurities and depairing mechanisms on the period and damping of the Tomasch oscillations. These oscillations in the density of states seem to provide a unique way of directly measuring the dispersion relation of electronic excitations in superconductors [ω versus $\mathrm{Re}(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}\propto$ pseudomomentum k^*, where $\tilde{\omega}$ and Δ are the usual renormalized diagonal and off-diagonal self-energies]. We shall discuss the theoretical dispersion curves for films with a small concentration of magnetic impurities and for dirty films ($l\ll\xi_0$) in the presence of a parallel magnetic field. The two situations lead to *quite different* dispersion curves at small values of k^*, for the same value of the depairing parameter. Using the Tomasch effect to plot out the dispersion curves becomes more difficult as ω decreases, since the damping of the oscillations [$\propto\mathrm{Im}(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}$] increases.

VERY recently, Tomasch observed[1] small periodic oscillations in the quasiparticle tunneling current between two superconductors, one of which was relatively thick ($d\gtrsim 10^4$ Å). McMillan and Anderson[2] (MA) have suggested that this oscillatory dependence on the applied voltage results from the interference of particle-like (or hole-like) excitations being scattered off the back surface of the thick film with hole-like (or particle-like) excitations at the front tunneling surface. The effect depends critically on the surface scattering being off-diagonal, as from a localized change in the order parameter. McMillan and Anderson have noted that a careful measurement of the period of the Tomasch oscillations provides a seemingly unique way of determining the renormalization parameter $Z(\omega)$, which modifies the quasiparticle spectrum in a nontrivial way in strong-coupling metals (like Pb).

In the present note, we use MA's theory to consider the effect of nonmagnetic impurities and pair-breaking perturbations (such as paramagnetic impurities or a parallel magnetic field) on the period and damping of the Tomasch oscillations. As will become more apparent later on, the period of these oscillations enables one to plot out the dispersion relation of electronic excitations in superconductors,[3] gapless or otherwise. Apart from a remark at the end, we shall limit ourselves to isotropic, weak-coupling superconductors for simplicity.

The impurity averaged Green's function (in 2×2 Nambu space) is

$$\hat{G}_\omega{}^0(R)=-\frac{m}{2\pi R}\exp(i\Omega(\omega)R/v_F)\times\left\{\cos k_FR(\tau_3)+i\sin k_FR\left(\frac{\tilde{\omega}+\tilde{\Delta}\tau_1}{\Omega(\omega)}\right)\right\}, \quad (1)$$

where $\Omega(\omega)\equiv(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}$, R is the relative coordinate, and $\hbar=1$. Finally, $\tilde{\omega}$ and $\tilde{\Delta}$ are the renormalized energy and order parameter, respectively, being given by coupled equations. For magnetic impurities, we have[4]

$$\tilde{\omega}=\omega+\frac{i}{2\tau}\frac{u}{(u^2-1)^{1/2}}\left(1+\frac{\tau}{\tau_s}\right),$$
$$\tilde{\Delta}=\Delta+\frac{i}{2\tau}\frac{1}{(u^2-1)^{1/2}}\left(1-\frac{\tau}{\tau_s}\right). \quad (2)$$

Here $u\equiv\tilde{\omega}/\tilde{\Delta}$, τ is the normal-state relaxation time due to nonmagnetic impurities, and τ_s is that from spin-exchange scattering (in Born approximation). For a parallel magnetic field H, we find[5] to lowest order in $\tau\Delta$ that

$$\tilde{\omega}=\omega+\frac{i}{2\tau}\frac{u}{(u^2-1)^{1/2}}\left(1+\zeta(\Delta\tau)\frac{3}{2}\frac{u^2}{u^2-1}\right),$$
$$\tilde{\Delta}=\Delta+\frac{i}{2\tau}\frac{1}{(u^2-1)^{1/2}}\left(1+\zeta(\Delta\tau)\frac{1}{2}\frac{2u^2+1}{u^2-1}\right), \quad (3)$$

where

$$\zeta=(\tfrac{2}{3})(\tau/\Delta)(ev_F/c)^2\langle\mathbf{A}^2(\mathbf{x})\rangle_{\mathrm{space}}. \quad (4)$$

Actually, in the presence of a vector potential $\mathbf{A}(\mathbf{x})$, the Green's function is given by

$$\hat{G}_\omega(\mathbf{R})=\left[1-i\left(\frac{e}{2mc}\right)\frac{\mathbf{R}\cdot\mathbf{A}}{R}\frac{\partial}{\partial R}\frac{\partial}{\partial\tilde{\omega}}+\cdots\right]\hat{G}_\omega{}^0(R).$$

The anisotropic correction is of order $(\tau\Delta)^{1/2}$, but the effect on the Tomasch oscillations can be shown to be order $\tau\Delta$ and hence negligible.

We might note that in both cases, one has a formally identical equation for u, namely

$$\omega/\Delta=u(1-i\zeta(u^2-1)^{-1/2}). \quad (5)$$

* This work was supported by the U. S. Air Force, through Grant No. AF-AFOSR-610-64, Theory of Solids.
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[1] W. J. Tomasch, Phys. Rev. Letters **15**, 672 (1965); **16**, 16 (1966).
[2] W. L. McMillan and P. W. Anderson, Phys. Rev. Letters **16**, 85 (1966).
[3] W. J. Tomasch and T. Wolfram, Phys. Rev. Letters **16**, 352 (1966).
[4] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].
[5] K. Maki, Progr. Thoret. Phys. (Kyoto) **29**, 603 (1963); **31**, 731 (1964).

TABLE I. This is a summary of the asymptotic behavior of the $k_R^*=\mathrm{Re}(\varpi^2-\tilde{\Delta}^2)^{1/2}/v_F$ and $k_I^*=\mathrm{Im}(\varpi^2-\tilde{\Delta}^2)^{1/2}/v_F$ for the two types of "gapless" superconductors: A stands for paramagnetic impurities and B for magnetic fields or currents. In the latter case, we have $\zeta=\tau(v_FK)^2/6\Delta$, where K is the center-of-mass momentum of the Cooper pairs. We have introduced the abbreviations $x\equiv\omega/\Delta$ and $x_0\equiv\omega_0/\Delta=(1-\zeta^{2/3})^{3/2}$. The high-frequency results are correct to order $(\omega/\Delta)^{-4}$, and to lowest nonvanishing order in ζ.

			$\mathrm{Re}(\varpi^2-\tilde{\Delta}^2)^{1/2}/\Delta$	$\mathrm{Im}(\varpi^2-\tilde{\Delta}^2)^{1/2}/\Delta-1/(2\tau\Delta)$
High frequency	A		$x(1-\frac{1}{2}x^{-2}-(\frac{1}{8}-\frac{3}{2}\zeta^2)x^{-4})$	$\frac{1}{2}\zeta(1+2x^{-2}+2x^{-4})$
	B		$x(1-\frac{1}{2}x^{-2}-(\frac{1}{8}+\frac{3}{2}\zeta^2)x^{-4})$	$-\frac{1}{2}\zeta x^{-2}(1+x^{-2})$
Low frequency	$\zeta<1;\ \omega\gtrsim\omega_0$	A	$(\frac{2}{3})^{1/2}x_0^{1/6}(x-x_0)^{1/2}$	$\zeta^{1/3}(1-\frac{1}{2}\zeta^{2/3})$
		B	$-2(\frac{2}{3})^{1/2}x_0^{1/6}(x-x_0)^{1/2}$	$2\zeta^{1/3}(1-\frac{1}{2}\zeta^{2/3})$
	$\zeta>1;\ \omega\gtrsim 0$	A	$\dfrac{\zeta}{(\zeta^2-1)^{1/2}}x$	$\frac{1}{2}\zeta+0(x^2)$
		B	$\dfrac{\zeta}{(\zeta^2-1)^{1/2}}(1-3\zeta^{-2})x$	$\frac{3}{2}\zeta^{-1}+0(x^2)$

For magnetic fields the depairing parameter ζ is given by (4), while for magnetic impurities, we have $\zeta\equiv(\tau_s\Delta)^{-1}$. As is well known, typical equilibrium properties depend only on u, and not on $\tilde{\omega}$ and $\tilde{\Delta}$ separately. Transport properties, on the other hand, involve a generalized frequency-dependent mean free path[6] defined by $l(\omega)=v_F/2\,\mathrm{Im}\Omega(\omega)$.

Following MA, we obtain an additional contribution to the usual density of states, which to lowest order in $(k_Fd)^{-1}$ is given by (we neglect vertex corrections)

$$\delta N(x,\omega)=\left(\frac{m^2}{2\pi^2}\right)\delta\Delta\,\mathrm{Im}\left\{\left(\frac{u}{u^2-1}\right)\times\int_1^\infty dy\frac{\exp(iAy)}{y}\right\}. \quad (6)$$

Here $A\equiv 2(d-x)\Omega(\omega)/v_F$ and $\delta\Delta$ is the magnitude of the order parameter change at the back wall. That the tunneling measurements only probe the $x=0$ surface is, of course, implicit in treating the back surface ($x=d$) perturbation to first order and neglecting any perturbation from the $x=0$ surface. In the oscillatory region ($|A|\gg 1$), we can use well-known asymptotic expansions to rewrite (6) as

$$\delta N(x=0,\omega)=\left(\frac{m^2}{2\pi^2}\right)\delta\Delta\frac{e^{-a_I}}{(a_R^2+a_I^2)^{1/2}}\times\left\{\mathrm{Re}\left(\frac{u}{u^2-1}\right)\cos(a_R-\tan^{-1}(a_I/a_R))-\mathrm{Im}\left(\frac{u}{u^2-1}\right)\sin(a_R-\tan^{-1}(a_I/a_R))\right\}, \quad (7)$$

where $A\equiv a_R+ia_I\equiv(k_R^*+ik_I^*)2d=k^*\times 2d$. A glance at $\hat{G}_\omega{}^0(R)$ in Eq. (1) will show that it is natural to regard k_R^* as a pseudomomentum, measured with respect to the Fermi momentum k_F. The peaks in the Tomasch oscillations occur approximately at ω_n corresponding to $k_R^*=n(\pi/d)$. Clearly k_I^* is related to the damping, and thus this interpretation of k_R^* breaks down if $k_I^*\gtrsim k_R^*$. The generalized mean free path $l(\omega)\equiv 1/2k_I^*$ also occurs in the thermal conductivity and ultrasonic attenuation coefficients[6] of bulk materials.

Using $k^*=\tilde{\Delta}(u^2-1)^{1/2}/v_F$, we may easily find explicit expressions for k^* for the two types of pair-breaking mechanisms. We note that the two cases are quite *different* for the same values of ω and ζ. In Table I, we present some asymptotic expansions of k_R^* and k_I^*. Figures 1 and 2 give some examples of dispersion curves, these being essentially ω versus k_R^*.[7] The straight lines correspond to the dispersion curve for a normal metal, $\omega=(k^2-k_F^2)/2m\simeq v_Fk$. The curves are really only meaningful (and measurable using the Tomasch effect) when $k_R^*\gg k_I^*$. The qualitative behavior of the damping (k_I^* versus ω) may be seen from the examples in Fig. 3.

As a special case useful for orientation, we note that in the presence of nonmagnetic scattering alone, $k_R^*=(\omega^2-\Delta^2)^{1/2}/v_F$, $k_I^*=1/2l$ ($l=v_F\tau$), and $u=\omega/\Delta$. As a result, we may reduce (7) to

$$\delta N(x=0,\omega)=\left(\frac{m^2}{2\pi^2}\right)\delta\Delta\left(\frac{\omega\Delta}{\omega^2-\Delta^2}\right)\times\frac{\cos[2dk_R^*-\tan^{-1}(k_I^*/k_R^*)]}{[(2dk_R^*)^2+(d/l)^2]^{1/2}}e^{-d/l}. \quad (8)$$

As MA pointed out, the oscillations are strongly damped when $l\lesssim d$.

From a practical point of view, observation of the Tomasch oscillations in films containing paramagnetic

[6] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964); V. Ambegaokar and A. Griffin, *ibid.* **137**, A1151 (1965).

[7] P. G. de Gennes and G. Sarma [J. Appl. Phys. **34**, 1380 (1963)] have used perturbation theory to construct the Bogoliubov-type quasiparticles (of energy E_n) in a superconductor with magnetic impurities in the extreme gapless region ($\zeta\gg 1$). However, E_n versus ϵ_n (where ϵ_n is the energy of the single-particle state of the alloy in the normal phase) has quite a different meaning from the dispersion relation we discuss here.

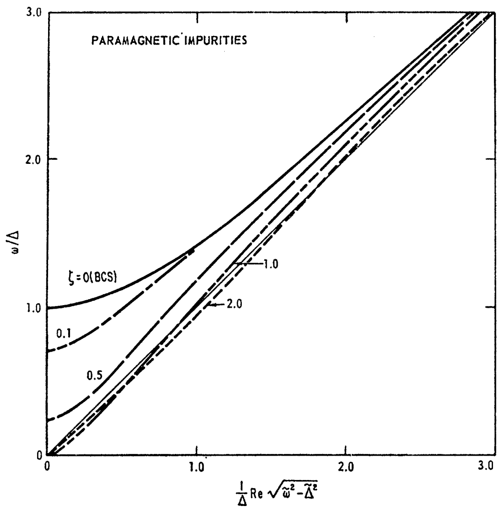

FIG. 1. The "dispersion relation" for electrons in a superconducting film with paramagnetic impurities. Here the depairing parameter is $\zeta=(\tau_s\Delta)^{-1}$ (where τ_s is the exchange scattering time) and $\mathrm{Re}(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}/v_F\equiv k_R^*$ is the pseudomomentum.

impurities is probably difficult when ζ is appreciable. The addition of these impurities automatically entails the shortening of the total electronic free path, and hence increased damping (see Fig. 3). On the other hand, the effect of strong depairing due to a magnetic field seems more accessible to study using the Tomasch effect. At high enough frequencies ($\omega\gtrsim\Delta$) we have $k_I^*\simeq 1/2l$, no matter what ζ is. Actually, the original work[5] of one of the authors (KM) was in the local limit $l\ll d$. However, calculation shows that the results of this note will not be altered very much when $l\gtrsim d$, as long as we are in the dirty limit $l\ll\xi_0$. This may be seen from some recent work[8] on the nonlocal limit.

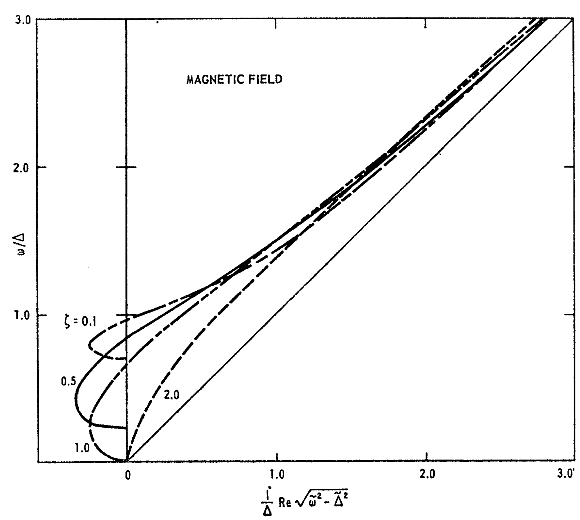

FIG. 2. The "dispersion relation" for electrons in a dirty superconducting film to which is applied a parallel magnetic field H. Here $\zeta=\tau(ev_F/c)^2d^2H^2/18\Delta$, where d is the thickness of the film.

In the case of strong-coupling superconductors with nonmagnetic impurities, one may easily check[9] that

$$k_R^*=\mathrm{Re}[\omega^2Z^2(\omega)-\varphi^2(\omega)]^{1/2}/v_F\,,$$
$$k_I^*=(1/2l)+\mathrm{Im}[\omega^2Z^2(\omega)-\varphi^2(\omega)]^{1/2}/v_F\,,$$

and

$$u=\omega Z(\omega)/\varphi(\omega)\,.$$

Here $Z(\omega)$ and $\varphi(\omega)$ are the usual renormalization and gap parameters of the pure material. This assumes that the phonon spectrum changes are negligible. The impurities in a strong-coupling superconductor simply change the amplitude of the Tomasch oscillations by a frequency-independent factor of $e^{-d/l}$ if $k_R^*\gg k_I^*$ [see Eq. (7)].

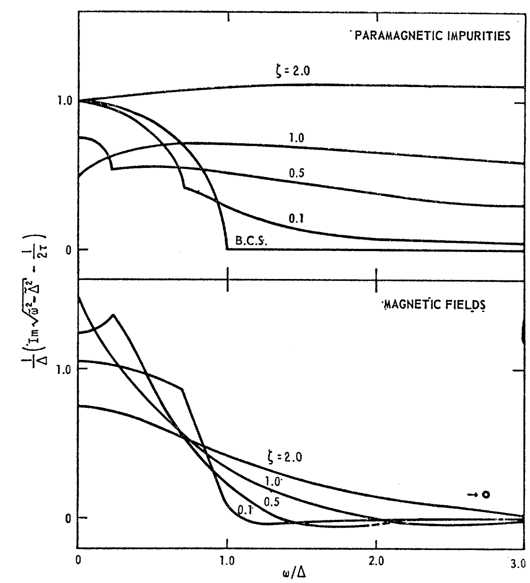

FIG. 3. The attenuation coefficient of electronic states as a function of their energy ω, for both kinds of "gapless" superconductor.

For simplicity, we have taken $|\Delta(\mathbf{r})|$ to be position-independent.[5] This is only strictly true when $d\ll(\xi_0 l)^{1/2}$, and is certainly not correct for the thicker films of interest ($d\simeq10^5$ Å). This fact will further complicate the study of the dispersion relation at low energies using the Tomasch oscillations.

While we do not want to critically discuss MA's calculation, two comments might be made. Instead of a delta function, we could have used a step-function perturbation, with $\delta\Delta=-\Delta$ for $x>d$. The end result is similar to that given by (7), with $\mathrm{Re}[u/(u^2-1)]$ and $\mathrm{Im}[u/(u^2-1)]$ replaced by $-\mathrm{Im}[u/(u^2-1)]$ and

[8] R. S. Thompson and A. Baratoff, Phys. Rev. Letters 15, 971 (1965).

[9] T. Tsuneto, Progr. Theoret. Phys. (Kyoto) 28, 857 (1962). See also Appendix.

$\text{Re}[u/(u^2-1)]$, respectively. In the second place, one might argue about the correctness of treating the change in the order parameter at the back wall as a *small* perturbation. In this connection, we note that even if we work to higher order, $\delta\Delta$ always enters in the combination $\delta\Delta \exp(-2dk_I^*)$. Thus, it would appear that a lowest order scattering-type calculation is valid if $2dk_I^* \gtrsim 1$. The case $2dk_I^* \ll 1$ seems more complicated.

We have discussed only one part of the excitation spectrum of a superconductor, corresponding to $k^*(\omega>0) = +(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}/v_F$. For energies below ϵ_F ($\omega<0$), the corresponding branch is given by $k^*(\omega<0)=k^*(-\omega)$. Finally there are branches corresponding to $k^*(\omega) = -(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}/v_F$. [It is useful to recall the BCS superconductor as a familiar example.] We note that the existence of two momenta $\pm\text{Re}\Omega(\omega)/v_F$ for a given energy ω is the basis of the description[2] of a superconducting excitation as a coherent mixture of particles and holes. The significance of the anomalous "switch-back" in the case of magnetic fields (see Fig. 2) is that the solutions $\pm\text{Re}\Omega(\omega)/v_F$ can no longer be simply distinguished as being larger or smaller than k_F. Moreover, we see in this case that if we study the spectral density as a function of ω for a fixed *small* value of momentum k_R^*, one may have three peaks for $\omega>0$ and three for $\omega<0$. As $k_R^* \to k_F$, two of the peaks merge while the third approaches the energy gap ω_0 (for $\omega<0$, at $-\omega_0$). Of course, the relative weights of these various resonances (or branches of the excitation spectrum) are quite different, depending especially on whether $k_R^* \lessgtr k_F$ and $\omega \lessgtr 0$.

The main purpose of this paper has been to point out what seems to be the natural excitation spectrum in a general class of superconductors (those in which the self-energies are momentum-independent). Even though the Tomasch oscillations may only give us the relatively high-frequency part of the ω versus k_R^* curves, the deviation from the BCS result is still of interest. Most measurable properties of superconductors with depairing perturbations are quite insensitive to the detailed nature of the excitation spectrum, insofar as the remarkable differences shown in Figs. 1 and 2 have no effect.[10] The Tomasch effect is a much more delicate probe of the electronic states, and thus this similarity between depairing mechanisms breaks down. Moreover, the sensitive dependence of the oscillations on bulk impurity scattering may provide a nice way of studying the strength of the electron-impurity interaction.

APPENDIX

A very successful theory of isotropic strong-coupling superconductors has been developed in the last few years.[11] In this Appendix we wish to briefly discuss the straightforward extension needed to include the effect of paramagnetic impurities. The Nambu Green's function is given by

$$\hat{G}^{-1}(\mathbf{k},\omega_l)=\omega_l 1-\epsilon_k\tau_3-\hat{\Sigma}(\mathbf{k},\omega_l)\,, \tag{A1}$$

where $\omega_l=i\pi(2l+1)/\beta$, $l=0, \pm1, \pm2, \cdots$. The matrix self-energy $\hat{\Sigma}(\mathbf{k},\omega_l)$ is approximated by the three lowest order self-energy diagrams arising from the electron-electron repulsive interaction, the electron-phonon interaction, and the electron-magnetic impurity coupling, respectively.

Introducing the renormalized Green's function

$$\hat{G}(k,\omega_l)=\frac{\tilde{\omega}(\omega_l)1+\epsilon_k\tau_3+\tilde{\Delta}(\omega_l)\tau_1}{\tilde{\omega}^2(\omega_l)-\epsilon_k^2-\tilde{\Delta}^2(\omega_l)}\,, \tag{A2}$$

one finds

$$\tilde{\omega}(\omega_l)=\omega_l-\Sigma_1(\omega_l)+\frac{i}{2}\left(\frac{1}{\tau}+\frac{1}{\tau_s}\right)\frac{\tilde{\omega}}{(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}}\,, \tag{A3}$$

$$\tilde{\Delta}(\omega_l)=\Sigma_2(\omega_l)+\frac{i}{2}\left(\frac{1}{\tau}-\frac{1}{\tau_s}\right)\frac{\tilde{\Delta}}{(\tilde{\omega}^2-\tilde{\Delta}^2)^{1/2}}\,, \tag{A3$'$}$$

where we have assumed that the impurity potentials are delta functions (s-wave scattering approximation). The Fourier transform of (A2) for $\epsilon_k \simeq v_F(k-k_F)$ is given by Eq. (1). The self-energies $\Sigma_{1,2}(\omega_l)$ are defined by

$$\Sigma_{\substack{1\\2}}(\mathbf{k},\omega_l)=\frac{1}{-i\beta}\sum_{l'=0,\pm1,\cdots}\int\frac{d\mathbf{k}'}{(2\pi)^3}iK(\mathbf{k}-\mathbf{k}',\,\omega_l-\omega_{l'})\times\text{Tr}\frac{1}{2}\left\{\begin{matrix}\hat{G}(\mathbf{k}',\omega_{l'})\\ \tau_1\hat{G}(\mathbf{k}',\omega_{l'})\end{matrix}\right\}\,, \tag{A4}$$

where the kernel is related to the electron-electron and electron-phonon interactions:

$$K(\mathbf{k},\omega_l)\equiv V_{ee}(\mathbf{k},\omega_l)+\sum_\lambda|V_{ep}(\mathbf{k},\lambda)|^2D_\lambda(\mathbf{k},\omega_l)\,, \tag{A5}$$

$D_\lambda(\mathbf{k},\omega)$ being the usual phonon Green's function for polarization λ. In general, the momentum dependence of this kernel is very weak.[11] To this approximation, we have

$$\Sigma_{\substack{1\\2}}(k_F,\omega_l)=\frac{\pi N(0)}{-i\beta}\sum_{l'}K(\omega_l-\omega_{l'})\times\frac{1}{(u^2(\omega_{l'})-1)^{1/2}}\left\{\begin{matrix}u(\omega_{l'})\\ 1\end{matrix}\right\}\,, \tag{A6}$$

with $N(0)$ the normal-state density of states and $u(\omega_l)\equiv\tilde{\omega}(\omega_l)/\tilde{\Delta}(\omega_l)$. Since the self-energies are independent of k, it is once again appropriate to identify $\text{Re}(\tilde{\omega}^2(\omega)-\tilde{\Delta}^2(\omega))^{1/2}/v_F$ as the momentum of an excitation with energy ω [corresponding to a pole of (A2)].

The coupled nonlinear integral equations (A3) and (A3′) must be solved numerically, even without im-

[10] K. Maki and P. Fulde, Phys. Rev. **140**, A1586 (1965).

[11] See, for example, J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), Chap. 7.

purity scattering. To see how the latter affects the ratio $u(\omega_l)$, we multiply (A3′) by $u(\omega_l)$ and subtract the resulting equation from (A3). One is left with an implicit equation for $u(\omega_l)$, namely

$$\frac{\omega_l-\Sigma_1(\omega_l)}{\Sigma_2(\omega_l)}=u(\omega_l)\left\{1-\frac{i}{\tau_s\Sigma_2(\omega_l)}\cdot\frac{1}{(u^2(\omega_l)-1)^{1/2}}\right\}. \quad \text{(A7)}$$

This may be compared with Eq. (5) for weak-coupling superconductors (here the kernel $K(k,\omega_l)\approx -g$, where $-g$ is the effective attractive interaction between electrons). An examination of (A6) and (A7) indicates that if the impurity scattering is nonmagnetic, then $u(\omega_l)$ is unaffected. This is an explicit proof of Anderson's theorem in strong coupling superconductors.[9]

When $1/\tau_s\neq 0$, one is faced with the task of solving (A6) in conjunction with (A7). A few remarks about the case of absolute zero might be useful. A finite amount of depairing will quickly smear out the phonon-induced structure[12] in the high-frequency part of the usual bulk density of states ($\propto dI/dV$),

$$\rho(\omega)=-\text{Re}\left\{\frac{u(\omega_l)}{(u^2(\omega_l)-1)^{1/2}}\right\}\Bigg|_{\omega_l\to\omega-i0^+}. \quad \text{(A8)}$$

The effect should be similar to thermal smearing at finite temperatures in the absence of depairing. The first resonance in d^2I/dV^2 should occur at $\omega_r\simeq\omega_D+\omega_{\max}$, where ω_D is the energy of the important phonons and $\omega_{\max}$ is the energy at which $\rho(\omega)$ is largest. The sharp threshold at the energy gap ω_0 [where $\rho(\omega)\propto(\omega-\omega_0)^{1/2}$] should give rise to a weaker structure in d^3I/dV^3 and higher derivatives. This might provide a way of measuring ω_0. More detailed calculations would be useful.

[12] D. J. Scalapino and P. W. Anderson, Phys. Rev. **133**, A921 (1964).