if the u's satisfy (41). (41) defines the u's as algebraic functions of Δ. Thus, in complex Δ space except at the poles of the u's and at points where $\psi'=0$, ψ' is an eigenstate of H. These exceptional points are finite in number. We can obtain a correct eigenfunction ψ'' at these points too by properly normalizing ψ' and approaching these exceptional points. Hence, Theorem 6. (In fact the above proves a generalization of Theorem 6 to complex Δ.)

We can also prove the following theorem, which clarifies but is not essential for later discussions.

Theorem 7: The p's are analytic in Δ in an open strip containing the semi-infinite real axis $\Delta<1$.

Proof[18]*:* (a) Starting from $\Delta=0$, and moving along the real axis towards $\Delta=-\infty$, let $\Delta=\Delta_1$ be the first singularity of the u's, if any is in the way. We can form a simple closed path that loops around Δ_1 and return to $\Delta=0$, which does not pass through and does not contain, inside of it, any other singularities of any u. Now $E(\Delta)$ is analytic along the real axis, by Theorem 4. Furthermore, it is a polynomial in u. Thus, E has no singularity on or in the path and it returns to the original value when Δ goes around the path back to $\Delta=0$. Thus, ψ' returns also to the ground-state wave function at $\Delta=0$, except for a possible multiplicative factor. This wave function is a determinant. Consider its values when $x_1=1$, $x_2=2$, $\cdots x_{m-1}=m-1$, but successively $x_m=m$, $m+1$, $m+2$, $\cdots$. Its values are in the ratio of 1, $\sum u$, $\sum u^2+\sum_{j>l} u_ju_l$, $\cdots$. Thus, all symmetrical polynomials of the u's return to their original values around the loop. Hence, the u's are merely permuted in going completely around the loop. Call that permutation $P(\Delta_1)$.

(b) For $0\leqq\Delta<1$, u_j is on the unit circle. By analytic continuation, it must remain so for $\Delta_1<\Delta<0$. Thus, $p_j=-i\ln u_j$ is analytic for $\Delta_1<\Delta<1$. For $0\leqq\Delta<1$, Theorem 1 shows that (18) is satisfied. Continuing all p's to values of $\Delta<0$, (18) remains satisfied until either we reach the point Δ_1, or the p's go outside of the limits defined in (8) and (9). The latter alternative, however, does not obtain, since before the p's reach the boundary, the corresponding point must go out of the surface of the cube (37). Part (b) of the proof of Theorem 3 demonstrates that that is not possible. Thus, (18) is satisfied for all $\Delta_1<\Delta<1$.

(c) Δ_1 is not a pole for the u's, since $|u|=1$ for $\Delta=\Delta_1+0$. Since each u_j is algebraic in Δ, it has a definite value at $\Delta=\Delta_1$. (18) shows that at $\Delta=\Delta_1$, all p's are unequal. Hence, all u's are unequal.

(d) Now tighten the loop of (a) around Δ_1. Since all u's are unequal at Δ_1, the permutation $P(\Delta_1)$ must be the identity. Thus, Δ_1 is not a branch point of any u. Contradiction.

[18] One can rearrange the theorems so that the topological theorem is not needed: After Theorems 1 and 2, 4, and 5 the concept of u of (39) is introduced, together with the ψ' of (43), leading to $H\psi'=E\psi'$ for complex Δ. One then proves Theorem 7, using in part (b) of the proof the discussions following Eq. (38). This proof of Theorem 7 then automatically establishes (18) for all $\Delta<1$, with all p's within the bounds (8) and (9).

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One-Dimensional Chain of Anisotropic Spin-Spin Interactions. II. Properties of the Ground-State Energy Per Lattice Site for an Infinite System

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The ground-state energy $2f$ per lattice site for an infinite system is studied as a function of Δ and of the magnetization y. Analyticity properties of $f(\Delta,y)$ are proved. The behavior of $f(\Delta,y)$ at and near $y=0$ and $y=1$ are investigated.

1. BASIC EQUATIONS

IN Paper I[1] it was shown that if $\Delta<1$, the ground state for a fixed $\mathfrak{N}$ (=No. of sites) and m (=No. of down spins) is of Bethe's form (I7), with p_j satisfying (I18), or

$$p_j=2\pi I_j(\mathfrak{N}^{-1})-\mathfrak{N}^{-1}\sum_{l=1}^{m}\Theta(p_j,p_l). \tag{1}$$

Since $p_j\neq p_i$ if $j>i$, by continuity argument with respect to Δ, we see that $p_1<p_2<p_3\cdots<p_m$ for all Δ. As $\mathfrak{N}$, $m\rightarrow\infty$ at a fixed ratio, the p's increase in

[1] C. N. Yang and C. P. Yang, preceding paper, Phys. Rev. 150, 321 (1966). Formulas and references there are referred to as (I18), etc. The notations are the same.

TABLE I. Values of Q and b for $y=0$ or 1. (b=limit in α integration.) μ is defined by $\cos\mu=-\Delta$, $0<\mu<\pi$. It is proved in Sec. 2 that y decreases monotonically with increasing Q.

	$\Delta<-1$	$\Delta=-1$	$-1<\Delta<1$
$y=1$	$Q=0$ $(b=0)$	$Q=0$ $(b=0)$	$Q=0$ $(b=0)$
$y=0$	$Q=\pi$ $(b=\pi)$	$Q=\pi$ $(b=\infty)$	$Q=\pi-\mu$ $(b=\infty)$

number, but always lie within the interval (I8) or (I9). Let us assume that the number of p's in an interval p to $p+dp$ approaches

$$\mathfrak{N}\rho(p)dp. \tag{2}$$

(We shall return to a rigorous proof of this assumption in a later paper.) (1) then becomes

$$p=2\pi f-\int\Theta(p,q)\rho(q)dq, \tag{3}$$

where $f=I/\mathfrak{N}$. Clearly,

$$df/dp=\rho. \tag{4}$$

Thus,

$$1=2\pi\rho-\int\frac{\partial\Theta}{\partial p}\rho(q)dq. \tag{5}$$

This integral equation was the one found and solved in Refs. I4, I5, I6, I8, I9, and I10 for the special cases illustrated in Fig. 1 of I.

In (3) and (5) we did not fix the limit of integration. The theorems of I show that the p's are distributed symmetrically with respect to $p=0$. Thus, we assume that in the limit we are now considering, the integration extends from $-Q$ to Q without any gaps, i.e.,

$$1=2\pi\rho-\int_{-Q}^{Q}\frac{\partial\Theta}{\partial p}(p,q)\rho(q)dq. \tag{6a}$$

(We shall return to a rigorous proof of this point in a later paper.)

We have obviously,

$$\tfrac{1}{2}(1-y)=\frac{m}{\mathfrak{N}}=\int_{-Q}^{Q}\rho(p)dp, \tag{6b}$$

and by (I11),

$$f(\Delta,y)=-\frac{\Delta}{4}+\frac{\Delta}{2}(1-y)-\int_{-Q}^{Q}\rho(p)\cos p\,dp. \tag{6c}$$

Equations (6) are the basic equations which define y and f as functions of Q. We shall show that y is a monotonically decreasing function of Q for Q between the limits tabulated in Table I. Thus, for given y in the closed interval (0,1) one can solve for Q uniquely, thereby obtaining $f(\Delta,y)$.

2. PROPERTIES OF THE INTEGRAL EQUATION (6a)

In this section we discuss Eqs. (6) with Q as a parameter.

A. Transformation $p\rightarrow\alpha$

The integral equation is simpler after the transformation $p\rightarrow\alpha$ introduced in (I21):

$$R(\alpha)=\frac{dp}{d\alpha}-\frac{1}{2\pi}\int_{-b}^{b}\frac{\partial\theta}{\partial\beta}R(\beta)d\beta, \tag{7a}$$

where

$$Rd\alpha=2\pi\rho dp \tag{8}$$

and b is the limit of the α integration that corresponds to the limit Q in the p integration. By definition, $dQ/db>0$. The functions $dp/d\alpha$, $\partial\theta/\partial\beta$ were given in (I21). They are retabulated in Table II for easy reference. Equations (8) and (6b) give

$$\pi(1-y)=\int_{-b}^{b}R(\alpha)d\alpha. \tag{7b}$$

Equation (6c) leads to

$$f(\Delta,y)=-\frac{\Delta}{4}-\frac{1}{C}\int Rd\alpha\left(\frac{dp}{d\alpha}\right), \tag{7c}$$

TABLE II. The functions $dp/d\alpha$ and $\partial\theta/\partial\beta$ in (7a) and the constant C in (7c). For $\Delta<-1$, $\lambda>0$ is defined by $-\Delta=\cosh\lambda$. For $-1<\Delta<1$, μ is defined to be between 0 and π, and $-\Delta=\cos\mu$.

	$\Delta<-1$	$\Delta=-1$	$-1<\Delta<1$
$\langle\alpha\vert\boldsymbol{\xi}\rangle=\dfrac{dp}{d\alpha}=$	$\dfrac{\sinh\lambda}{\cosh\lambda-\cos\alpha}>0$	$\dfrac{4}{1+4\alpha^2}>0$	$\dfrac{\sin\mu}{\cosh\alpha-\cos\mu}>0$
$2\pi\langle\alpha\vert\mathbf{K}\vert\beta\rangle=\dfrac{\partial\theta}{\partial\beta}=$	$\dfrac{\sinh2\lambda}{\cosh2\lambda-\cos(\alpha-\beta)}$	$\dfrac{2}{1+(\alpha-\beta)^2}$	$\dfrac{\sin2\mu}{\cosh(\alpha-\beta)-\cos2\mu}$
$C=$	$\dfrac{2\pi}{\sinh\lambda}$	4π	$\dfrac{2\pi}{\sin\mu}$

TABLE III. Eigenvalues and eigenfunctions of $-\mathbf{K}$. For definitions see the caption of Table II. The integration limit discussed here is defined as $b_0(\Delta)$ in the text.

	$\Delta<-1$ $b=\pi$	$\Delta=-1$ $b=\infty$	$-1<\Delta<1$ $b=\infty$
Eigenvalue of $-\mathbf{K}$	$-e^{-2\lambda\lvert n\rvert}$	$-e^{-\lvert\gamma\rvert}$	$-\dfrac{\sinh[(\pi-2\mu)\gamma]}{\sinh[\pi\gamma]}$
	(n = all integers)	(γ = real No.)	(γ = real No.)
Eigenfunction	$e^{in\alpha}$	$e^{i\gamma\alpha}$	$e^{i\gamma\alpha}$

where C was introduced in (I21p) and retabulated in Table III.

We shall sometimes write Eqs. (7) in operator notation:

$$R(\alpha)=\langle\alpha|\mathbf{R}\rangle,$$

$$dp/d\alpha=\langle\alpha|\xi\rangle,$$

$$\frac{1}{2\pi}\frac{\partial\theta}{\partial\beta}=\langle\alpha|\mathbf{K}|\beta\rangle.$$

Then

$$\mathbf{R}=\xi-\mathbf{KR},\tag{9a}$$

$$\pi(1-y)=\boldsymbol{\eta}^T\mathbf{R},\tag{9b}$$

$$f(\Delta,y)=-\frac{\Delta}{4}-\frac{1}{C}\xi^T\mathbf{R},\tag{9c}$$

where $\boldsymbol{\eta}$ is defined so that

$$\langle\alpha|\boldsymbol{\eta}\rangle=1,$$

and the superscript T means the transpose. These operators are defined within the range $\alpha=-b$ to $\alpha=b$.

It is obvious that (9) can be expressed as a variational principle: $\mathbf{R}$ is a vector that minimizes the quadratic functional

$$\tfrac{1}{2}\mathbf{R}^T\mathbf{R}+\tfrac{1}{2}\mathbf{R}^T\mathbf{KR}-\mathbf{R}^T\xi.$$

[See Sec. IIB for a discussion of the eigenvalue of $\mathbf{K}$ showing that $1+\mathbf{K}$ is positive definite.] The minimum value of the functional is

$$-\tfrac{1}{2}\mathbf{R}^T\xi=\tfrac{1}{2}C[f(\Delta,y)+\tfrac{1}{4}\Delta].$$

B. Spectrum of K and Existence of Resolvent J

For $\Delta<-1$, and $b=\pi$, the kernel $-\mathbf{K}$ is cyclic. For $-1\leqq\Delta<1$ and $b=\infty$, the integration extends throughout the real α axis and

$$\langle\alpha|-\mathbf{K}|\beta\rangle=\text{function of }\alpha-\beta.$$

For these cases the eigenvalues and eigenfunctions of $-\mathbf{K}$ are trivially computable and are exhibited in Table III.

We define a function b_0 of Δ:

$$\begin{aligned}b_0&=\pi &&\Delta<-1,\\ b_0&=\infty &&-1\leqq\Delta<1.\end{aligned}\tag{10}$$

Thus, for $b=b_0$, there exists a number c_1 so that the eigenvalues of $-\mathbf{K}$ are all $\leqq c_1<1$. For $0\leqq b<b_0$, *the eigenvalues of* $-\mathbf{K}$ *must therefore also be* $\leqq c_1<1$, because shrinking the limit of an integral operator never extends the range covered by the eigenvalues.

Thus, the Fredholm equation (9a) has a unique solution $\mathbf{R}$ for any b and Δ if $\Delta<1$, $0\leqq b\leqq b_0$. We shall in this paper only consider values of b in this interval.

We can therefore define a resolvent operator $\mathbf{J}$ so that

$$\mathbf{1}=(\mathbf{1}+\mathbf{K})(\mathbf{1}+\mathbf{J})\tag{11}$$

or

$$0=\mathbf{K}+\mathbf{J}+\mathbf{KJ}=\mathbf{K}+\mathbf{J}+\mathbf{JK}.\tag{12}$$

$\mathbf{J}$ of course depends on Δ and b.

The Fredholm equation (9a) can be iterated to give

$$\mathbf{R}=[\mathbf{1}-\mathbf{K}+\mathbf{K}^2-\mathbf{K}^3+\cdots]\xi.\tag{13}$$

C. Proof that $R(\alpha)>0$ and $dy/db<$ for $0\leqq\Delta<1$

For $0\leqq\Delta<1$, we have, by inspection of Table II, that

$$-\langle\alpha|\mathbf{K}|\beta\rangle\geqq0,\quad\langle\alpha|\xi\rangle>0.$$

Thus, $R(\alpha)>0$. Furthermore, when b increases, the range of integration increases and it is obvious from (7b) that $dy/db<0$.

D. $R_0(\alpha)\equiv R(\alpha)$ when $b=b_0$

It is easy to compute R when $b=\infty$, $-1<\Delta<1$, by taking the Fourier transformation of (9a). One thus obtains for $-1<\Delta<1$, $b=\infty$

$$R_0(\alpha)=\int_{-\infty}^{\infty}\frac{e^{i\gamma\alpha}d\gamma}{2\cosh\mu\gamma}=\frac{\pi}{2\mu\cosh(\pi\alpha/2\mu)}>0.\tag{14a}$$

Similarly, when $\Delta=-1$, $b=\infty$,

$$R_0(\alpha)=\frac{\pi}{\cosh(\pi\alpha)}>0.\tag{14b}$$

For $\Delta<-1$, $b=\pi$, one obtains by the same method

$$R_0(\alpha)=\sum_{n=-\infty}^{\infty}\frac{e^{in\alpha}}{2\cosh n\lambda}>0.\tag{14c}$$

The inequality in this equation is proved in Appendix B.

E. $J_0 \equiv J$ when $b = b_0$

From (12) it follows that

$$\mathbf{J} = -\mathbf{K}/(1+\mathbf{K}). \tag{15}$$

For $b = b_0$, the eigenvalues of $\mathbf{K}$ have been exhibited in Table III. Thus, the eigenvalues of $\mathbf{J}_0$ (defined to be $\mathbf{J}$ for $b = b_0$) are known. We have then directly

$-1 < \Delta < 1$,

$$\langle\alpha|\mathbf{J}_0|\beta\rangle = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\gamma(\alpha-\beta)}d\gamma \times \frac{\sinh[(\pi-2\mu)\gamma]}{\sinh[(\pi-2\mu)\gamma]+\sinh[\pi\gamma]}; \tag{16a}$$

$\Delta = -1$,

$$\langle\alpha|\mathbf{J}_0|\beta\rangle = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\gamma(\alpha-\beta)}d\gamma \frac{1}{1+e^{|\gamma|}}; \tag{16b}$$

and

$\Delta < -1$,

$$\langle\alpha|\mathbf{J}_0|\beta\rangle = -\frac{1}{2\pi}\sum_n e^{i(\alpha-\beta)n}\frac{1}{1+e^{2\lambda|n|}}. \tag{16c}$$

It will be shown (in Appendix C) from these that

$$\langle\alpha|\mathbf{J}_0|\beta\rangle > 0 \quad \text{for} \quad \Delta > 0, \tag{17}$$

$$\langle\alpha|\mathbf{J}_0|\beta\rangle < 0 \quad \text{for} \quad \Delta < 0. \tag{18}$$

F. Equation for $R' = (R$ Extended$)$

In the integral equation

$$\langle\alpha|\mathbf{R}\rangle = \langle\alpha|\xi\rangle - \int_{-b}^{b}\langle\alpha|\mathbf{K}|\beta\rangle d\beta\langle\beta|\mathbf{R}\rangle, \tag{19}$$

the functions $\langle\alpha|\xi\rangle$ and $\langle\alpha|\mathbf{K}|\beta\rangle$ are defined for all α (Table II). Thus, we can regard (19) as defining $\langle\alpha|\mathbf{R}\rangle$ for *all* α between $-b_0$ and b_0. To avoid confusion we shall indicate with a prime the extended operators and vectors. Thus,

$$\mathbf{R}' = \xi' - \mathbf{K}'\mathbf{B}'\mathbf{R}', \tag{20}$$

where $\langle\alpha|\mathbf{R}'\rangle = \langle\alpha|\mathbf{R}\rangle$ for $-b \leqq \alpha \leqq b$; and $\mathbf{B}'$ is a projection operator:

$$\langle\alpha|\mathbf{B}'\zeta'\rangle = \langle\alpha|\zeta'\rangle \quad \text{for} \quad -b \leqq \alpha \leqq b$$

and

$$\langle\alpha|\mathbf{B}'\zeta'\rangle = 0 \quad \text{for} \quad b < \alpha \leqq b_0 \quad \text{or} \quad -b_0 \leqq \alpha < -b.$$

Thus,

$$(1'+\mathbf{K}')\mathbf{R}' = \xi' + \mathbf{K}'(1'-\mathbf{B}')\mathbf{R}'. \tag{21}$$

Now

$$(1'+\mathbf{J}_0')(1'+\mathbf{K}') = 1' \quad (\mathbf{J}_0' = \mathbf{J}_0). \tag{22}$$

Hence,

$$\mathbf{R}' = (1'+\mathbf{J}_0')\xi' - \mathbf{J}_0'(1'-\mathbf{B}')\mathbf{R}'.$$

But

$$(1'+\mathbf{J}_0')\xi' = \mathbf{R}_0, \quad [\mathbf{R}_0' = \mathbf{R}_0]. \tag{23}$$

Thus,

$$\mathbf{R}' = \mathbf{R}_0' - \mathbf{J}_0'(1'-\mathbf{B}')\mathbf{R}'. \tag{24a}$$

This is an integral equation for $\langle\alpha|\mathbf{R}'\rangle$ with α *outside* the interval $(-b, b)$, but within $(-b_0, b_0)$. We can also write down equations in terms of such $\langle\alpha|\mathbf{R}'\rangle$ for y and f:

(i) $-1 \leqq \Delta < 1$. $b_0 = \infty$.

We integrate (20) over all α. Now

$$\int_{-\infty}^{\infty} d\alpha\langle\alpha|\xi\rangle = 2(\pi-\mu) \quad [\text{see Table II and (A1)}],$$

and

$$\int_{-\infty}^{\infty} d\alpha\langle\alpha|\mathbf{K}|\beta\rangle = \frac{\pi-2\mu}{\pi} \quad (\text{see Table III}).$$

Hence,

$$\int_{-\infty}^{\infty} d\alpha\langle\alpha|\mathbf{R}'\rangle = 2(\pi-\mu) - (\pi-2\mu)(1-y).$$

Subtracting (9b) from this equation, we obtain

$$2(\pi-\mu)y = \boldsymbol{\eta}'^T(1'-\mathbf{B}')\mathbf{R}' = \int_{-\infty}^{-b} d\alpha\langle\alpha|\mathbf{R}'\rangle + \int_{b}^{\infty} d\alpha\langle\alpha|\mathbf{R}'\rangle. \tag{24b$'$}$$

(ii) $\Delta < -1$. $b_0 = \pi$.

We integrate (20) over all α between $(-\pi, \pi)$. Now we have

$$\int_{-\pi}^{\pi} d\alpha\langle\alpha|\xi\rangle = 2\pi \quad [\text{Cf. (A2)}],$$

$$\int_{-\pi}^{\pi} d\alpha\langle\alpha|\mathbf{K}|\beta\rangle = 1.$$

Thus,

$$\int_{-\pi}^{\pi} d\alpha\langle\alpha|\mathbf{R}'\rangle = 2\pi - \pi(1-y).$$

Subtracting (9b) from this we obtain

$$2\pi y = \boldsymbol{\eta}'^T(1'-\mathbf{B}')\mathbf{R}' = \int_{-\pi}^{-b} d\alpha\langle\alpha|\mathbf{R}'\rangle + \int_{b}^{\pi} d\alpha\langle\alpha|\mathbf{R}'\rangle. \tag{24b$''$}$$

To obtain an expression for f we use (9c) and (24a)

$$-C[\Delta/4+f(\Delta,y)] = \xi'^T\mathbf{B}'\mathbf{R}' = \xi'^T\mathbf{R}' - \xi'^T(1'-\mathbf{B}')\mathbf{R}' = \xi'^T\mathbf{R}_0' - \xi'^T\mathbf{J}_0'(1'-\mathbf{B}')\mathbf{R}' - \xi'^T(1'-\mathbf{B}')\mathbf{R}'.$$

Using (23) we obtain, since $J_0 = J_0^T$,

$$-C[\Delta/4+f(\Delta,y)] = \xi'^T\mathbf{R}_0' - \mathbf{R}_0'^T(1'-\mathbf{B}')\mathbf{R}'. \tag{25}$$

Putting $b=b_0$ in (24b′) and (24b″) one obtains $\mathbf{1}'=\mathbf{B}'$ and $y=0$. Thus, (25) leads to an expression for $f(\Delta,0)$. Subtracting that expression from (25) we obtain

$$C[f(\Delta,y)-f(\Delta,0)]=\mathbf{R}_0'^T(\mathbf{1}'-\mathbf{B}')\mathbf{R}', \qquad (24c)$$

which is valid for all $\Delta<1$.

G. Proof that $R(\alpha)>0$ and $dy/db<0$ for $\Delta<0$

For $\Delta<0$, the eigenvalues of $\mathbf{K}_0$ are between $+1$ and 0 (Table III). Hence the eigenvalues of $\mathbf{J}_0$ are between $-1/2$ and 0. So must be the eigenvalues of $(\mathbf{1}'-\mathbf{B}')\mathbf{J}_0'(\mathbf{1}'-\mathbf{B}')$. Now

$$(\mathbf{1}'-\mathbf{B}')^2=\mathbf{1}'-\mathbf{B}'.$$

Hence writing $(\mathbf{1}'-\mathbf{B}')\mathbf{R}'=\mathbf{S}'$ we obtain

$$\mathbf{S}'=(\mathbf{1}'-\mathbf{B}')\mathbf{R}_0'-(\mathbf{1}'-\mathbf{B}')\mathbf{J}_0'(\mathbf{1}'-\mathbf{B}')\mathbf{S}'. \qquad (26)$$

This equation for $\mathbf{S}'$ can be iterated. Using (24) we then obtain

$$\mathbf{R}'=\mathbf{R}_0'-\mathbf{J}_0'(\mathbf{1}'-\mathbf{B}')\mathbf{R}_0'+[\mathbf{J}_0'(\mathbf{1}'-\mathbf{B}')]^2\mathbf{R}_0'-[\mathbf{J}_0'(\mathbf{1}'-\mathbf{B}')]^3\mathbf{R}_0'+\cdots. \qquad (27)$$

In the α representation, all elements of $\mathbf{R}_0'=\mathbf{R}_0$ are positive (Sec. 2D), and all elements of $\mathbf{J}_0'=\mathbf{J}_0$ are negative [(18)]. Thus, (27) shows that

$$\langle\alpha|\mathbf{R}'\rangle>0. \qquad (28)$$

Furthermore, each term on the right-hand side of (27) decreases as b increases. Hence,

$$d\langle\alpha|\mathbf{R}'\rangle/db<0. \qquad (29)$$

Equations (24b′) and (24b″) then show that

$$dy/db<0.$$

3. ANALYTICITY OF $f(\Delta,y)$ IN y AND IN Δ

At $b=0$, it is clear that $\mathbf{R}'=\boldsymbol{\xi}'$, and that $y=1$. At $b=b_0$, $\mathbf{1}'-\mathbf{B}'=0$. It follows from (24b′) and (24b″) that $y=0$. For $0\leqq b<b_0$, y is monotonic in b with $dy/db<0$. It is clear that the solution $R(\alpha)$ of the integral equation (7a) is analytic in b for $0\leqq b<b_0$. Hence, y and f are analytic in b in this semiopen interval. Thus,

$$f(\Delta,y) \text{ is analytic in } y \text{ for } 0<y\leqq 1, \quad \Delta<1. \qquad (30)$$

By using the variable p or the variable a introduced in (I25), one can also prove similarly that R is analytic in Δ for fixed b. Hence, y and f are analytic in Δ for fixed b. Thus,

$$f(\Delta,y) \text{ is analytic in } \Delta \text{ for } 0<y\leqq 1, \quad \Delta<1. \qquad (30')$$

Thus $2f(\Delta,y)$ (= the ground-state energy per bond) to the left of the point A *in Fig.* I1, *an even function of* y, *can only be nonanalytic in* y *or* Δ *along the line* $y=0$.

Actually along that line, to the left of B (*i.e., for* $\Delta<-1$), *the function* $f(\Delta,y)$ *for* $y\geqq0$ *and fixed* Δ *can be continued to* $y<0$ *analytically.* [To see this we remark that, for $\Delta<-1$, the discussion preceding (30) extends also to the case $b=b_0=\pi$.] But the continuation no longer is equal to $f(\Delta,y)$. On the other hand, between A and B (i.e., $-1\leqq\Delta<1$) the function $f(\Delta,y)$ for $y\geqq0$, fixed Δ, cannot in general be continued to $y<0$ analytically. The explicit behavior of $f(\Delta,y)$ at and near $y=0$ will be studied in Secs. 4 and 5, and in Appendix E.

4. $f(\Delta,0)$

A. Explicit Formulas for $f(\Delta,0)$

As mentioned in I, the value of $f(\Delta,0)$ for $\Delta=-1$ was given by Bethe[I4] and Hulthén,[I5] for $\Delta<-1$ by Orbach[I6] and Walker,[I7] and for $1\leqq\Delta$, it is obvious[I1] that $f(\Delta,0)=-\Delta/4$.

We are now in a position to calculate $f(\Delta,0)$ for all Δ, and to discuss it as a complex function of Δ. The calculation is straightforward. One substitutes the explicit expressions for $R_0(\alpha)$, given in (14), into (7c), obtaining

$1\leqq\Delta$,

$$f(\Delta,0)=-\Delta/4; \qquad (31a)$$

$-1<\Delta<1$,

$$f(\Delta,0)=\frac{\cos\mu}{4}-\frac{\sin\mu}{\mu}\times\int_{-\infty}^{\infty}\frac{\mu\sin\mu dx}{2[\cosh(\pi x)][\cosh(2\mu x)-\cos\mu]}; \qquad (31b)$$

$\Delta=-1$,

$$f(\Delta,0)=\tfrac{1}{4}-\ln 2; \qquad (31c)$$

$\Delta<-1$,

$$f(\Delta,0)=\frac{\cosh\lambda}{4}-\frac{\sinh\lambda}{\lambda}\left[\frac{\lambda}{2}+2\lambda\sum_{n=1}^{\infty}\frac{1}{1+e^{2\lambda n}}\right]. \qquad (31d)$$

The definitions of λ, μ were given in (I21).

B. Analytic Continuation of Integral in (31b)

We define, for $0<\mu<2\pi$,

$$Y(\mu)=\int_{-\infty}^{\infty}\frac{\mu\sin\mu dx}{2[\cosh(\pi x)][\cosh(2\mu x)-\cos\mu]}. \qquad (32)$$

(a) For $0<\mu<2\pi$, the integral is well defined and analytic in μ. The integrand has poles at $x=i(\frac{1}{2}+n)$ and $x=\pm i/2+\pi ni/\mu$. As μ is changed, these poles move. By distorting the integration path one can study the analytic continuation of $Y(\mu)$ to complex μ. It can thus be proved that Y is analytic in the whole complex μ plane except for a cut on the negative real axis. By the same method one can also prove that

$$Y(\mu^*)=[Y(\mu)]^*, \tag{33}$$

$$Y(\mu)-Y(-\mu)=-2\pi\sum_{n=1}^{\infty}\left[\sin\frac{\pi^2 n}{\mu}\right]^{-1} \quad \text{for} \quad \mu\neq\text{real}, \tag{34}$$

$$Y(-i\lambda)=\int_{-\infty}^{\infty}\frac{\lambda\sinh\lambda dx}{2(\cosh\pi x)(\cosh\lambda-\cos 2x\lambda)}+\pi i\sum_{1}^{\infty}\frac{1}{\sinh(\pi^2 n/\lambda)} \quad (\lambda=\text{real}>0), \tag{35}$$

where the first term on the right is real, the second pure imaginary.

(b) We can convert the integral in (35) into a sum by completing the path of integration to enclose the whole lower half-complex x plane. The residues at $x=-i/2\pm\pi n/\lambda$ cancel. The residue at $x=-i/2-in$ ($n>0$) gives a contribution

$$-\frac{\lambda\sinh\lambda(-1)^n}{2\sinh(\lambda n+\lambda)\sinh\lambda n}=\frac{-\lambda(-1)^n}{e^{2n\lambda}-1}+\frac{\lambda(-1)^n}{e^{2n\lambda+2\lambda}-1}.$$

The residue at $x=-i/2$ gives a contribution

$$\tfrac{1}{2}\lambda\coth\lambda=\frac{\lambda}{2}+\frac{\lambda}{e^{2\lambda}-1}.$$

Thus, (35) gives for real $\lambda>0$

$$\text{real part of} \quad Y(-i\lambda)=\frac{\lambda}{2}-2\lambda\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{2n\lambda}-1}.$$

By (A5) we have

$$\text{real part of} \quad Y(-i\lambda)=\frac{\lambda}{2}+2\lambda\sum_{n=1}^{\infty}\frac{1}{1+e^{2\lambda n}}. \tag{36}$$

(c) The series $\sum_{n=1}^{\infty}(\sin nz)^{-1}$ is convergent for all $z\neq$real. The real axis forms a natural boundary across which no analytic continuation is possible. Using (34) we see that $Y(\mu)$ has a natural boundary along the negative real axis.

C. Asymptotic Expansion of $Y(\mu)$ near $\mu=0$

(a) Consider a pie-shaped section S in the lower complex μ plane:

$$|\mu|<\pi, \quad -\delta_0\leqq\arg\mu\leqq 0, \quad \text{where} \quad \delta_0=\pi-\epsilon. \tag{37}$$

Consider the straight line P in the complex x plane:

$$x=te^{i(\pi-\epsilon)/2} \quad t=-\infty\to\infty.$$

For μ in S, the line P is always free of poles of the integrand in (32). Thus, we can distort the integration path to P. Now along P the function

$$\frac{\mu\sin\mu}{\cosh(2\mu x)-\cos\mu} \tag{38}$$

and each of its derivatives with respect to μ is bounded. Thus, by the generalized mean-value theorem, for any l,

$$(38)=c_0(x)+c_1(x)\mu+\cdots+c_l(x)\mu^l+d_{l+1}\mu^{l+1},$$

where $|d_{l+1}|<$constant. The functions $c_n(x)$ are real rational functions of x, with denominators which are powers of $(4x^2+1)$. Thus,

$$Y(\mu)=\sum_{n=0}^{l}\mu^n\int_P\frac{c_n(x)dx}{2\cosh(\pi x)}+\mu^{l+1}\int_P\frac{d_{l+1}(x)dx}{2\cosh(\pi x)},$$

and we arrive at an asymptotic expansion of $Y(\mu)$ in the section S:

$$Y(\mu)=\sum_{n=0}^{l}\mu^n h_n+O(\mu^{l+1}), \quad (\text{for } \mu\to 0) \tag{39}$$

where h_n is real.

Using (33) we see that the asymptotic expansion (39) holds also in the complex-conjugate region of S.

We can start with an integral for $dy/d\mu$ and go through the same reasoning as above, obtaining

$$\frac{dy}{d\mu}=\sum_{n=0}^{l}n\mu^{n-1}h_n+O(\mu^l) \quad (\text{for } \mu\to 0). \tag{40}$$

(b) To obtain the coefficients h_n explicitly we express (32), for real values of μ, in a different representation, obtained by writing the two factors in the integrand as Fourier integrals with the aid of (A1):

$$Y(\mu)=\int_0^{\infty}dy\left[1-\frac{\tanh y}{\tanh(y\pi/\mu)}\right]=\int_0^{\infty}dy[1-\tanh y]-2\int_0^{\infty}\frac{\tanh y}{e^{2y\pi/\mu}-1}dy. \tag{41}$$

The first integral is ln2. To evaluate the second, let

$$\tanh y=\sum_{n=1}^{\infty}\alpha_n y^{2n-1} \tag{42}$$

be the power-series expansion of tanh$y$ near $y=0$. By the generalized mean-value theorem, for *all* real y,

$$\tanh y=\sum_{n=1}^{l}\alpha_n y^{2n-1}+y^{2l}\beta_l,$$

where β_l is proportional to the value of some high-order derivative of tanh$y$ at $y'=\theta y$. Clearly,

$$|\beta_l|< \text{a constant dependent on } l, \text{ not } y.$$

Thus,

$$Y(\mu)=\ln 2-2\sum_{n=1}^{l}\alpha_n\int_0^\infty \frac{y^{2n-1}}{e^{2y\pi/\mu}-1}dy+O(\mu^{2l+1})$$

$$=\ln 2-\tfrac{1}{2}\sum_{n=1}^{l}\left(\frac{1}{n}\right)\alpha_n B_n\mu^{2n}+O(\mu^{2l+1}), \tag{43}$$

where

$$B_n=\text{Bernoulli's number}=4n\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}dx.$$

Comparing (43) with (39) we obtain h_n. Thus we conclude that (43) is valid for μ in the union of the two pie-shaped sections S and S^*. Furthermore, (40) shows that (43) can be differentiated term by term.

Now

$$B_n=\frac{2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\left(\frac{1}{m}\right)^{2n}>\frac{2(2n)!}{(2\pi)^{2n}}.$$

The radius of convergence of (42) is $\pi/2$. It can be shown that

$$|\alpha_n|>\text{constant }(2/\pi)^{2n}.$$

Thus, the asymptotic series in (43) has a radius of convergence$=0$.

D. Analyticity of $f(\Delta,0)$ in Δ

The Δ plane, cut along $(-\infty,-1)$ and $(1,\infty)$, (R^0 in Fig. 1), is mapped by $\Delta=-\cos\mu$ to the strip $0<\text{Re}\mu<\pi$ in the μ plane. Defining the $f(\Delta,0)$ of (31b) as $F(\Delta)$ in the cut Δ plane:

$$F(\Delta)=-\frac{\Delta}{4}-\frac{\sin\mu}{\mu}Y(\mu), \tag{44}$$

we find, by the results of Sec. 4B, the following.

(i) $F(\Delta)=f(\Delta,0)$ for $-1\leqq\Delta\leqq 1$. (44a)

(ii) $F(\Delta)$ is analytic in the cut Δ plane (R^0 in Fig. 1). $F(\Delta^*)=[F(\Delta)]^*$.

(iii) For $\Delta<-1$,

$$F(\Delta\pm i0)=f(\Delta,0)\pm\pi i\sum_{n=1}^{\infty}\left(\sinh\frac{\pi^2 n}{\lambda}\right)^{-1}\frac{\sinh\lambda}{\lambda}, \tag{44b}$$

where $\lambda>0$ is defined by $\Delta=-\cosh\lambda$. [This follows from (35), (36), and (31d).]

(iv) The mapping $\mu\rightarrow\Delta$ divides the μ plane into many regions each of which is mapped to the whole Δ plane (Fig. 2). Each of these regions in the Δ plane is one Riemann sheet of $F(\Delta)$. Some of these sheets are illustrated in Fig. 1. Notice that the natural boundary in the μ plane (=negative real axis in the μ plane) becomes a natural boundary in each sheet R^1, R^2, $\cdots$, etc., and R^{-1}, R^{-2}, etc., extending from $\Delta=-1$ to $\Delta=1$.

(v) In the neighborhood of $\Delta=1$ in R^0, $F(\Delta)$ has a branch point. It is straightforward from (31b) to evaluate $F(\Delta)$ as a power series in $\sqrt{(1-\Delta)}$. The result

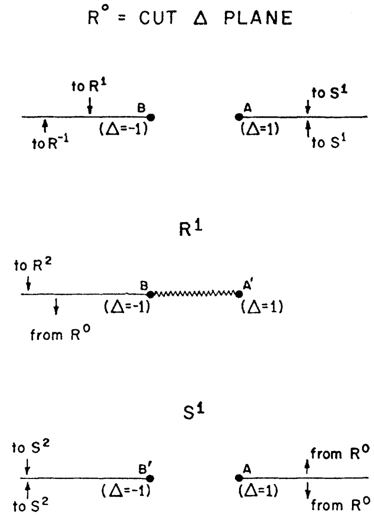

FIG. 1. Riemann sheets of $F(\Delta)$. S^2, S^3, etc., are similar to S^1. R^2, R^3, R^{-1}, R^{-2}, etc., are similar to R^1. The wavy line represents a natural boundary across which $F(\Delta)$ cannot be continued. See Fig. 2.

is

$$F(\Delta)=-\frac{1}{4}-\frac{\sqrt{2}}{6\pi}[\sqrt{(1-\Delta)}]^3+\left(\frac{1}{16}-\frac{1}{2\pi^2}\right)\times[\sqrt{(1-\Delta)}]^4+\cdots. \tag{45}$$

Unlike the neighborhood of $\Delta=-1$ discussed above in (iii), continuation of $F(\Delta)$ to $F(\Delta+i0)$ for $\Delta>1$ in the cut plane R^0 does not lead to anything resembling the value of $f(\Delta,0)$ given by (31a).

(vi) In R^0 the function $F(\Delta)$ has an asymptotic expansion (43) near the point $\Delta=-1$ [valid actually on R^0, R^1, and R^{-1} since Sec. 2C yielded the expansion in the μ plane around point B with only the exclusion of the natural boundary]. Now near $\Delta=-1$, μ^2 is a power series in $(1+\Delta)$. Thus, in R^0, (43) gives an asymptotic expansion of $F(\Delta)$ in powers of $(1+\Delta)$.

Equations (44a) and (44b) thus show that the asymptotic expansion of $f(\Delta,0)$ in powers of $(1+\Delta)$ is the same for $\Delta>-1$ and for $\Delta<-1$. (*The series has a radius of convergence*$=0$.) Therefore, $f(\Delta,0)$ and all of its derivatives with respect to Δ are continuous at $\Delta=-1$.

5. $f(\Delta,y)$ FOR $y=0+$

A. Case $\Delta<-1$

For $y=0+$, $b=\pi-\epsilon$, Eqs. (27), (24b″), and (24c) give

$$2\pi y=\eta(1-\mathbf{B})\{1-\mathbf{J}_0(1-\mathbf{B})+[\mathbf{J}_0(1-\mathbf{B})]^2-[\quad]^3+\cdots\}\mathbf{R}_0, \tag{46}$$

$$f(\Delta,y)=f(\Delta,0)+C^{-1}\mathbf{R}_0(1-\mathbf{B})\{\cdots\}\mathbf{R}_0,$$

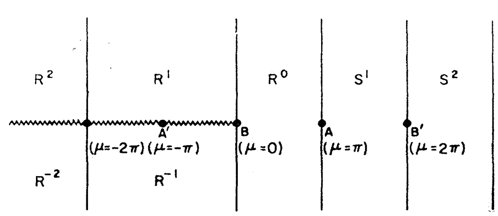

FIG. 2. Riemann sheets of Fig. 1 in μ plane.

where we have deleted all primes for simplicity of writing. Now by (14c) and (16c), $\langle\alpha|\mathbf{J}_0|\beta\rangle$ is periodic in α and in β, and $\langle\alpha|\mathbf{R}_0\rangle$ is periodic in α, all with period 2π. $\langle\alpha|\boldsymbol{\eta}\rangle=1$ is of course periodic. Thus, the combined interval ($-\pi$ to $-b$) and (b to π) can be replaced by one interval ($\pi-\epsilon$ to $\pi+\epsilon$) where $b=\pi-\epsilon$. We thus have

$$2\pi y=\boldsymbol{\eta}''\{1''-\mathbf{J}_0''+\mathbf{J}_0''^2-\cdots\}\mathbf{R}_0'',$$
$$f(\Delta,y)=f(\Delta,0)+C^{-1}\mathbf{R}_0''\{\cdots\}\mathbf{R}_0'', \tag{47}$$

where the double prime means that the α space extends from

$$(\pi-\epsilon) \quad \text{to} \quad (\pi+\epsilon).$$

Write

$$\langle\pi+\sigma|\mathbf{R}_0''\rangle=\sum_{n=-\infty}^{\infty}\frac{(-1)^n\cos n\sigma}{2\cosh n\lambda}$$
$$=e_0+e_2\sigma^2+e_4\sigma^4+\cdots, \tag{48a}$$

$$-\langle\pi+\sigma|J_0''|\pi+\tau\rangle=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\cos(\sigma-\tau)}{1+e^{2\lambda|n|}}$$
$$=f_0+f_2(\sigma-\tau)^2+f_4(\sigma-\tau)^4+\cdots. \tag{48b}$$

Then

$$2\pi y=e_0 2\epsilon+e_0f_0 4\epsilon^2+O(\epsilon^3),$$
$$\frac{2\pi}{\sinh\lambda}[f(\Delta,y)-f(\Delta,0)]=e_0 2\pi y+e_2e_0\tfrac{2}{3}\epsilon^3+O(\epsilon^4)$$
$$=2\pi e_0 y+\frac{2\pi^3}{3}\frac{e_2}{e_0^2}y^3+O(y^4), \tag{49}$$

where the e's are defined in (48a).

For $\Delta<-1$, *$f(\Delta,y)$ as a function of y has thus a cusp at $y=0$* [f is even in y]. The function $f(\Delta,y)$ can be analytically continued from $y>0$ to $y<0$ (cf. Sec. 3). But the continuation is not equal to $f(\Delta,y)$ for $y<0$.

The physical meaning of the cusp and the lack of the term y^2 for the magnetic problem and for the quantum-lattice-gas problem will be discussed in a later paper.

B. Case $\Delta=-1$

Equation (24a) means

$$\langle\alpha|\mathbf{R}'\rangle=R_0(\alpha)-\int_{-\infty}^{-b}J_0(\alpha-\beta)d\beta\langle\beta|\mathbf{R}'\rangle$$
$$-\int_b^\infty J_0(\alpha-\beta)d\beta\langle\beta|\mathbf{R}'\rangle,$$

where we have written $\langle\alpha|\mathbf{J}_0'|\beta\rangle=J_0(\alpha-\beta)$, according to (16b). Clearly $\langle\alpha|\mathbf{R}'\rangle$ is even in α. Thus,

$$\langle\alpha|\mathbf{R}'\rangle=R_0(\alpha)-\int_b^\infty J_0(\alpha-\beta)d\beta\langle\beta|\mathbf{R}'\rangle$$
$$-\int_b^\infty J_0(\alpha+\beta)d\beta\langle\beta|\mathbf{R}'\rangle. \tag{50}$$

Write $\langle\alpha|\mathbf{R}'\rangle=S(\alpha-b)$. Then

$$S(\sigma)=R_0(b+\sigma)-\int_0^\infty J_0(\sigma-\tau)d\tau\, S(\tau)$$
$$-\int_0^\infty J_0(2b+\sigma+\tau)d\tau\, S(\tau). \tag{51}$$

For large b, $J_0(2b+\sigma+\tau)$ for $\sigma\geqq0$, $\tau\geqq0$ is $0(b^{-2})$. (See Appendix C.) We shall treat the last integral as a perturbation, since without it the rest of the equation is of the Wiener-Hopf type:

$$S_0(\sigma)+\int_0^\infty J_0(\sigma-\tau)d\tau\, S_0(\tau)=R_0(b+\sigma), \tag{52a}$$

$$S_1(\sigma)+\int_0^\infty J_0(\sigma-\tau)d\tau\, S_1(\tau)$$
$$=-\int_0^\infty J_0(2b+\sigma+\tau)d\tau\, S_0(\tau), \tag{52b}$$

$$S_2(\sigma)+\int_0^\infty J_0(\sigma-\tau)d\tau\, S_2(\tau)$$
$$=-\int_0^\infty J_0(2b+\sigma+\tau)d\tau\, S_1(\tau),\quad \text{etc.}, \tag{52c}$$

$$S=S_0+S_1+S_2+\cdots. \tag{53}$$

In terms of S, (24b′) and (24c) become

$$(\pi-\mu)y=\int_0^\infty S(\sigma)d\sigma, \tag{54}$$

$$C[f(\Delta,y)-f(\Delta,0)]=2\int_0^\infty R_0(b+\sigma)S(\sigma)d\sigma. \tag{55}$$

To solve (52a) we write

$$R_0(b+\sigma)=\frac{2\pi e^{-\pi(b+\sigma)}}{1+e^{-2\pi(b+\sigma)}}$$
$$=2\pi[\zeta e^{-\pi\sigma}-\zeta^3e^{-3\pi\sigma}+\zeta^5e^{-5\pi\zeta}-\cdots], \tag{56}$$

where

$$\zeta=e^{-\pi b}. \tag{57}$$

Thus,

$$S_0(\sigma)=\sum_{n=0}^\infty T_n(\sigma)2\pi\zeta^{2n+1}(-1)^n, \tag{58}$$

where

$$T_n(\sigma)+\int_0^\infty J_0(\sigma-\tau)d\tau\, T_n(\tau)=e^{-(2n+1)\pi\sigma}. \tag{59}$$

This is a Wiener-Hopf equation. The transform of T_n

is known through Appendix D:

$$\tilde{T}_n(\omega)=\int_0^\infty e^{i\omega\sigma}T_n(\sigma)d\sigma=\frac{G_+(\omega)G_-[-i(2n+1)\pi]}{(2n+1)\pi-i\omega}=\text{independent of } b. \quad (60)$$

Now

$$\int_0^\infty S_0(\sigma)d\sigma=\sum_{n=0}^\infty 2\pi(-1)^n e^{-\pi b(2n+1)}\tilde{T}_n(0), \quad (61)$$

$$\int_0^\infty R_0(b+\sigma)S_0(\sigma)d\sigma=\sum_{n,m=0}^\infty 4\pi^2(-1)^{n+m}e^{-\pi b(2n+2m+2)}\times\tilde{T}_n[i\pi(2m+1)]. \quad (62)$$

Thus, for large b, if we first neglect $S_1+S_2+\cdots$, we obtain

$$y\cong 2[\zeta\tilde{T}_0(0)-\zeta^3\tilde{T}_1(0)+\zeta^5\tilde{T}_2(0)-\cdots], \quad (63)$$

$$f(\Delta,y)-f(\Delta,0)\cong 2\pi[\zeta^2\tilde{T}_0(i\pi)-\zeta^4\tilde{T}_0(3i\pi)-\zeta^4\tilde{T}_1(i\pi)+0(\zeta^6)]. \quad (64)$$

The correction terms $S_1+S_2+\cdots$ will introduce terms of the form

$$\frac{\zeta}{b}=\frac{\zeta}{(\ln\zeta)}\text{ (const)}$$

into y, and terms of the form

$$\frac{\zeta^2}{b^2}=\frac{\zeta^2}{(\ln\zeta)^2}\text{ (const)}$$

into f. We thus obtain

$$\zeta=y[2\tilde{T}_0(0)]^{-1}+O[y/(\ln y)], \quad (65)$$

$$f(\Delta,y)-f(\Delta,0)=y^2\frac{\pi\tilde{T}_0(i\pi)}{2[\tilde{T}_0(0)]^2}+O[y^2/(\ln y)]. \quad (66)$$

The coefficient of y^2 in this formula can be evaluated by using (60). It is

$$\frac{\pi}{2}\frac{G_+(i\pi)}{[G_+(0)]^2}\frac{G_-(-i\pi)}{[G_-(-i\pi)]^2}\frac{\pi^2}{2\pi}=\frac{\pi^2}{4}\frac{G_+(i\pi)}{[G_+(0)]^2[G_-(-i\pi)]}. \quad (67)$$

Using (D5) and (D6) this becomes

$$\frac{\pi^2}{4}\frac{1}{[G_+(0)]^2}=\frac{\pi^2}{4}\frac{1}{G_+(0)G_-(0)}=\frac{\pi^2}{8}. \quad (68)$$

This result has been conjectured by Griffith[18] on the basis of a numerical solution of (52a).

C. Case $-1<\Delta<1$

The method here is the same as above. In fact (50)–(55) are applicable to the present case as well. [$J_0(2b+\sigma+\tau)$ now falls off exponentially with b. See Appendix C.] In place of (56) we have

$$R_0(b+\sigma)=\frac{\pi}{2\mu\cosh s(b+\sigma)}=\frac{\pi}{\mu}[\zeta e^{-s\sigma}-\zeta^3e^{-3s\sigma}+\cdots], \quad (56')$$

where

$$\zeta=e^{-sb},\quad s=\pi/(2\mu). \quad (57')$$

Thus,

$$S_0(\sigma)=\frac{\pi}{\mu}\sum_{n=0}^\infty(-1)^n\zeta^{2n+1}T_n(\sigma) \quad (58')$$

where

$$T_n(\sigma)+\int_0^\infty J_0(\sigma-\tau)d\tau\, T_n(\tau)=\exp[-(2n+1)s\sigma]. \quad (59')$$

Similarly,

$$\tilde{T}_n(\omega)=\int_0^\infty e^{i\omega\sigma}T_n(\sigma)d\sigma=\frac{G_+(\omega)G_-[-i(2n+1)s]}{(2n+1)s-i\omega}, \quad (60')$$

$$(\pi-\mu)y\cong\frac{\pi}{\mu}[\zeta\tilde{T}_0(0)-\zeta^3\tilde{T}_1(0)+\cdots]. \quad (63')$$

$$C[f(\Delta,y)-f(\Delta,0)]\cong 2\left(\frac{\pi}{\mu}\right)^2[\zeta^2\tilde{T}_0(is)-\zeta^4\tilde{T}_0(3is)-\zeta^4\tilde{T}_1(is)+O(\zeta^6)]. \quad (64')$$

The corrections due to $S_1+S_2+\cdots$ will be treated in Appendix E. One obtains

$$\zeta=y\frac{\mu(\pi-\mu)}{\pi\tilde{T}_0(0)}[1+O(y^2)+O(y^{4\mu(\pi-\mu)^{-1}})], \quad (65')$$

$$f(\Delta,y)-f(\Delta,0)=\frac{\sin\mu}{\pi}(\pi-\mu)^2\frac{\tilde{T}_0(is)}{[\tilde{T}_0(0)]^2}\times y^2[1+O(y^2)+O(y^{4\mu(\pi-\mu)^{-1}})]. \quad (66')$$

Now

$$\frac{\tilde{T}_0(is)}{[\tilde{T}_0(0)]^2}=\frac{G_+(is)}{[G_+(0)]^2}\frac{G_-(-is)}{[G_-(-is)]^2}\frac{s^2}{2s}=\frac{G_+(is)s}{[G_+(0)]^2G_-(-is)2}=\frac{s}{2[G_+(0)]^2}=\frac{\pi^2}{8\mu(\pi-\mu)}.$$

Thus,

$$f(\Delta,y)-f(\Delta,0)=\frac{\pi(\pi-\mu)\sin\mu}{8\mu}\times y^2[1+O(y^2)+O(y^{4\mu(\pi-\mu)^{-1}})]. \quad (69)$$

The coefficient of y^2 is monotonically decreasing with increasing μ. Some special values are as follows:

$$\text{Coefficient of } y^2 \text{ in } f(\Delta,y)-f(\Delta,0)=\begin{cases}\pi^2/8, & \Delta=-1\\ \pi^2/8, & \Delta=-1+\\ \pi/8, & \Delta=0\\ 0, & \Delta=1-\\ 0, & \Delta\geq 1.\end{cases}$$

6. $f(\Delta,y)$ FOR $y=1-$

It is easy to evaluate f and y for small b, by direct iteration of (6a) in the variables p, q. One thus obtains

$$f(\Delta,y)=-\frac{\Delta}{4}+\frac{\Delta-1}{2}(1-y)+\frac{\pi^2}{48}(1-y)^3+O[(1-y)^4]. \quad (70)$$

7. $f(0,y)$

For $\Delta=0$, (6a) gives $2\pi\rho=1$. Thus,

$$f(0,y)=-\frac{1}{\pi}\cos\frac{\pi y}{2}. \quad (71)$$

This result is well known.[I10]

The results of this paper have been briefly announced in an earlier publication.[2]

APPENDIX A

We list here some useful formulas:

$$\int_{-\infty}^{\infty}\frac{e^{i\alpha\gamma}d\alpha}{\cosh\alpha-\cos\mu}=\frac{2\pi}{\sin\mu}\frac{\sinh[(\pi-\mu)\gamma]}{\sinh\pi\gamma}. \quad (A1)$$

$$\int_{-\infty}^{\infty}\frac{e^{i\alpha\gamma}d\alpha}{\cosh\alpha}=\frac{\pi}{\cosh(\pi\gamma/2)}. \quad (A1')$$

$$\int_{-\pi}^{\pi}\frac{e^{in\alpha}d\alpha}{\cosh\lambda-\cos\alpha}=\frac{2\pi}{\sinh\lambda}e^{-\lambda|n|}. \quad (A2)$$

$$\int_0^{\infty}\frac{dx}{1+e^x}=\ln 2. \quad (A3)$$

$$\int_{-\infty}^{\infty}\frac{dx}{(1+4x^2)\cosh(\pi x)}=\ln 2. \quad (A4)$$

$$\frac{1}{y-1}-\frac{1}{y^2-1}+\frac{1}{y^3-1}-\cdots=\frac{1}{y+1}+\frac{1}{y^2+1}+\frac{1}{y^3+1}+\cdots \quad (y>1). \quad (A5)$$

Proof:

$$\frac{1}{y^n-1}-\frac{2}{y^{2n}-1}=\frac{1}{y^n+1}.$$

Sum over n and one obtains (A5).

APPENDIX B

We shall now prove the inequality in (14c). To do this we first convert $R_0(\alpha)$ into an integral: Expand the denominator in the summand of (14c)

$$(2\cosh n\lambda)^{-1}=e^{-n\lambda}-e^{-3n\lambda}+e^{-5n\lambda}-\cdots, \quad (B1)$$

then sum over n, yielding

$$R_0(\alpha)-\tfrac{1}{2}=\left[\frac{\sinh\lambda}{\cosh\lambda-\cos\alpha}-1\right]-[\text{same with }\lambda\to3\lambda]+[\text{same with }\lambda\to5\lambda]-\cdots$$

$$=-\frac{1}{4i}\int\left[\frac{\sinh n\lambda}{\cosh n\lambda-\cos\alpha}-1\right]\frac{dn}{\cos(\pi n/2)}, \quad (B2)$$

where the integration loops around all positive odd integers n counterclockwise. Detour to $n=+0+2iy$:

$$R_0(\alpha)=\frac{1}{2}+\frac{1}{2}\int_{-\infty-i0}^{\infty-i0}\frac{dy}{\cosh\pi y}\left[\frac{i\sin 2y\lambda}{\cos 2y\lambda-\cos\alpha}-1\right]$$

$$=\frac{i}{2}\int_{-\infty-i0}^{\infty-i0}\frac{dy}{\cosh\pi y}\frac{\sin 2y\lambda}{\cos 2y\lambda-\cos\alpha}. \quad (B3)$$

Take the average of this equation with its complex conjugate

$$R_0(\alpha)=\frac{i}{4}\left[\int_{-\infty-i0}^{\infty-i0}+\int_{\infty+i0}^{-\infty+i0}\right]\frac{dy}{\cosh\pi y}\frac{\sin 2y\lambda}{\cos 2y\lambda-\cos\alpha}. \quad (B4)$$

Thus, $R_0(\alpha)$ is a contour integral. Its value can be readily evaluated:

$$R_0(\alpha)=\frac{\pi}{2\lambda}\sum_{n=-\infty}^{\infty}\operatorname{sech}\left[\frac{\pi}{2\lambda}(\alpha+2\pi n)\right]>0. \quad (B5)$$

APPENDIX C

For $0<\Delta$, $-\langle\alpha|\mathbf{K}|\beta\rangle>0$, thus $\mathbf{J}=-\mathbf{K}+\mathbf{K}^2-\mathbf{K}^3+\cdots$ has all elements >0.

For $-1<\Delta<0$, we write the denominator of the integrand in (16a) as $2\sinh(\pi-\mu)\gamma\cosh\mu\gamma$ and then express $(\cosh\mu\gamma)^{-1}$ as a Fourier integral through (A1′). The γ integration can then be performed by first writing $(\pi-\mu)\gamma=\pi\xi$ and using the Fourier inverse of (A1).

[2] C. N. Yang and C. P. Yang, Phys. Letters 20, 9 (1966); 21, 719 (1966).

Thus, we obtain

$$\langle\alpha|\mathbf{J}_0|\beta\rangle=-\frac{\sin\nu}{8\mu(\pi-\mu)}\int_{-\infty}^{\infty}\frac{d\eta}{\{\cosh[\pi(\alpha-\beta-\eta)(\pi-\mu)^{-1}]-\cos\nu\}\cosh[\pi\eta(2\mu)^{-1}]}<0, \tag{C1}$$

where

$$\frac{\pi-\nu}{\pi}=\frac{\pi-2\mu}{\pi-\mu}. \tag{C2}$$

For $\Delta=-1$, we write (16b) as

$$\langle\alpha|\mathbf{J}_0|\beta\rangle=-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\exp[i\gamma(\alpha-\beta)-\frac{1}{2}|\gamma|]}{\cosh(\gamma/2)}d\gamma$$

and then express $[\cosh(\gamma/2)]^{-1}$ as a Fourier integral through (A1′). The γ integration is then trivial and we arrive at

$$\langle\alpha|\mathbf{J}_0|\beta\rangle=\frac{-1}{4\pi}\int_{-\infty}^{\infty}\frac{d\eta}{\cosh(\pi\eta)}\frac{4}{1+4(\alpha-\beta-\eta)^2}<0. \tag{C3}$$

For $\Delta<-1$, we proceed to treat the sum in (16c) similarly.

$$\begin{aligned}\langle\alpha|\mathbf{J}_0|\beta\rangle&=-\frac{1}{4\pi}\sum_n\frac{\exp[in(\alpha-\beta)-\lambda|n|]}{\cosh(\lambda n)}\\&=-\frac{1}{4\pi^2}\sum_n\int_{-\infty}^{\infty}\frac{\exp[in(\alpha-\beta)-\lambda|n|+in2\eta\lambda\pi^{-1}]}{\cosh\eta}d\eta\\&=-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{d\eta}{\cosh\eta}\left\{\frac{1}{1-\exp[i(\alpha-\beta)-\lambda+i2\eta\lambda\pi^{-1}]}+\text{c.c.}-1\right\}<0,\end{aligned} \tag{C4}$$

since the curly bracket is always >0.

To find the asymptotic form for $\langle\alpha|\mathbf{J}_0|\beta\rangle$ for large $\alpha-\beta$, we take first the case $-1<\Delta<1$. The integrand in (16a) has poles at

$$\gamma_n=\frac{\pi i}{\pi-\mu}n,\quad\text{and}\quad\gamma_n'=\frac{\pi i}{2\mu}(2n-1), \tag{C5}$$

$$(n=\text{any integer}).$$

(But $\gamma=0$ is not a pole.) For $\alpha-\beta>0$, we close the contour around the upper half γ plane and obtain $\langle\alpha|\mathbf{J}_0|\beta\rangle$ as a sum over the residues at the poles (C5) for $n>0$. If $(\pi-\mu)/\mu$ is irrational, all poles are simple. Otherwise, some are simple poles, some double. Let the successive simple poles along the positive imaginary axis be $i\xi_1$, $i\xi_2$, $\cdots$; the successive double poles, $i\eta_1$, $i\eta_2$, $\cdots$. Then for $\alpha>\beta$,

$$\langle\alpha|\mathbf{J}_0|\beta\rangle=\sum_i g_{1i}e^{-\xi_i(\alpha-\beta)}+\sum_i[g_{2i}+(\alpha-\beta)g_{3i}]e^{-\eta_i(\alpha-\beta)}, \tag{C6}$$

where g_{1i}, g_{2i}, and g_{3i} are numerical constants. (C6) gives the asymptotic form of $\langle\alpha|\mathbf{J}_0|\beta\rangle$ for large $\alpha-\beta$. Notice ξ_i and η_i are all >0 and are integral multiples of $\pi/(\pi-\mu)$, or $\pi/2\mu$.

To find the asymptotic form of $\mathbf{J}_0$ for $\Delta=-1$, we use (16b) and integrate by parts:

$$\begin{aligned}-\pi\langle\alpha|\mathbf{J}_0|\beta\rangle&=\int_0^{\infty}f(\gamma)\cos(\gamma\phi)d\gamma\\&=-\frac{1}{\varphi^2}f'(0)-\frac{1}{\varphi^2}\int_0^{\infty}f''(\gamma)\cos\gamma\varphi d\gamma\\&=-\frac{1}{\varphi^2}f'(0)+\frac{1}{\varphi^4}f'''(\gamma)\\&\qquad+\frac{1}{\varphi^4}\int_0^{\infty}f^{(4)}(\gamma)\cos\gamma\phi d\gamma\\&=\cdots,\end{aligned} \tag{C7}$$

where

$$f(\gamma)=1/(1+e^{\gamma})\quad\text{and}\quad\varphi=\alpha-\beta.$$

Carrying this procedure to the dominant term we obtain

$$\langle\alpha|\mathbf{J}_0|\beta\rangle=-\frac{1}{4\pi(\alpha-\beta)^2}+O\left(\frac{1}{(\alpha-\beta)^4}\right). \tag{C8}$$

APPENDIX D

Equation (59) is a Wiener-Hopf equation. Its solution can be found by a Wiener-Hopf factorization.[3] One

[3] M. G. Krein, Usp. Mat. Nauk (N.S.) **13**, No. 5 (83), pp. 3–120. [English transl.: Am. Math. Soc. Translations, Series 2, **22**, 163 (1962).]

defines

$$\tilde{J}_0(\omega)=\int_{-\infty}^{\infty} e^{i\omega\sigma}J_0(\sigma)d\sigma$$
$$=-\frac{\sinh(\pi-2\mu)\omega}{\sinh[(\pi-2\mu)\omega]+\sinh(\pi\omega)},\ (-1<\Delta<1)$$
$$=-\frac{1}{1+e^{|\omega|}}\quad(\Delta=-1). \tag{D1}$$

One then factorizes

$$\frac{1}{1+\tilde{J}_0(\omega)}=G_+(\omega)G_-(\omega), \tag{D2}$$

where

$G_+(\omega)$ $[G_-(\omega)]$ is analytic in the open upper [lower] half-plane, continuous and different from zero in the upper (lower) half-plane plus the real axis, (D3)

and

$$G_+(\infty)=1. \tag{D4}$$

For our problem,

$$G_+(\omega)=G_-(-\omega). \tag{D5}$$

Therefore,

$$\frac{1}{[G_+(0)]^2}=1+\tilde{J}_0(0)=\frac{\pi}{2(\pi-\mu)},\quad(-1<\Delta<1);$$
$$=\tfrac{1}{2},\quad(\Delta=-1). \tag{D6}$$

For a general Wiener-Hopf equation, Krein[3] described a method of solution which involves lengthy calculations. For the special case of Eq. (59) where the inhomogeneous term is a pure exponential, it can be shown, by a variation of his arguments, that the solution is given by (60).

APPENDIX E

We concentrate on the case $-1<\Delta<1$. To find the correction due to S_1, we introduce some notations:

$$s=\pi/2\mu,\quad t=\pi/(\pi-\mu),$$
$$\zeta=e^{-bs},\quad \theta=e^{-bt}. \tag{E1}$$

$$J_0(\alpha)=u_1e^{-s\alpha}+v_1e^{-t\alpha}+u_2e^{-3s\alpha}+v_2e^{-2t\alpha}+\cdots, \tag{E2}$$

where we have assumed $(\pi-\mu)/\mu$ to be irrational and have used the expansion (C6) with no double poles. The coefficients u_1, v_1, etc., are numerical constants:

$$u_1=+\frac{1}{2\mu}\left[\cot\frac{(\pi-\mu)\pi}{2\mu}\right], \tag{E3}$$

$$v_1=-\frac{1}{2(\pi-\mu)}\left[\tan\frac{\pi\mu}{\pi-\mu}\right],\quad\text{etc.} \tag{E3$'$}$$

In the following, the b dependence of all quantities will be only through their dependence on ζ and θ. O(qu.) means "of quartic order in θ and ζ."

$$R_0(b+\sigma)=\frac{\pi}{\mu}\zeta[e^{-s\sigma}-\zeta^2e^{-3s\sigma}+O(\zeta^4)], \tag{E4}$$

$$J_0(2b+\sigma+\tau)=u_1\zeta^2e^{-s\sigma-s\tau}+v_1\theta^2e^{-t\sigma-t\tau}+O(\text{qu.}), \tag{E5}$$

$$S_0(\sigma)=\frac{\pi}{\mu}\zeta(W_s-\zeta^2W_{3s}+O(\zeta^4)], \tag{E6}$$

where $W_x(\sigma)$ satisfies

$$W_x(\sigma)+\int_0^\infty J_0(\sigma-\tau)W_x(\tau)=e^{-x\sigma},\quad(x>0). \tag{E7}$$

Thus,

$$T_n=W_{(2n+1)s}.$$

Substituting (E5) and (E6) into (52b) we can solve for S_1:

$$S_1(\sigma)=-\frac{\pi}{\mu}\zeta[u_1\zeta^2(s;s)W_s+v_1\theta^2(t;s)W_t+O(\text{qu.})], \tag{E8}$$

where

$$(y;x)=\int_0^\infty e^{-y\sigma}W_x(\sigma)d\sigma. \tag{E9}$$

Using (60′) we find

$$(y;x)=\frac{G_+(iy)G_-(-ix)}{x+y}=\frac{G_+(iy)G_+(ix)}{x+y}=(x;y),$$
$$(x+y>0). \tag{E10}$$

It is clear that

$$S_2(\sigma)=\zeta O(\text{quartic polynomial in }\zeta\text{ and }\theta),$$
$$S_3(\sigma)=\zeta O(\text{6th degree polynomial in }\zeta\text{ and }\theta),\text{ etc.} \tag{E11}$$

Thus, by (24b′)

$$(\pi-\mu)y=\int_0^\infty S(\sigma)d\sigma=\frac{\pi}{\mu}\zeta[(0;s)-\zeta^2(0;3s)$$
$$-u_1\zeta^2(s;s)(0;s)-v_1\theta^2(t;s)(0;t)+O(\text{qu.})]. \tag{E12}$$

By (24c),

$$\tfrac{1}{2}C[f(\Delta,y)-f(\Delta,0)]$$
$$=\int_0^\infty R_0(b+\sigma)S(\sigma)d\sigma$$
$$=\left(\frac{\pi\zeta}{\mu}\right)^2[(s;s)-2\zeta^2(3s;s)-u_1\zeta^2(s;s)^2$$
$$-v_1\theta^2(t;s)^2+O(\text{qu.})]. \tag{E13}$$

Thus,

$$\zeta=y_1+y_1^3\left[\frac{(0;3s)}{(0;s)}+u_1(s;s)\right]+y_0\theta^2\frac{(t;s)(0;t)}{(0;s)}v_1+y_1O(\text{qu.}),\qquad\text{(E14)}$$

$$\theta^2=\zeta^{4\mu/(\pi-\mu)}=y_1^{4\mu/(\pi-\mu)}+O(\text{qu.}),\qquad\text{(E15)}$$

where

$$y_1=\frac{\mu(\pi-\mu)}{\pi(0;s)}y.\qquad\text{(E16)}$$

(E13) gives, in terms of y,

$$f(\Delta,y)-f(\Delta,0)=A_0y^2\{1+y^2d_1+y^{4\mu/(\pi-\mu)}d_2+O(y^4)+O(y^{8\mu/(\pi-\mu)})\},\qquad\text{(E17)}$$

where A_0 was given before in (69), and

$$d_1=\left[\frac{\mu(\pi-\mu)}{\pi(0;s)}\right]^2\left[\frac{2(0;3s)}{(0;s)}+u_1(s;s)-2\frac{(3s;s)}{(s;s)}\right],\qquad\text{(E18)}$$

$$d_2=\left[\frac{\mu(\pi-\mu)}{\pi(0;s)}\right]^{4\mu/(\pi-\mu)}v_1\left[\frac{2(t;s)(0;t)}{(0;s)}-\frac{(t;s)^2}{(s;s)}\right].\qquad\text{(E19)}$$

Writing $G_+(ix)=g_x$ we obtain from (E10)

$$d_2=\left[\frac{\pi-\mu}{2g_sg_0}\right]^{4\mu/(\pi-\mu)}v_1\frac{2s^2}{(t+s)^2t}g_t^2.\qquad\text{(E20)}$$

Now by explicit construction,

$$g_x=\frac{\Gamma(1+x)}{\Gamma[1+x(\pi-\mu)/\pi]\Gamma[\tfrac{1}{2}+x(\mu/\pi)]}\left[2\pi\left(1-\frac{\mu}{\pi}\right)\right]^{1/2}\times\exp\left[x\ln\left(1-\frac{\mu}{\pi}\right)-\frac{x\mu}{\pi}\ln\frac{\pi-\mu}{\mu}\right]>0.\qquad\text{(E21)}$$

When $\mu/(\pi-\mu)=\text{irrational}$, $d_2\neq0$. Hence if $\mu/(\pi-\mu)=\text{irrational}$,

$$\lim_{y\to0}\left[\left(\frac{d}{dy}\right)^n f(y,\Delta)\right]=\text{finite}\quad\text{for}\quad n<2+\frac{4\mu}{\pi-\mu}$$
$$=\pm\infty\quad\text{for}\quad n>2+\frac{4\mu}{\pi-\mu}.\qquad\text{(E22)}$$

It is not difficult to show that (E22) *is valid also for* $4\mu/(\pi-\mu)=\text{rational}$, but $\neq$ an integer.

For integral values of $4\mu/(\pi-\mu)$, sometimes $f(y,\Delta)$ is analytic at $y=0$. An example is when $\Delta=0$, $4\mu/(\pi-\mu)=4$. See Sec. 7. Other integral values of $4\mu/(\pi-\mu)$ are under investigation.

(E22) shows that for $4\mu/(\pi-\mu)\neq\text{integer}$, the zero-temperature susceptibility $\chi(\mathcal{H})$ as a function of the magnetic field has some high-order derivative (with respect to $\mathcal{H})\to\pm\infty$ when $\mathcal{H}\to0$. This will be discussed in a later paper.