temperatures on the T-62 He^3 vapor-pressure scale with an error less than 0.0027°K, over the temperature range 0.82 to 1.0833°K. Since this range of temperatures is a very awkward region in which to calibrate an apparatus lacking a He^3 vapor bulb, the critical field of gallium provides an excellent secondary standard.

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One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System

C. N. Yang

Institute for Advanced Study, Princeton, New Jersey

and

State University of New York, Stony Brook, New York

AND

C. P. Yang

Physics Department, Ohio State University, Columbus, Ohio

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Bethe's hypothesis is proved for the ground state of a one-dimensional cyclic chain of anisotropic nearest-neighbor spin-spin interactions. The proof holds for any fixed number of down spins.

I. INTRODUCTION

THE eigenvalue spectrum of the Hamiltonian

$$H=-\tfrac{1}{2}\sum\{\sigma_x\sigma_x'+\sigma_y\sigma_y'+\Delta\sigma_z\sigma_z'\} \qquad (1)$$

is of current interest. In (1) $\boldsymbol{\sigma}$ are the Pauli spin matrices at a particular site ($\sigma_x^2=\sigma_y^2=\sigma_z^2=1$), $\boldsymbol{\sigma}'$ are the Pauli spin matrices at a neighboring site. Δ is a real numerical constant ($\Delta=1$ corresponds to the isotropic ferromagnetic problem, $\Delta=-1$ the isotropic antiferromagnetic problem[1]). The sum extends over all nearest neighbors in a 1-dimensional linear, 2-dimensional square, or 3-dimensional simple cubic lattices with *cyclic* boundaries.

The significance of (1) in the theory of ferromagnetism and the theory of antiferromagnetism is well known. (1) is also the problem to consider for the quantum lattice gas.[2] [In particular, the ground-state energy and the thermodynamical properties of a system with the Hamiltonian (1) can be transformed to give the ground-state energy and the thermodynamical properties of a quantum lattice gas. This quantum lattice gas is a Bose gas moving on a lattice with (a) a quantum kinetic energy, not in the form of an operator $(-\hbar^2/2\mathfrak{M})\nabla^2$, but in the form of a double difference, (b) a hard core preventing two atoms from occupying the same site, and (c) an energy of interaction equal to -2Δ for nearest neighbors. See Table I.]

Let y be the magnetization per site,

$$y=\text{eigenvalue of } (1/\mathfrak{N})\sum\sigma_z\,, \qquad (2)$$

where $\mathfrak{N}$ = total number of sites in the lattice. One is particularly interested in the function

$$f(\Delta,y)=\lim_{\mathfrak{N}\to\infty}\frac{1}{\mathfrak{N}z}\quad(\text{lowest eigenvalue of } H \text{ for fixed } y)\,, \qquad (3)$$

which is half of the ground-state energy per bond for a fixed y. Here z is the number of nearest neighbors at each site. The existence of the limiting function $f(\Delta,y)$ was proved in Ref. 1. A number of general properties of f was also established there. In particular, inequalities were given between the f for one-, two-, and three-dimensional lattices.

The purpose of this and subsequent papers is to study properties of the Hamiltonian (1) for the one-dimensional linear cyclic chain.

This problem was studied by approximate methods by Bloch.[3] Bethe[4] then proposed that the eigenfunctions are of a certain specific form (to be called Bethe's hypothesis). The particular case $\Delta=-1$ (antiferro-

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TABLE I. Physical problems for different values of Δ.

	Quantum lattice gas with	Hamiltonian (1) is equivalent to
$\Delta > 0$	Attractive interaction outside of hard core	Anisotropic ferromagnetic Hamiltonian ($\Delta = 1$ corresponds to isotropic case)
$\Delta < 0$	Repulsive interaction outside of hard core	Anisotropic antiferromagnetic Hamiltonian (Ref. 1) ($\Delta = -1$ corresponds to isotropic case)

magnetic isotropic case) was considered in detail by Hulthén,[5] who gave an evaluation of $f(-1, 0)$ using Bethe's hypothesis. Later, Orbach[6] extended these considerations and obtained an integral equation which he numerically solved to evaluate $f(\Delta,0)$ for $\Delta \leqq -1$, again using Bethe's hypothesis. The integral equation was later solved by series expansion by Walker,[7] who obtained $f(\Delta,0)$ for $\Delta \leqq -1$ as a series. Griffiths[8] investigated the problem of $f(-1, y)$ and des Cloizeaux and Pearson[9] the excited states at $\Delta = -1$, $y = 0$. (See Fig. 1.) Lieb, Schultz, and Mattis[10] and Katsura[10] studied the case $\Delta = 0$.

In this series of papers we study the problem for general values of Δ and y. In the process we also establish *rigorously* the validity of Bethe's hypothesis for the ground state. These papers will use the same notation as Ref. 1 and will form a self-contained series.

2. BETHE'S HYPOTHESIS

We generalize in this section Bethe's hypothesis to the general case of $\Delta < 1$.

Consider an eigenfunction ψ of H with m down spins and $\mathfrak{N} - m$ up spins. Clearly,

$$y = 1 - 2(m/\mathfrak{N}). \tag{4}$$

We assume

$$2m \leqq \mathfrak{N}, \quad \text{or} \quad y \geqq 0. \tag{5}$$

Let $x_1, x_2, \cdots, x_m$ (in ascending order) be the sites with down spins. ($1 \leqq x_j \leqq \mathfrak{N}$). Bethe's hypothesis says that

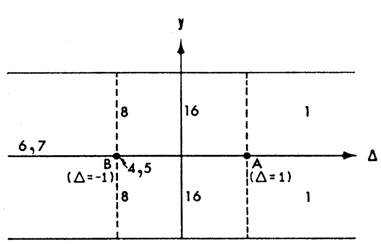

FIG. 1. Δ and y values where $f(\Delta,y)$ has been discussed in the literature. The numbers are the reference numbers quoted in this paper. The dotted line through A represents the isotropic ferromagnetic case. That through B represents the isotropic antiferromagnetic case.

there are m *unequal real* numbers $p_1 \cdots p_m$ such that the wave function ψ is a sum of $m!$ terms each of which is of the exponential form

$$(\text{constant}) \exp i[p_{P1}x_1 + p_{P2}x_2 + \cdots], \tag{6}$$

where $(P1, P2, P3, \cdots Pm)$ is a permutation of 1, 2, 3, $\cdots m$. In other words,

$$\psi = \sum_P A_P \exp i[\sum_j p_{Pj} x_j]. \tag{7}$$

It will be further assumed[11] that the p's are within the following range:

$$-\pi < p_j < \pi, \quad \text{for} \quad \Delta \leqq -1; \tag{8}$$

$$-(\pi - \mu) < p_j < \pi - \mu, \quad \text{for} \quad -1 \leqq \Delta < 1; \tag{9}$$

where

$$0 \leqq \mu < \pi, \quad \cos\mu = -\Delta. \tag{10}$$

Clearly, $\cos p_j > \Delta$. We plot the range of p_j in Fig. 2.

For large $\mathfrak{N}$ and m, but with $m/\mathfrak{N} = $ fixed, the number of A_P's is larger than the number of spin arrangements. (7) is therefore not in general a hypothesis without further conditions on the A_P's. These conditions are stated below in (16) and (17) and form an integral part of Bethe's hypothesis.

We now examine the following points:

(a) Consider the equation $H\psi = E\psi$ at a configuration in which no down spins are nearest neighbors of each other. Write

$$H = -(\Delta/2)\mathfrak{N} - \tfrac{1}{2} \sum [\sigma_x\sigma_x' + \sigma_y\sigma_y' + \Delta\sigma_z\sigma_z' - \Delta].$$

The square bracket operating on any state for which the two spins in question are both up or both down gives zero. It is then easy to see that $H\psi = E\psi$ is satisfied for the configuration studied if

$$E = -(\Delta/2)\mathfrak{N} + \sum_j (2\Delta - 2\cos p_j). \tag{11}$$

(One can see this most easily by taking $m = 2$, then $m = 3$, etc.)

(b) Consider the equation $H\psi = E\psi$ at a configuration in which among the down spins there is exactly one pair of nearest neighbors. Using (11) one sees that $H\psi = E\psi$ is satisfied if

$$\frac{A_P}{A_{P'}} = -\frac{2\Delta e^{ip} - 1 - e^{ip+iq}}{2\Delta e^{iq} - 1 - e^{ip+iq}},$$

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[11] The original Bethe hypothesis was broader than that stated here. Our more restrictive form makes it easier to prove the validity of the hypothesis for the ground state.

where P and P' are any two permutations so that

$$p_{P1},\ p_{P2}\cdots=\cdots p,\ q\cdots \tag{12}$$

and

$p_{P'1},\ p_{P'2}\cdots=$ same as above except with p and q switched.

(These points are again easily proved first for $m=2$, then for $m=3$, etc.) Define[12]

$$\Theta(p,q)=+2\tan^{-1}\left[\frac{\Delta\sin[(p-q)/2]}{\cos[(p+q)/2]-\Delta\cos[(p-q)/2]}\right]. \tag{13}$$

Notice

$$\Theta(p,q)=-\Theta(q,p). \tag{14}$$

Then

$$A_P/A_{P'}=-e^{-i\Theta(p,q)}. \tag{15}$$

Equations (14) and (15) lead to a solution of A_P in terms of A_0 (i.e., the A_P for $P=$identity):

$$A_P/A_0=\pm\exp\{-i\sum\Theta(p_j,p_l)\}, \tag{16}$$

where the sign is + for P = even and − for odd and the summation extends over all pairs p_j, p_l for which $j>l$ and j stands to the left of l in the sequence $P1$, $P2$, $P3$, $\cdots$. (j and l need not be consecutive.)

(c) Consider the equation $H\psi=E\psi$ at other configurations. It is easy to prove that (15) ensures that $H\psi=E\psi$ is satisfied.

(d) The cyclic boundary condition must be imposed on (7). Using (7) the condition is fulfilled if for all P

$$A_P=A_{P''}\exp(p_{P1}\mathfrak{N}),$$

where

$$P''1,\ P''2,\cdots=P2,\ P3,\ \cdots Pm,\ P1.$$

Because of (16) this condition is in turn fulfilled if

$$\exp(ip_j\mathfrak{N})=(-1)^{m-1}\exp[-i\textstyle\sum_l\Theta(p_j,p_l)],\quad j=1\to m. \tag{17}$$

One of the possible sets of solutions[13] of this equation, upon taking the logarithm, is

$$\mathfrak{N}p_j=2\pi I_j-\sum_{l=1}^{m}\Theta(p_j,p_l), \tag{18}$$

where

$$I_1,I_2,\cdots I_m=\left(-\frac{m-1}{2}\right),\left(-\frac{m-1}{2}+1\right),\cdots\left(\frac{m-1}{2}\right). \tag{19}$$

Notice[14] that for

$$\begin{aligned}&m=\text{even}, && I_j=\text{half-odd integer},\\ &m=\text{odd}, && I_j=\text{integer}.\end{aligned} \tag{20}$$

Thus for every set of p_j satisfying (18), (8), *and* (9), *we can construct an eigenfunction* (7) *for the Hamiltonian* (1), by taking[15] $A_0\neq 0$ and substituting (16) into (7).

3. SOME PROPERTIES OF THE FUNCTION Θ

It is convenient, to study $\Theta(p,q)$ in the interval (8) and (9), to apply the following transformations[16] $p\leftrightarrow\alpha$:

$$\Delta<-1:\ \Delta=-\cosh\lambda,\quad \lambda>0, \tag{21a}$$

$$e^{ip}=\frac{e^{\lambda}-e^{-i\alpha}}{e^{\lambda-i\alpha}-1}, \tag{21b}$$

$$-\pi<p<\pi\leftrightarrow-\pi<\alpha<\pi,$$

$$p(-\alpha)=-p(\alpha),$$

$$\cos p=-\cosh\lambda+\frac{\sinh^2\lambda}{\cosh\lambda-\cos\alpha},$$

$$\sin p=\frac{\sinh\lambda\,\sin\alpha}{\cosh\lambda-\cos\alpha},$$

$$\frac{dp}{d\alpha}=\frac{\sin p}{\sin\alpha}=\frac{\sinh\lambda}{\cosh\lambda-\cos\alpha}>0, \tag{21c}$$

$$\Theta(p,q)=2\tan^{-1}\left[(\coth\lambda)\tan\frac{\beta-\alpha}{2}\right]\equiv\theta(\alpha,\beta). \tag{21d}$$

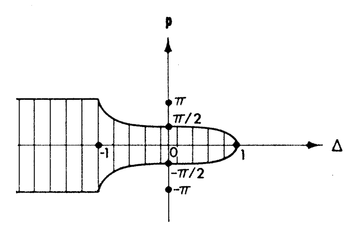

FIG. 2. Interval in which p lies.

[12] Θ is a single-valued real analytic function of Δ, p and q if the latter two are in the open interval given for p_j in (8) and (9). $\Theta(0,0)=0$. These conditions define uniquely the branch of $\tan^{-1}$ to take in (13). The function Θ becomes more visualizable after the transformation (21) to be discussed later. The range of values of Θ will also be given there.

[13] (18) is the same as the solution chosen by Bethe (Ref. 4), Hulthén (Ref. 5), and Orbach (Ref. 6) in their special cases. The notation here is, however, different from that in their papers. The main points in the difference are (a) we use $\tan^{-1}$ instead of $\cot^{-1}$ in (13). This difference results in our $\Delta I=1$ in (19), while in Orbach, the corresponding $\Delta\lambda=2$. (b) Our range of p as given in (8) is shifted by π from the previous convention. This is because our Hamiltonian (1) at $\Delta\leq-1$ is related to Orbach's by a unitary transformation. (See Ref. 1.) Our definition (13) and the range (8) and (9) are chosen to facilitate continuity arguments with respect to Δ which we shall need later on for proving Bethe's hypothesis.

[14] By making I_j half-odd integral for the case $m=$even, one can treat all values of m together. Notice that, however, this method works in the case of the quantum lattice gas only for bosons.

[15] Provided (7) is not identically zero for all x_j where $1\leq x_j<x_2\cdots<x_m\leq\mathfrak{N}$. This provision is probably satisfied for all $\Delta<1$, $m\leq\mathfrak{N}/2$. We have so far, however, only succeeded in proving it, for each fixed $\mathfrak{N}$, for sufficiently small m. However, by a roundabout argument in Sec. 5 we circumvent the necessity of an explicit proof.

[16] The transformation for the case $\Delta<-1$ was used by Walker (Ref. 7), and for $\Delta=-1$ by Hulthén (Ref. 5).

$-2\pi<\Theta<2\pi$, $\Theta=$continuous in p and q.

$$\frac{\partial\theta}{\partial\beta}=-\frac{\partial\theta}{\partial\alpha}=\frac{\sinh 2\lambda}{\cosh 2\lambda-\cos(\alpha-\beta)}>0. \quad (21e)$$

$-1<\Delta<1$: $\Delta=-\cos\mu$, $0<\mu<\pi$, (21f)

$$e^{ip}=\frac{e^{i\mu}-e^{\alpha}}{e^{i\mu+\alpha}-1}, \quad (21g)$$

$$-(\pi-\mu)<p<(\pi-\mu)\leftrightarrow-\infty<\alpha<+\infty,$$

$$p(-\alpha)=-p(\alpha),$$

$$\cos p=-\cos\mu+\frac{\sin^2\mu}{\cosh\alpha-\cos\mu},$$

$$\sin p=\frac{\sin\mu\sinh\alpha}{\cosh\alpha-\cos\mu},$$

$$\frac{dp}{d\alpha}=\frac{\sin p}{\sinh\alpha}=\frac{\sin\mu}{\cosh\alpha-\cos\mu}>0, \quad (21h)$$

$$\Theta(p,q)=2\tan^{-1}\left[(\cot\mu)\tanh\frac{\beta-\alpha}{2}\right]\equiv\theta(\alpha,\beta), \quad (21i)$$

$$-|\pi-2\mu|<\theta<|\pi-2\mu|,$$

$$\frac{\partial\theta}{\partial\beta}=-\frac{\partial\theta}{\partial\alpha}=\frac{\sin 2\mu}{\cosh(\alpha-\beta)-\cos 2\mu}. \quad (21j)$$

$\Delta=-1$: $\alpha=\frac{1}{2}\tan p/2$, (21k)

$$-\pi<p<\pi\leftrightarrow-\infty<\alpha<+\infty,$$

$$\frac{dp}{d\alpha}=4\cos^2\frac{p}{2}=\frac{4}{1+4\alpha^2}>0, \quad (21l)$$

$$\Theta(p,q)=2\tan^{-1}(\beta-\alpha)\equiv\theta(\alpha,\beta), \quad (21m)$$

$$-\pi<\theta<\pi,$$

$$\frac{\partial\theta}{\partial\beta}=-\frac{\partial\theta}{\partial\alpha}=\frac{2}{1+(\alpha-\beta)^2}. \quad (21n)$$

We notice that for all cases

$$\cos p=\Delta+\frac{2\pi}{C}\frac{dp}{d\alpha}, \quad (21o)$$

where

$$C=\frac{2\pi}{\sinh\lambda},\ \frac{2\pi}{\sin\mu}\ \text{ or }\ 4\pi \quad (21p)$$

for the three cases, respectively.

Using these, and also the original form of Θ in (13), it is easy to see the following:

(a) Reference 12 is correct.

(b) Θ can be extended to the boundary of the open square (8) and (9) for p and q. For $\Delta<-1$, there are no singularities of Θ in the closed square $-\pi\leqq p\leqq\pi$, $-\pi\leqq q\leqq\pi$. For $-1\leqq\Delta<1$, the only singularities of Θ in the closed square $-(\pi-\mu)\leqq p\leqq(\pi-\mu)$, $-(\pi-\mu)\leqq q\leqq\pi-\mu$ are at

$$p=q=\pi-\mu \quad\text{and}\quad p=q=-(\pi-\mu), \quad (22)$$

at which Θ is discontinuous.

(c) $\Theta(\pi-\mu,q)=2\mu-\pi$, (for $-1\leqq\Delta<1$), (23)

except at $q=\pi-\mu$, where Θ is discontinuous.

(d) $\Theta(-p,-q)=-\Theta(p,q)=\Theta(q,p)$. (24)

It is useful, for discussing Δ dependence, to make a further transformation (for all $\Delta<1$):

$$p\leftrightarrow\alpha\leftrightarrow a,$$

where

$$a=C\alpha/(2\pi)=\int_0^p\frac{dp}{\cos p-\Delta}. \quad (25)$$

The intervals (8) and (9) become

$$\frac{-\pi}{\sinh\lambda}<a<\frac{\pi}{\sinh\lambda}\qquad \Delta<-1, \quad (26a)$$

$$-\infty<a<\infty\qquad -1\leqq\Delta<1. \quad (26b)$$

Within this range a is analytic in p and Δ.

4. PROOF OF EXISTENCE OF SOLUTION FOR (18)

Consider the function [$\Delta<1$, p_j satisfying (8) and (9)]:

$$Z(p_1\cdots p_m,\Delta)=\sum_j r(a_j)-2\pi(\mathfrak{N}^{-1})\sum_j I_ja_j+\tfrac{1}{2}\sum_{i,j}\mathfrak{N}^{-1}\Omega(a_i-a_j), \quad (27)$$

where C was defined in (21p), a in (25), and

$$r(x)=\int_0^x p\,da,\quad \Omega(x)=\int_0^x\theta\left(\frac{2\pi a}{C},0\right)da. \quad (28)$$

Clearly

$$\Omega(a_i-a_j)=\int_{\alpha_j}^{\alpha_i}\theta(\alpha,\alpha_j)d\alpha=\int_{p_i}^{p_j}\Theta(p,p_j)\frac{da}{dp}dp, \quad (29)$$

and

$$\Omega(a_i-a_j)=\Omega(a_j-a_i).$$

Thus, *Z is analytic in all p_j and Δ* for $\Delta<1$ and p_j in (8) and (9).

One has also by straightforward differentiation

$$\frac{\partial Z}{\partial p_j}=\frac{da_j}{dp_j}\left[p_j-2\pi(\mathfrak{N}^{-1})I_j+\sum_l\mathfrak{N}^{-1}\Theta(p_j,p_l)\right]. \quad (30)$$

Thus (18) is the condition for an extremum of Z at fixed Δ.

We are now in a position to prove

Theorem 1: For $m \leqq \mathfrak{N}/2$, $0 \leqq \Delta < 1$, (18) has a unique solution S so that each p_j is in (9). Each p_j is an analytic function of Δ. For any Δ, $p_i \neq p_j$ unless $i=j$.

Proof: (a) At $\Delta=0$, $\Theta=0$. (18) has then a unique solution satisfying this theorem.

(b) For $0 \leqq \Delta < 1$, Z as a function of $a_1, a_2, \cdots a_m$ has a positive-definite second-derivative matrix [*Proof:*

$$r''(x) > 0, \quad \Omega''(x) \geqq 0, \tag{31}$$

as is easily verified from (28), and (21j). Each term $\Omega(a_i - a_j)$ gives therefore a contribution to the second-derivative matrix that is positive (but not definite).] Z can thus have only one stationary point. To prove that it does have a minimum (for each Δ) at finite values of a, consider successively larger *closed* cubes C_i in p_j space approaching the open cube (9). We shall show that the position P_i of the minimum of Z in these closed cubes C_i cannot always lie on the boundary of C_i: If they always do, there would be an accumulation point P [on the boundary of the open cube (9)] of these minima P_i.

(α) Now suppose P is on the "surface" of the closed cube of (9). In other words at P, there is one p, say, p_j which is $=\pi-\mu$, all other $p < \pi-\mu$. We can approach P through a series of minima P_i at each of which $\partial Z/\partial p_j \leqq 0$, or

$$p_j - 2\pi(\mathfrak{N}^{-1})I_j + \textstyle\sum_l \mathfrak{N}^{-1}\Theta(p_j,p_l) \leqq 0. \tag{32}$$

Approaching P we obtain, by (23) and the continuity of Θ,

$$(\pi-\mu) - 2\pi(\mathfrak{N}^{-1})I_j + \mathfrak{N}^{-1}(2\mu-\pi)(m-1) \leqq 0.$$

This is a contradiction since

$$I_j \leqq \tfrac{1}{2}(m-1), \quad \mu < \pi, \quad 2m \leqq \mathfrak{N}.$$

Similarly, P cannot be such that one $p = -(\pi-\mu)$.

(β) Suppose P is on an "edge" of the closed cube of (9). For example, at P, $p_j = p_l = \pi-\mu$, all other $p < \pi-\mu$. In this case we use the fact that at each P_i, since P_i is a minimum in a closed cube,

$$\frac{\partial Z}{\partial a_j} + \frac{\partial Z}{\partial a_l} \leqq 0.$$

That is,

$$p_j + p_l - 2\pi\mathfrak{N}^{-1}(I_j + I_l) + \mathfrak{N}^{-1} \times \textstyle\sum_n [\Theta(p_j,p_n) + \Theta(p_l,p_n)] \leqq 0. \tag{33}$$

Using (24) and approaching P we obtain

$$2(\pi-\mu) - 2\pi\mathfrak{N}^{-1}(I_j+I_l) + \mathfrak{N}^{-1}\, 2(m-2)(2\mu-\pi) \leqq 0. \tag{34}$$

This is again a contradiction since $I_j + I_l \leqq \frac{1}{2}(m-1) + [\frac{1}{2}(m-1) - 1]$.

(γ) Similarly we can prove that P is not on a "super-edge" of the closed cube of (9), etc.

(c) Thus some of the P_i are not on the boundary of the closed cube C_i. Such a P_i must give an absolute minimum of Z. Hence, Z has a unique minimum in the open cube (9), for each value of Δ. That minimum gives the unique solution of (18).

(d) Since the second-derivative matrix of Z has an inverse, one can evaluate $dp_j/d\Delta$ for every Δ in the interval $0 \leqq \Delta < 1$. This evaluation is also possible for complex values of Δ in the neighborhood of the interval. Thus p_j is an analytic function of Δ.

(e) (18) shows directly that if $p_i = p_j$, $I_i = I_j$, hence $i=j$.

Theorem 2: The solution discussed in Theorem 1 satisfies

$$p_j = -p_{m-j+1}, \quad j = 1 \to m. \tag{35}$$

Proof: For $m=$ even, consider p_j $(j=1 \to m/2)$ as dependent on p_j $[j=(m/2)+1 \to m]$ through (35). For $m=$ odd, put $p_{(m+1)/2}=0$, and use (35) to eliminate half of the p's. Z as a function of the independent p's clearly has a positive-definite second-derivative matrix. We can prove that the minimum of Z does not lie at infinite values of a, just as in Theorem 1. Thus, Z has a minimum with respect to the independent p's satisfying (35). Using (24) one sees that (18) is satisfied at this minimum. But by Theorem 1 (18) has only one solution. Hence, Theorem 2.

Theorem 3: For $m \leqq \mathfrak{N}/2$, $\Delta \leqq 0$, (18) has solutions S forming a continuous curve in the real k_j $(j=1 \to m) \times \Delta$ space with k_j satisfying (8) and (9). The curve extends from $\Delta=0$ down to all $\Delta<0$. At each point S on the curve

$$p_i \neq p_j \quad \text{unless} \quad i=j.$$

Furthermore,

$$p_j = -p_{m-j+1}, \quad j=1 \to m. \tag{36}$$

Proof: (a) Consider the case $m=$ even. The case $m=$ odd can be treated similarly. Consider the cube $\mathfrak{C}$: $j=(m/2)+1 \to m$.

$$0 \leqq p_j \leqq (\pi-\mu)(1-\mathfrak{N}^{-1}) \quad -1 \leqq \Delta \leqq 0, \tag{37a}$$

$$0 \leqq p_j \leqq \pi(1-\mathfrak{N}^{-1}) \quad \Delta < -1. \tag{37b}$$

For every point in the cube $\mathfrak{C}$, we can construct a full set of p's satisfying (36). Clearly this full set lies in (8) and (9). Thus, Z is an analytic function of p and Δ in (cube $\mathfrak{C}$)$\times\Delta$. For

$$j = (m/2)+1 \to m,$$

$$\frac{\partial Z}{\partial p_j} = 2\frac{da_j}{dp_j}[p_j - 2\pi(\mathfrak{N}^{-1})I_j + \sum_l \mathfrak{N}^{-1}\Theta(p_j,p_l)]. \tag{38}$$

(b) At every point P in the cube $\mathfrak{C}$ there is a vector $v_j = -\mathfrak{N}[\quad]$ of (38). A stationary point of Z is a point where $v=0$. v is continuous in both P and Δ. Now on the boundary of $\mathfrak{C}$, the vector v is $\neq 0$ and always points *inward*. To prove this we discuss three points:

(α) For $p_j=0$, $v_j = [2\pi I_j] > 0$.

(β) If $-1\leqq\Delta\leqq 0$ and $p_j=(\pi-\mu)(1-\mathfrak{N}^{-1})$, then by (21.10), (21.14), and (23),

$$\Theta(p_j,p_l)\geqq\Theta(\pi-\mu,\, p_l)=2\mu-\pi.$$

Thus,

$$\begin{aligned} v_j&\leqq-(\pi-\mu)(\mathfrak{N}-1)+2\pi I_j-(m-1)(2\mu-\pi)\\ &\leqq-(\pi-\mu)(\mathfrak{N}-1)+\pi(m-1)-(m-1)(2\mu-\pi)\\ &<0.\end{aligned}$$

(γ) If $\Delta<-1$, $0\leqq p<\pi$,

$$\begin{aligned}\Theta(p,p_l)+\Theta(p,\,-p_l)&\geqq 2\Theta(p,0)\\ &=-4\tan^{-1}[(\coth\lambda)\tan(\alpha/2)]>-2\pi,\end{aligned}$$

where the $\geqq$ sign can be verified by using (21.5) to calculate the derivative of its left-hand side with respect to p_l. Thus, if $p_j=\pi(1-\mathfrak{N}^{-1})$,

$$v_j<-\pi(\mathfrak{N}-1)+2\pi I_j+\pi m\leqq 0.$$

(c) Thus with respect to the vector $\mathbf{v}(P)$, the boundary of the cube $\mathfrak{C}$ has an *index* of 1. It follows from a theorem in topology[17] that there are solutions of $v=0$ which form a continuous curve in the product space of $\mathfrak{C}$ with Δ. We can then use (36) to construct a continuous curve in the product space of p_j with Δ. By (24) one easily verifies that (18) is satisfied on the curve.

(d) Obviously $p_i=p_j$ implies $I_i=I_j$, hence $i=j$.

5. PROOF THAT BETHE'S HYPOTHESIS IS VALID FOR THE GROUND STATE

We shall now use continuity arguments with respect to Δ to study the ground state. To do this we need

Theorem 4: The ground state of the Hamiltonian (1) for finite $\mathfrak{N}$ and m(m= no. of spins down) is nondegenerate for any real Δ. The ground-state energy is analytic in Δ for all real Δ.

Proof: The Hamiltonian is a matrix operator between the $\mathfrak{N}![m!(\mathfrak{N}-m)!]^{-1}$ spin arrangements. The off-diagonal elements of this matrix are -1 or 0. The diagonal elements can be all made negative if we subtract a large constant from H, i.e., there is a large number A so that $A-H$ has all elements $\geqq 0$, and all diagonal elements >0. A nonvanishing off-diagonal element connects every two spin arrangements with one pair of neighboring spins ↑↓ switched. Clearly, for large enough powers of $A-H$ all elements will be >0. Consider one such *odd* power: $(A-H)^n$. The largest eigenvalue of $(A-H)^n$ cannot be degenerate, since any corresponding wave function can be normalized so that all its elements are >0.

Now the eigenvalues are solutions of a polynomial equation with coefficients which are polynomials in Δ. Any nondegenerate solution must be analytic. Thus, Theorem 4 is proved.

Theorem 5: At $\Delta=0$, the solution of (18) is unique and gives through (20) the ground state of H.

Proof: At $\Delta=0$, $\Theta=0$. Thus (18) gives $\mathfrak{N}p_j=2\pi I_j$. (16) gives $A_P/A_0=\pm1$, + for even and − for odd permutations P. Thus, (7) becomes a determinantal wave function.

Now at $\Delta=0$, all eigenstates of H are known.[10] It is easily seen that the solution above is the ground state.

If the provision referred to in Ref. 15 is satisfied always along the solution S of Theorems 1 and 3, we have a wave function ψ for every point on S, with an eigenvalue E given by (11). The (E,Δ) plot forms a continuous curve extending over every real $\Delta<1$. Continuity in Δ and Theorems 4 and 5 would then lead to

Theorem 6: For any real $\Delta<1$ and for $2m\leqq\mathfrak{N}$, the ground state is given by Bethe's hypothesis as stated in (20). Furthermore,

$$p_j=-p_{m+1-j}\quad j=1\rightarrow m.$$

Proof: We need only examine the provision of Ref. 15. The main idea is to show that the point where the provision is not valid is discrete and therefore could be rendered harmless. This is done by showing that each element of the wave function (7) is algebraic in Δ:

(a) Put
$$u_j=\exp(ip_j). \tag{39}$$

Then

$$\exp[-i\Theta(p_j,p_l)]=\frac{2\Delta u_j-1-u_ju_l}{2\Delta u_l-1-u_ju_l}. \tag{40}$$

Now (18) implies (17) which becomes

$$u_j^{\mathfrak{N}}=(-1)^{m-1}\prod_l\frac{2\Delta u_j-1-u_ju_l}{2\Delta u_l-1-u_ju_l}. \tag{41}$$

[Notice, however, that solutions u_j of (41) may not satisfy (18).] One can eliminate all u's but one from (41), obtaining an equation

$$\mathcal{P}(u,\Delta)=0, \tag{42}$$

which is satisfied by u_1, by u_2, $\cdots$ by u_m. $\mathcal{P}$ is a polynomial of u and Δ.

Thus, each u is an algebraic function of Δ. u has no more than a finite number of cuts and poles. Furthermore, it has only a finite number of Riemann sheets.

(b) Now (16) and (40) show that A_P/A_0 is a rational function of Δ and the u's. Thus, after (16) is substituted into (7) and A_0 put $=1$, we obtain a ψ every element of which is a rational function of Δ and the u's. Define

$$\psi'=\psi\prod_{j,l}(2\Delta u_l-1-u_ju_l). \tag{43}$$

Every element of ψ' is *a polynomial of* Δ *and the* u's. At $\Delta=0$, $\Theta=0$ and all the u's are, by (18) and (39), on the unit circle and have positive real parts. Thus, at $\Delta=0$ the product in (43) is not zero and ψ' is the genuine (i.e., nonvanishing) ground-state wave function.

We have thus ψ' and E, both as polynomials in Δ and the u's so that

$$H\psi'=E\psi', \tag{44}$$

[17] P. Alexandroff and H. Hopf, *Topologie* (Springer-Verlag, Berlin, 1935).

if the u's satisfy (41). (41) defines the u's as algebraic functions of Δ. Thus, in complex Δ space except at the poles of the u's and at points where $\psi'=0$, ψ' is an eigenstate of H. These exceptional points are finite in number. We can obtain a correct eigenfunction ψ'' at these points too by properly normalizing ψ' and approaching these exceptional points. Hence, Theorem 6. (In fact the above proves a generalization of Theorem 6 to complex Δ.)

We can also prove the following theorem, which clarifies but is not essential for later discussions.

Theorem 7: The p's are analytic in Δ in an open strip containing the semi-infinite real axis $\Delta<1$.

Proof[18]*:* (a) Starting from $\Delta=0$, and moving along the real axis towards $\Delta=-\infty$, let $\Delta=\Delta_1$ be the first singularity of the u's, if any is in the way. We can form a simple closed path that loops around Δ_1 and return to $\Delta=0$, which does not pass through and does not contain, inside of it, any other singularities of any u. Now $E(\Delta)$ is analytic along the real axis, by Theorem 4. Furthermore, it is a polynomial in u. Thus, E has no singularity on or in the path and it returns to the original value when Δ goes around the path back to $\Delta=0$. Thus, ψ' returns also to the ground-state wave function at $\Delta=0$, except for a possible multiplicative factor. This wave function is a determinant. Consider its values when $x_1=1$, $x_2=2$, $\cdots x_{m-1}=m-1$, but successively $x_m=m$, $m+1$, $m+2$, $\cdots$. Its values are in the ratio of 1, $\sum u$, $\sum u^2+\sum_{j>l} u_j u_l$, $\cdots$. Thus, all symmetrical polynomials of the u's return to their original values around the loop. Hence, the u's are merely permuted in going completely around the loop. Call that permutation $P(\Delta_1)$.

(b) For $0\leqq\Delta<1$, u_j is on the unit circle. By analytic continuation, it must remain so for $\Delta_1<\Delta<0$. Thus, $p_j=-i\ln u_j$ is analytic for $\Delta_1<\Delta<1$. For $0\leqq\Delta<1$, Theorem 1 shows that (18) is satisfied. Continuing all p's to values of $\Delta<0$, (18) remains satisfied until either we reach the point Δ_1, or the p's go outside of the limits defined in (8) and (9). The latter alternative, however, does not obtain, since before the p's reach the boundary, the corresponding point must go out of the surface of the cube (37). Part (b) of the proof of Theorem 3 demonstrates that that is not possible. Thus, (18) is satisfied for all $\Delta_1<\Delta<1$.

(c) Δ_1 is not a pole for the u's, since $|u|=1$ for $\Delta=\Delta_1+0$. Since each u_j is algebraic in Δ, it has a definite value at $\Delta=\Delta_1$. (18) shows that at $\Delta=\Delta_1$, all p's are unequal. Hence, all u's are unequal.

(d) Now tighten the loop of (a) around Δ_1. Since all u's are unequal at Δ_1, the permutation $P(\Delta_1)$ must be the identity. Thus, Δ_1 is not a branch point of any u. Contradiction.

[18] One can rearrange the theorems so that the topological theorem is not needed: After Theorems 1 and 2, 4, and 5 the concept of u of (39) is introduced, together with the ψ' of (43), leading to $H\psi'=E\psi'$ for complex Δ. One then proves Theorem 7, using in part (b) of the proof the discussions following Eq. (38). This proof of Theorem 7 then automatically establishes (18) for all $\Delta<1$, with all p's within the bounds (8) and (9).

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One-Dimensional Chain of Anisotropic Spin-Spin Interactions. II. Properties of the Ground-State Energy Per Lattice Site for an Infinite System

C. N. YANG
Institute for Advanced Study, Princeton, New Jersey
and
State University of New York, Stony Brook, New York
AND
C. P. YANG
Ohio State University, Columbus, Ohio

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The ground-state energy $2f$ per lattice site for an infinite system is studied as a function of Δ and of the magnetization y. Analyticity properties of $f(\Delta,y)$ are proved. The behavior of $f(\Delta,y)$ at and near $y=0$ and $y=1$ are investigated.

1. BASIC EQUATIONS

IN Paper I[1] it was shown that if $\Delta<1$, the ground state for a fixed $\mathfrak{N}$ (=No. of sites) and m (=No. of down spins) is of Bethe's form (I7), with p_j satisfying (I18), or

$$p_j=2\pi I_j(\mathfrak{N}^{-1})-\mathfrak{N}^{-1}\sum_{l=1}^{m}\Theta(p_j,p_l). \qquad (1)$$

Since $p_j\neq p_i$ if $j>i$, by continuity argument with respect to Δ, we see that $p_1<p_2<p_3\cdots<p_m$ for all Δ. As $\mathfrak{N}$, $m\to\infty$ at a fixed ratio, the p's increase in

[1] C. N. Yang and C. P. Yang, preceding paper, Phys. Rev. **150**, 321 (1966). Formulas and references there are referred to as (I18), etc. The notations are the same.