Superconducting Transition and Critical Field of Pure Gallium Single Crystals*

WILLIAM D. GREGORY,† THOMAS P. SHEAHEN,‡ AND J. F. COCHRAN§
Department of Physics and Center for Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 1 April 1966; revised manuscript received 31 May 1966)

The superconducting critical temperature T_c and critical-field curve for pure gallium have been measured using a He^3 vapor-pressure thermometer. The critical field $H_c(T)$ has been measured in the temperature range 0.82–1.083°K. Within experimental error the results of the measurements can be expressed as $H_c(T)=56.48[(1-t^2)-0.0314\sin\pi(1-t^2)]$, or $H_c(T)=100.86(1-t)-36.18(1-t)^2$, or $(1-t^2)=(H_c/57.807)+0.0424\sin(\pi H_c/57.807)$, where $t=T/T_c$, and $T_c=1.0833\pm0.0005$°K.

The derivative of the critical field at T_c is $(dH_c/dT)_{T_c}=-93.8\pm0.5$ G/°K, or $(dh/dt)_1=-1.72$ in reduced coordinates. The superconducting enegy gap at 0°K is estimated to be $2\Delta_0/kT_c=3.32$, and the mean square anisotropy of the gap, $\langle a^2\rangle$, is estimated to be 0.04 from the deviation of the critical field curve from the BCS form.

I. INTRODUCTION

INTERCOMPARISON of a number of pure gallium single crystals indicated that they all had the same superconducting transition temperature within 10^{-3} °K, that the critical temperature was the same whether the specimen was warmed or cooled through the transition, and that the phase change was completed in a temperature interval smaller than 10^{-3} °K. These observations suggested that gallium could be used as a thermometer for temperatures below its critical temperature $T_c=1.0833$°K where a measurement of the critical field would suffice to determine the temperature.

In recent years the critical field and critical temperature of gallium have been reported four times, and there is disagreement as to the values of T_c: Calorimetric measurements by Phillips[1] and by Seidel and Keesom,[2] and magnetic measurements by Cochran and Mapother,[3] give 1.078, 1.081, and 1.086°K, respectively, when converted to the 1962 He^3 vapor pressure scale.[4] Zavaritskii[5] measured the critical field but did not determine T_c.

In order to resolve this discrepancy and establish gallium as a thermometric substance useful for determining temperatures to $\pm10^{-3}$ °K in the range 0.8–1.1°K, we have measured the critical field of gallium as a function of the vapor pressure of He^3.

* This work was supported in part by the Advanced Research Projects Agency under contract No. SD-90, and in part by the National Science Foundation.

† Present address: Department of Physics, Georgetown University, Washington, D. C. 20007.

‡ Present address: Bell Telephone Laboratories, Whippany, New Jersey 07981

§ Present address: Simon Fraser University, Burnaby, British Columbia, Canada.

[1] N. E. Phillips, Phys. Rev. **134**, A385 (1964).

[2] G. Seidel and P. H. Keesom, Phys. Rev. **112**, 1083 (1958).

[3] J. F. Cochran and D. E. Mapother, Phys. Rev. **121**, 1688 (1961).

[4] R. H. Sherman, S. G. Sydoriak, and T. R. Roberts, Los Alamos Scientific Report No. LAMS-2701, Los Alamos, New Mexico (unpublished).

[5] N. K. Zavaritskii, Zh. Eksperim. i Teor. Fiz. **37**, 1506 (1959) [English transl.: Soviet Phys.—JETP **10**, 1069 (1960)].

II. APPARATUS

A. Principle of the Measurement

The gallium single crystals were attached by one end to a He^3 vapor bulb and formed the core of a pair of coils. The change ΔM in mutual inductance between the coils at a frequency of 23 Hz was used to detect the superconducting phase change. A carbon resistor was also attached to each specimen and served as a high-resolution thermometer. ΔM was plotted as a function of the resistance of the carbon thermometer as the temperature of a gallium single crystal was changed slowly through the transition corresponding to some fixed magnetic field. The temperature of the specimen was then held constant within 10^{-4} °K at the transition temperature corresponding to this field, and the pressure of He^3 in the vapor bulb was measured.

B. The Cryostat

Figure 1 is a diagram of the main parts of the cryostat. Two specimens plus associated mutual inductance coils were attached to the He^3 vapor bulb. The He^3 vapor bulb formed the bottom of a pot which was filled with liquid He^4 and pumped to attain a limiting temperature of 0.8°K. The temperature of this He^4 bath could be maintained constant to within 10^{-5} °K by combined throttling of the pumps and regulation of the current through a carbon resistor which was attached to the He^3 pot by means of epoxy resin[6] and which served as a heater. This heater is not shown in Fig. 1.

Each specimen was cemented to a $\frac{1}{4}$-in. oxygen-free high-conductivity (OFHC) copper rod using an epoxy resin.[6] This specimen-support rod was firmly clamped in a heavy copper flange which was in turn screwed tightly into the base of the He^3 vapor bulb. The primaries of the mutual inductance coils consisted of 3600 turns wound uniformly over a 5-in. length. Each

[6] Stycast #2850 FT, Manufactured by Emerson and Cummings, Inc., Canton, Massachusetts.

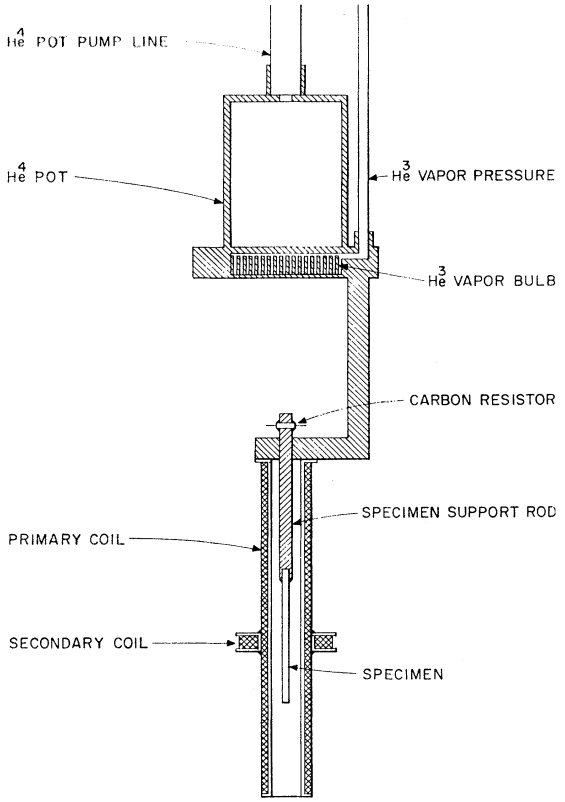

FIG. 1. Inner parts of the cryostat.

secondary coil was wound of 5000 turns distributed over $\frac{1}{2}$ cm, and was attached to its primary by means of a few turns of Teflon tape. An empty pair had a mutual inductance of 20 mH.

The volume of the He^3 vapor bulb was 10 cm^3, and its bottom was slotted as shown in Fig. 1 in order to provide a large surface area for heat exchange between the walls of the bulb and liquid He^3. The total surface area was 250 cm^2. The maximum charge[7] of He^3 was 1.4 cm^3. The vapor bulb was connected to the manometers at room temperature through a $\frac{3}{16}$-in. o.d. stainless steel tube having a wall 0.020 in. thick. This vapor-pressure line contained a radiation shield maintained at 4.2°K (not shown in Fig. 1).

A $\frac{1}{2}$-W, 470-Ω (nominal) Speer carbon composition resistor was fastened into each specimen-support rod with eposy resin. These resistance thermometers were used to trace out the variation of mutual inductance with temperature as the specimens warmed or cooled through the superconducting phase transition.

The parts of the apparatus shown in Fig. 1 were enclosed in a vacuum jacket surrounded by liquid helium at atmospheric pressure. Leads to the mutual inductance coils and the carbon resistor were thermally anchored both to the bath at 4.2°K and to the He^4 pot.

[7] Vapor-Pressure Standard He^3 from Monsanto Research Corporation Mound Laboratory, Miamisburg, Ohio.

C. Mutual Inductance and Carbon Resistance Measurements

The resistance of the carbon thermometer was measured using a 33-Hz Wheatstone bridge. The off-balance bridge signal was amplified, rectified in a phase-sensitive detector, and the resulting dc signal applied to the X-axis of an X-Y recorder. This X-axis signal was proportional to the difference between the resistance of the thermometer and a fiducial resistance set on a decade box which formed the variable arm of the Wheatstone bridge. The value set on the decade box was chosen to be the resistance of the carbon thermometer at the transition temperature.

Mutual inductances were measured using a bridge of the type described by Pillinger *et al.*[8] The bridge was balanced at a temperature near the appropriate transition temperature but with the specimen in the superconducting state. The small off-balance signal which resulted as the specimen became normal was amplified and fed into a phase-sensitive detector. The resulting dc signal, proportional to the mutual inductance change which occured during the phase transition, was applied to the Y terminals of the X-Y recorder. Thus the variation of mutual inductance with temperature could be automatically traced out as the specimen was warmed or cooled through the superconducting phase transition in a constant magnetic field. The 23-Hz magnetic field produced by the mutual inductance primary was made so small that no effect on the superconducting transition was observable upon doubling the primary current. Typically, the primary current used was 55 μA, corresponding to 0.02 G rms. As expected, the transition curves were eventually broadened and shifted as the primary current was made larger and larger.

The carbon resistance thermometers were calibrated by holding the temperature of the specimens constant to $\pm 10^{-4}$ °K, usually at the temperature corresponding to the midpoint of the superconducting transition, and measuring the vapor pressure of He^3 in the vapor bulb.

D. Vapor Pressure Measurements

Vapor pressures of the He^3 in the vapor bulb were measured using a butyl pthalate oil manometer calibrated against a mercury manometer. All pressures were corrected to 0°C and standard gravity. The data were not corrected for thermo-molecular pressure gradients in the vapor-pressure line because such corrections[9] were ten times smaller than the uncertainty in the vapor pressure measurements over the range 0.8 to 1.1°K.

[8] W. L. Pillinger, P. S. Jastram, and J. G. Daunt, Rev. Sci. Instr. **29**, 159 (1958).

[9] T. R. Roberts and S. G. Sydoriak, Phys. Rev. **102**, 304 (1956).

E. The Specimens

Gallium single crystals were grown in Lucite molds in the fashion described by Yaqub and Cochran,[10] and from gallium of comparable purity.[11] Size-effect measurements[10] indicate that in such gallium the electronic mean free path due to impurity scattering is in excess of 1 cm.

Two crystals were used for the work reported in this paper. One crystal was a cylinder, $\frac{1}{4}$-in. in diameter, 4.7 in. long, and the gallium A axis was parallel to the cylinder axis within 1°. The second crystal was of square cross section, $\frac{1}{8}$ in. on a side, 2 in. long, and the gallium A axis was parallel to the specimen axis within $\frac{1}{2}$°. Crystalline conditions and orientation were determined using Laue back-reflection x-ray photographs.

F. Magnetic Fields

The superconducting transition temperature in zero magnetic field was measured with the cryostat surrounded by a Mu-metal shield.[12] This shield reduced the magnetic field of the earth to a residuum of $\pm 10^{-3}$ G within a sphere 3 in. in diameter centered at the specimens. The Mu-metal shield was not used when the transition temperature was measured in a finite magnetic field. Instead, the earth's magnetic field was canceled to 0.01 G, within a sphere 6 in. in diameter centered at the specimens, by means of two pairs of Helmholtz coils.

Magnetic fields for the critical field measurements were generated by a solenoid 9 cm in diameter and 32 cm long. These dimensions and the solenoid current were used to calculate the magnetic field at the midpoint of each specimen, using the coordinates of the midpoint relative to the solenoid axis and center, but assuming axial symmetry of the field. The field was calculated to be uniform to 5% over the length of the $\frac{1}{4}$-in. crystal and uniform to 0.5% over the length of the $\frac{1}{8}$-in. crystal.

III. RESULTS

A. Zero-Field Transition

The change in mutual inductance caused by the superconducting transition is shown in Fig. 2(a) for the coils containing the $\frac{1}{4}$-in.-diam crystal, and in Fig. 2(b) for those containing the $\frac{1}{8}$-in. square crystal. In each case the transition is reversible and occurs over a finite temperature interval of approximately 0.5×10^{-3} °K. The length-to-diameter ratio for the $\frac{1}{4}$-in. round crystal is 19, and the length-to-side ratio for the $\frac{1}{8}$-in. square crystal is 16, so these two specimens have nearly the same demagnetizing coefficients. Subsequent experiments on a 6-in.-long rod having a square cross section $\frac{1}{16}$ in. on a side revealed a transition similar in shape to those shown in Fig. 2, but narrower in temperature by more than a factor of 10. We conclude that the smearing out of the transition over a finite temperature interval is largely a geometrical effect, possibly due to our having cemented one end of the specimen to the apparatus, and we assume that the critical temperature in zero magnetic field is that temperature which corresponds to a 50% change in mutual inductance. The midpoint of the transition for each specimen corresponded, within experimental error, to the same vapor pressure of He^3.

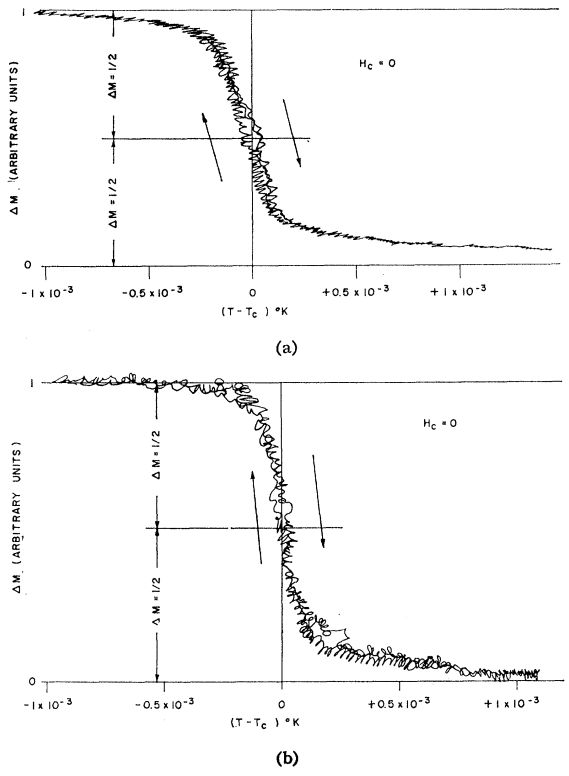

FIG. 2. The zero-field superconducting transitions in (a) the $\frac{1}{4}$-in.-diam, 4.7-in.-long cylinder, and (b) in the $\frac{1}{8}$-in.-side, 2-in.-long specimen.

The above definition of the critical temperature gives the same value, within 0.5×10^{-3} °K, for both specimens. Unfortunately, we were not able to make a direct comparison of the transitions by interchanging the carbon resistors used on the two specimens because the calibration of a carbon resistor changes when it is cycled to room temperature and back to liquid helium temperature. However, we did measure the transition for the $\frac{1}{4}$-in. specimen using the resistor attached to the $\frac{1}{8}$-in. specimen, and vice versa. The midpoints of the transition curves for the two specimens, when measured using a single carbon thermometer, lay within an increment corresponding to 0.3×10^{-3} °K. This result indicates that the midpoints of the transitio ns were at the same temperature within 0.3×10^{-3} °K, and also

[10] M. Yaqub and J. F. Cochran, Phys. Rev. **137**, A1182 (1965).
[11] 99.9999% pure gallium from the Aluminum Co. of America, Pittsburgh, Pennsylvania.
[12] Fabricated and annealed by the Allegheny Ludlum Steel Company, Pittsburgh, Pennsylvania.

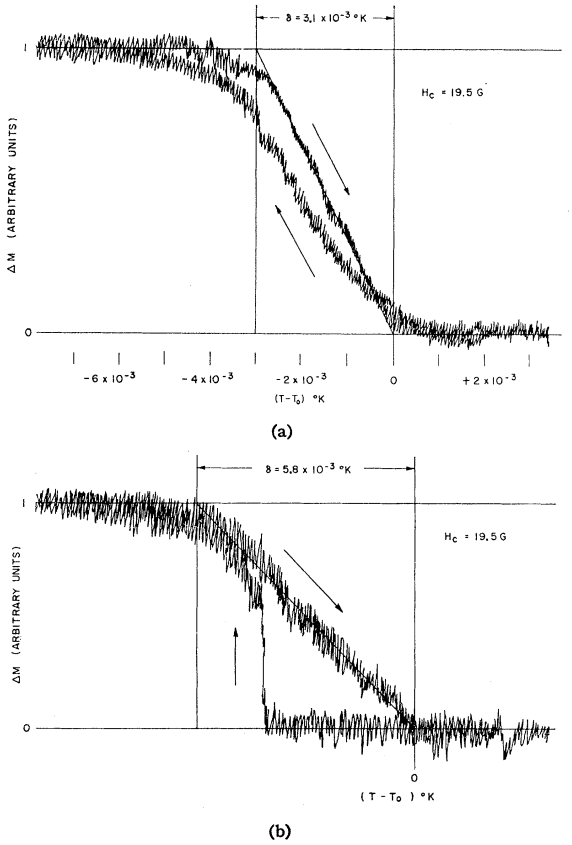

FIG. 3. The superconducting transition in a field of 19.5 G for (a) the $\frac{1}{4}$-in.-diam, 4.7-in.-long cylinder, and (b) the $\frac{1}{8}$-in.-side, 2-in.-long specimen.

shows that there were no large temperature gradients in the copper assembly from which they were suspended.

The results of the He^3 vapor-pressure measurements corresponding to the critical temperature of gallium are given in Table I.

B. Transition in a Magnetic Field

Figures 3(a) and 3(b) are typical of the temperature variation of mutual inductance observed at the superconducting transition in the presence of an applied magnetic field. The transition occurs over a temperature interval which, at 19.5 G, is 3×10^{-3} °K compared to the zero-field transition width of 0.5×10^{-3} °K. The transitions for both specimens exhibited some time-independent hysteresis. The hysteresis observed for the $\frac{1}{8}$-in. specimen, Fig. 3(b), is clearly associated with supercooling. For the $\frac{1}{4}$-in. specimen [Fig. 3(a)], the hysteresis seems to be due to a genuinely different, but reproducible, arrangement of normal and superconducting regions in the intermediate state depending upon whether the specimen is warmed or cooled through the transition. The data obtained as a transition proceeded from the superconducting to the normal state

TABLE I. The vapor pressure of He^3 at the critical temperature of gallium. The specimens were warmed to 300°K between each of the three runs. Mean pressure $\bar{P}_c=12.806$ Torr. Mean temperature $\bar{T}_c=1.0833$ °K.

Run No.	Vapor pressure of He^3 (Torr)	Temperature on $T_{62}He^3$ scale (T_c in °K)	$(T_c-\bar{T}_c)$ (°K)
1	12.811	1.0834	+0.0001
	12.780	1.0828	−0.0005
	12.803	1.0833	0.0000
2	12.806	1.0833	0.0000
	12.832	1.0838	+0.0005
3	12.812	1.0834	+0.0001
	12.801	1.0832	−0.0001

have been assumed to be characteristic of the equilibrium intermediate state structure of the specimen. The width δ of this intermediate state has been defined by the straight-line extrapolation of the variation of mutual inductance with temperature as shown in Fig. 3. The transition temperature corresponding to the applied field H_c is assumed to be the intercept of this straight line with the normal-state mutual inductance.

The intermediate-state width δ has been plotted as a function of field in Fig. 4. The range of fields shown in this diagram is sufficiently small that to a good approximation dH_c/dT may be taken to be constant. It then follows that the temperature width of the intermediate state should be a linear function of the field if the specimen can be characterized by a demagnetizing coefficient.[13] The transition width is indeed observed to be a linear function of the field, and the demagnetizing coefficients obtained from the data of Fig. 4 are 0.015 and 0.022, respectively, for the $\frac{1}{4}$- and $\frac{1}{8}$-in. specimens. The values calculated[14] for ellipsoids of revolution

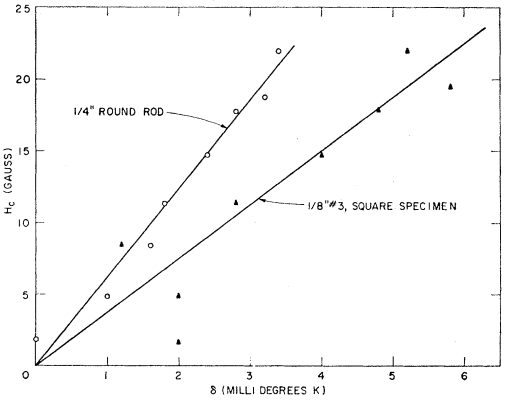

FIG. 4. The relation between the width of the superconducting transition and the critical field.

[13] D. Shoenberg, *Superconductivity* (Cambridge University Press, Cambridge, England, 1952), 2nd ed.
[14] E. C. Stoner, Phil. Mag. **36**, 803 (1945).

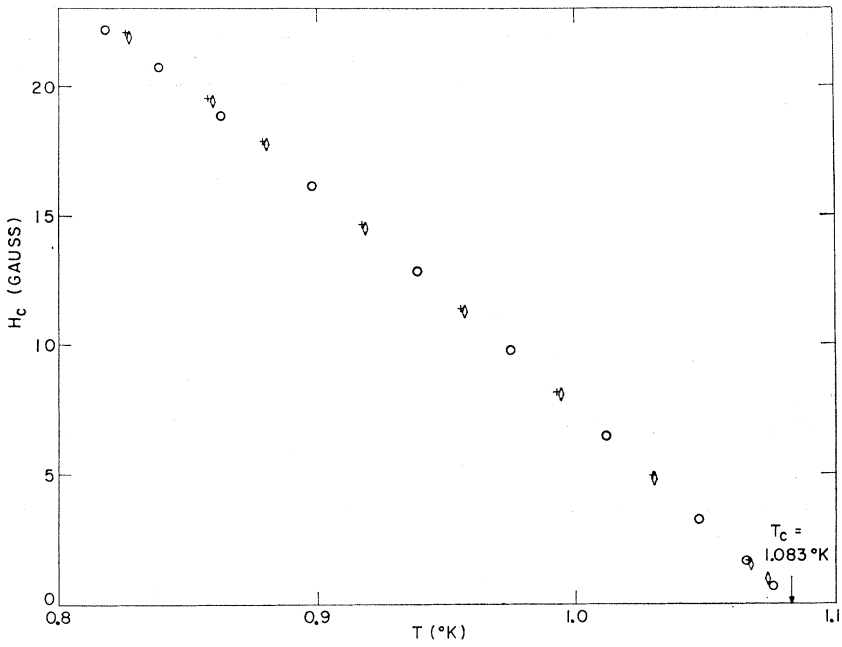

FIG. 5. The critical-field curve for gallium: ○=Run No. 1, ⅛-in. specimen; +=Run No. 2, ⅛-in. specimen; ◇=Run No. 2, ¼-in. specimen.

having the same length-to-diameter ratio are 0.007 and 0.01. The difference between calculated and observed demagnetizing coefficients demonstrates the inadequacy of approximating a specimen having the shape of a truncated cylinder by an ellipsoid of the same length-to-diameter ratio.

C. The Critical-Field Curve

The temperature dependence of the critical-field curve above 0.8°K is shown in Fig. 5. Least-squares fitting of 33 data points leads to the following representations of the critical field over the range $0.816 \leq T \leq 1.0833$°K:

$$H_c(T)=56.48[(1-t^2)-0.0314\sin\{\pi(1-t^2)\}], \quad (1)$$

$$H_c(T)=100.86(1-t)-36.18(1-t)^2, \quad (2)$$

where $t=T/1.0833$°K. The standard deviations were computed to be 0.074 G for both (1) and (2), i.e., the data are equal well represented by either equation.

In order to present temperature as a function of field, we have fitted the data to the form

$$(1-t^2)=H_c/57.807+0.0424\sin(\pi H_c/57.807), \quad (3)$$

with a standard deviation of 0.080 G. Higher-order polynomials and other functional forms improve the fit only slightly. The *maximum* deviation from (1) over the entire range was 0.22 G, equivalent to 0.0027°K.

The derivative of the critical-field curve at $T=T_c$ is −94.0 G/°K from Eq. (1), −93.3 G/°K from Eq. (2), and −94.2 G/°K from Eq. (3). The specific-heat discontinuity is related to the slope of the critical-field curve through

$$(\Delta C)_{T_c}=\frac{V_m T_c}{4\pi}\left[\frac{dH_c}{dT}\right]^2_{T_c}. \quad (4)$$

Using Eq. (4) and the average of (1), (2), and (3) above for $(dH_c/dT)_{T_c}$, one obtains $\Delta C=0.884\pm0.010$ mJ/mole °K. This value is somewhat lower than previously reported measurements of ΔC, but is in excellent agreement with the value obtained calorimetrically for a single crystal of gallium (see Table II).

In case the T-62 He^3 temperature scale should be superseded, we give the results of fitting the critical field by least squares to a polynomial in the vapor pressure of He^3:

$$H_c=1.815078(P_c-P)-0.004084(P_c-P)^2+0.006320(P_c-P)^3, \quad (5a)$$

where $P_c=12.806$ mm Hg; or

$$P=12.830-0.58215H_c+0.007043H_c^2. \quad (5b)$$

IV. DISCUSSION

Of all the measurements of gallium, Phillips[1] and Zavaritskii[5] reached low enough temperatures so that H_c was within 1% of H_0 ($T_{\min}=0.1$°K, $t^2=0.01$). Phillips's value of H_0, corrected for a reduced molar volume,[15] is used in determining deviations from a parabolic dependence of critical field on temperature displayed by our data.

Figure 6 shows the deviations from a parabolic law

$$h\equiv H_c(T)/H_0=1-t^2\equiv1-(T/T_c)^2,$$

[15] The molar volume of gallium is 11.65 cm³ at 2.35°K; see Ref. 10.

TABLE II. Thermodynamic parameters of superconducting gallium.

Source	T_c (T_{62})[a]	H_0 (G)	$(dH_c/dT)_{T_c}$(G/°K)	$(dh/dt)_1$	$-D_0$[b]	$(\Delta C)_{T_c}$(mJ/m°K)
BCS theory				−1.74	0.037	0.93 (for Ga)
this work	1.0833		−93.8	−1.72		0.884 (mag)
Sheahen[c]						0.892 (cal)
Cochran and Mapother[d]	1.086		−99.5			0.99 (mag)
Seidel and Keesom[e]	1.081	59.8[f]	(−96.5)[g]	(−1.77)[g]	0.036	0.93 (cal)
Phillips[h]	1.078[i]	59.2[f]	(−97.1)[g]	(−1.77)[g]	0.035	0.96 (cal)
Zavaritskii[j]	1.08	59.5	−92			
Goodman and Mendoza[k]	1.090	50.3				

[a] Author's quoted values converted to 1962 He^3 scale.
[b] Maximum deviation from a parabola (reduced coordinates).
[c] T. P. Sheahen, Ph.D. thesis, MIT, 1966 (unpublished).
[d] Reference 3.
[e] Reference 2.
[f] Author's quoted values corrected for molar volume of 11.65 cc at low temperatures (see Ref. 10).
[g] Not stated by these authors but calculated from their data.
[h] Reference 1.
[i] Phillips changed his salt-pill parameter from $\Delta=0.033$°K (theoretical) to $\Delta=0.042$°K, to fit his low-temperature data. $T_c=1.087$°K if $\Delta=0.033$°K is used.
[j] Reference 5.
[k] B. B. Goodman and E. Mendoza, Phil. Mag. **42**, 594 (1951).

where we have used $T_c=1.0833$°K and $H_0=59.2$ G. Also shown are the calorimetric deviation plot of Phillips, the weak-coupling BCS curve with no anisotropy,[16] and the deviation curve computed by Clem[17] for a weak-coupling superconductor with a mean-square anisotropy of $\langle a^2\rangle=0.04$.

The discrepancy between our magnetic data and the calorimetric data[1] may be due to the use of an impure polycrystalline specimen for the calorimetric work. Gallium has an extremely anisotropic coefficient of thermal expansion,[18] so that the grains of a polycrystalline specimen would be under stress at helium temperatures. Assuming that the magnetic data do represent the thermodynamic critical field of pure unstrained gallium, the analysis of Clem[17] may be used to estimate the anisotropy of the superconducting energy gap from the deviation of the critical field curve from the form predicted by the BCS theory. One obtains an anisotropy

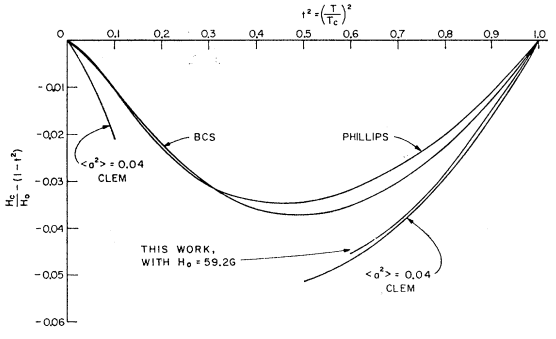

FIG. 6. The deviation of the critical field from a parabolic dependence on temperature, $h\equiv H_c(T)/H_0$, $t\equiv T/T_c$. Our magnetic data lie close to the calculated curve of Clem (Ref. 17) for $\langle a^2\rangle=0.04$, while Phillips (Ref. 1) calorimetric data lie closer to the BCS curve.

of approximately 20%, i.e., the mean-squared anisotropy $\langle a^2\rangle$ of Clem's theory is 0.04.

Although the most reliable values of H_0 reported[1,2,5] (Table II) agree within 1%, it must be emphasized that a small change in the choice of H_0 radically affects this estimate of the anisotropy. For example, our deviation plot can be contrived to fall along the isotropic BCS curve by the choice $H_0=57.4$ G, which is 3% lower than the best experimental values. The value of 57.8 G of (3) should be understood strictly as a fitting parameter which best represents the data, but which is only a crude estimate of H_0. Nevertheless, it is a much better estimate than would arise from any polynomial representation such as (2).

The energy gap at 0°K ($2\Delta_0$) is given by another relation of Clem:

$$\left[\frac{\Delta_0}{kT_c}\right]_{\langle a^2\rangle\neq 0}=\left[\frac{\Delta_0}{kT_c}\right]_{\langle a^2\rangle=0}(1-[3/2]\langle a^2\rangle). \quad (6)$$

Assuming the BCS value of $[\Delta_0/kT_c]_{\langle a^2=0\rangle}0=1.764$, we find, using (6) with $\langle a^2\rangle=0.04$, that $\Delta_0/kT_c=1.66$ or $2\Delta_0=3.60$°K.

V. CONCLUSIONS

The superconducting transition temperature of pure gallium single crystals is 1.0833 ± 0.0005°K on the T-62 He^3 vapor-pressure scale. This result is substantially in agreement with that of Seidel and Keesom[2] (see also Table II), since their data were not corrected for the earth's magnetic field.

The critical-field-deviation curve indicates that the gallium energy gap may have anisotropy up to 20%, subject to considerable uncertainty due to uncertainties in H_0.

Under the proper experimental conditions the transition from the superconducting to the normal state can be used in conjunction with Eq. (3) to obtain

[16] J. C. Swihart, IBM J. Res. Develop. **6**, 14 (1962).
[17] J. R. Clem, Ph.D. thesis, University of Illinois, 1965 (unpublished).
[18] R. W. Powell, Proc. Roy. Soc. (London) **A209**, 525 (1951).

temperatures on the T-62 He^3 vapor-pressure scale with an error less than 0.0027°K, over the temperature range 0.82 to 1.0833°K. Since this range of temperatures is a very awkward region in which to calibrate an apparatus lacking a He^3 vapor bulb, the critical field of gallium provides an excellent secondary standard.

ACKNOWLEDGMENTS

One of us (T.P.S.) wishes to thank the National Science Foundation for a Graduate Fellowship held during part of this work. The computations were performed on an IBM 7094 at the MIT computation center.

PHYSICAL REVIEW VOLUME 150, NUMBER 1 7 OCTOBER 1966

One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System

C. N. YANG
Institute for Advanced Study, Princeton, New Jersey
and
State University of New York, Stony Brook, New York

AND

C. P. YANG
Physics Department, Ohio State University, Columbus, Ohio
(Received 8 April 1966)

Bethe's hypothesis is proved for the ground state of a one-dimensional cyclic chain of anisotropic nearest-neighbor spin-spin interactions. The proof holds for any fixed number of down spins.

I. INTRODUCTION

THE eigenvalue spectrum of the Hamiltonian

$$H=-\tfrac{1}{2}\sum\{\sigma_x\sigma_x'+\sigma_y\sigma_y'+\Delta\sigma_z\sigma_z'\} \quad (1)$$

is of current interest. In (1) $\boldsymbol{\sigma}$ are the Pauli spin matrices at a particular site ($\sigma_x^2=\sigma_y^2=\sigma_z^2=1$), $\boldsymbol{\sigma}'$ are the Pauli spin matrices at a neighboring site. Δ is a real numerical constant ($\Delta=1$ corresponds to the isotropic ferromagnetic problem, $\Delta=-1$ the isotropic antiferromagnetic problem[1]). The sum extends over all nearest neighbors in a 1-dimensional linear, 2-dimensional square, or 3-dimensional simple cubic lattices with *cyclic* boundaries.

The significance of (1) in the theory of ferromagnetism and the theory of antiferromagnetism is well known. (1) is also the problem to consider for the quantum lattice gas.[2] [In particular, the ground-state energy and the thermodynamical properties of a system with the Hamiltonian (1) can be transformed to give the ground-state energy and the thermodynamical properties of a quantum lattice gas. This quantum lattice gas is a Bose gas moving on a lattice with (a) a quantum kinetic energy, not in the form of an operator $(-\hbar^2/2\mathfrak{M})\nabla^2$, but in the form of a double difference, (b) a hard core preventing two atoms from occupying the same site, and (c) an energy of interaction equal to -2Δ for nearest neighbors. See Table I.]

Let y be the magnetization per site,

$$y=\text{eigenvalue of } (1/\mathfrak{N})\sum\sigma_z, \quad (2)$$

where $\mathfrak{N}=$ total number of sites in the lattice. One is particularly interested in the function

$$f(\Delta,y)=\lim_{\mathfrak{N}\to\infty}\frac{1}{\mathfrak{N}z}\ (\text{lowest eigenvalue of } H \text{ for fixed } y), \quad (3)$$

which is half of the ground-state energy per bond for a fixed y. Here z is the number of nearest neighbors at each site. The existence of the limiting function $f(\Delta,y)$ was proved in Ref. 1. A number of general properties of f was also established there. In particular, inequalities were given between the f for one-, two-, and three-dimensional lattices.

The purpose of this and subsequent papers is to study properties of the Hamiltonian (1) for the one-dimensional linear cyclic chain.

This problem was studied by approximate methods by Bloch.[3] Bethe[4] then proposed that the eigenfunctions are of a certain specific form (to be called Bethe's hypothesis). The particular case $\Delta=-1$ (antiferro-

[1] C. N. Yang and C. P. Yang, Phys. Rev. **147**, 303 (1966).
[2] T. Matsubara and H. Matsuda, Progr. Theoret. Phys. (Kyoto) **16**, 569 (1956); **17**, 19 (1957); R. T. Whitlock and P. R. Zilsel, Phys. Rev. **131**, 2409 (1963); P. R. Zilsel, Phys. Rev. Letters **15**, 476 (1965).
[3] F. Bloch, Z. Physik **61**, 206 (1930); **74**, 295 (1932).
[4] H. A. Bethe, Z. Physik **71**, 205 (1931).