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Differential Paramagnetic Effect and Harmonic Generation in an Enclosed Superconductor*

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The low-frequency behavior of a cylindrical enclosed superconductor (superconducting coaxial cable) is discussed, with particular attention to the case in which the inner conductor is in the intermediate state. Preliminary observations of an effect which is equivalent to the familiar differential paramagnetic effect (DPE) are reported. A qualitative explanation is developed by a detailed analysis of the ideal case, which corresponds to the limit in which the phase boundaries follow the ac field perfectly. For the specimen studied, in which the DPE is reduced from the ideal value, a model is proposed based on the assumption that the intermediate state is composed of only a few normal and superconducting regions. In addition to accounting for the lack of reproducibility of the DPE on different trials, this model makes it possible, in principle, to relate the DPE directly to the velocity of propagation of the phase boundaries, although the data are not yet adequate for this purpose. The DPE was observed by means of a coaxial pair of coils wound directly on the inner superconductor. Under conditions such that the boundary motion led to switching of the inner conductor from the intermediate to the normal state or from the intermediate to the superconducting state, a strong harmonic content was observed. A brief analysis of this harmonic content is presented. Phase-boundary motion was observed directly in an auxiliary two-frequency experiment in which the low-frequency boundary motion was monitored by a high-frequency signal.

I. INTRODUCTION

IN a previous paper,[1] referred to hereinafter as I, we discussed the static magnetic properties of the "enclosed superconductor" which is illustrated in Fig. 1. This system presents us with an interesting special case of a geometry in which the superconducting intermediate state will occur. As was shown in I, the effect of the outer superconductor can be described as giving an effective demagnetizing coefficient to the inner superconductor for axial fields, even when both are infinite in length. The magnetic behavior is analogous to that of an ellipsoid, with one important difference; namely, the simple geometry leads one to expect that the intermediate state of the inner superconductor will consist of a small number of large normal and superconducting regions, instead of a finely divided array of small regions.

We have begun experimental studies of the dc and ac magnetic behavior of such a system. The dc measurements, which will be reported shortly, are in many respects in good agreement with the predictions of I, although we have not established the structure of the intermediate state. Our ac measurements, some of which are reported briefly here and the remainder of which will be reported in detail at a later date, have raised several interesting points which have not been discussed previously. The purpose of this paper is to present an analysis of the ac effects observed, with particular attention to their relation to the propagation of superconductor-to-normal phase boundaries, and we shall show that the enclosed arrangement has an important advantage over other geometries in this respect.

* This work was supported by the National Science Foundation.
[1] R. I. Gayley, Cryogenics 5, 89 (1965).

II. THE IDEAL DIFFERENTIAL PARAMAGNETIC EFFECT

When a superconductor goes from the normal to the superconducting state in the presence of an applied field, the flux threading the specimen is expelled. Imagine a long, thin rod with a pair of closely wound coils and with a dc field applied axially by some external solenoid. If the dc field is just slightly more than the critical field of the rod, then a small field applied from one of the coils can switch the specimen into the superconducting state. The large amount of flux expelled from the rod will induce a large voltage spike in the coils, and the closely wound pair of coils will behave as if there were a strongly paramagnetic substance in their core. For this one small increment of field, a very large emf is generated. This is what Hein and Falge[2] have called the differential paramagnetic effect (DPE).

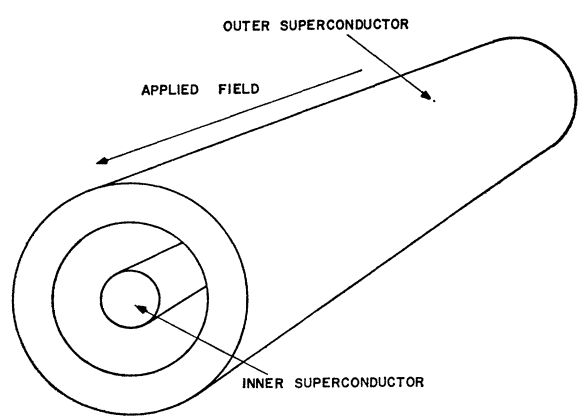

FIG. 1. The cylindrical enclosed superconductor. The outer superconductor, which has the higher transition temperature, gives the inner superconductor an effective demagnetizing coefficient.

[2] Robert A. Hein and Raymond L. Falge, Jr., Phys. Rev. 123, 407 (1961). See also M. C. Steele, *ibid.* 87, 1137 (1952), and Ref. 3.

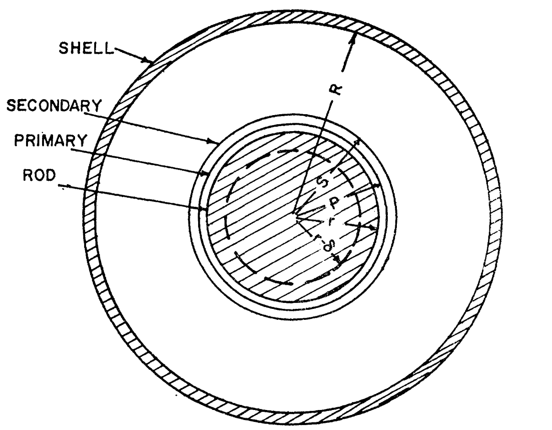

FIG. 2. A cross sectional view of the system under study, showing the locations of the search coils. The normal-state penetration of the ac field into the rod is indicated schematically by the dashed circle.

If, instead of a rod, we have an ellipsoid or other specimen of nonzero demagnetizing coefficient, then there is a range of fields over which the DPE is observed, and the strength of the DPE is simply related to the demagnetizing coefficient. This effect was first observed and explained by Shoenberg,[3] who studied a spherical specimen. We have observed what we consider to be the same effect in an enclosed superconductor, although in this case a discussion in terms of demagnetizing coefficient is not convenient because the coils are inside the shell. Instead, we will proceed in a more straightforward way to analyze the emf in the secondary coil.

By an enclosed superconductor we mean one that is enclosed by another superconductor, in particular, the coaxial arrangement of Fig. 1. The transition temperature of the outer cylindrical shell is supposed to be higher than that of the inner rod, so that at any temperature the critical field of the shell is higher than that of the rod. If an axial magnetic field is applied while the entire system is normal, and the system is then cooled, the shell will trap whatever flux lies within its hole.[4]

On further cooling, as we have shown in I, there will be a broad temperature range in which the rod will be in the intermediate state, even though the rod, by itself, has a zero demagnetizing coefficient. This may be described by saying that the effect of the shell is to give to the rod an effective demagnetizing coefficient

$$D=A/(A+a)\,, \tag{1}$$

where A is the cross-sectional area of the rod and a is the cross-sectional area of the space between the rod and the shell.

The system to be considered is shown in Fig. 2, with the dimensions defined as indicated. We assume that a field H_T has been trapped by the shell and that we have cooled to a temperature such that the rod is in the intermediate state.

In the absence of an ac field, we suppose that a fraction x_0 of the rod is normal. We assume that the shell acts so as to keep the net flux in its interior constant, so that x_0 is given by the conservation of-flux-equation (in Gaussian units, with $H=B$ *in vacuo*)

$$\pi R^2 H_T=\pi H_2(R^2-r^2)+\pi H_2(r^2)x_0\,. \tag{2}$$

The field H_2 produced by the shell when superconducting will be equal to H_C, the critical field of the rod. The first and second terms on the right are the flux contained between the shell and the rod and the flux contained in the normal part of the rod, respectively.

In order to treat the ac behavior of this system, we must consider the normal-state skin effects. It will turn out that for the ideal case, in which the phase boundaries follow the fluctuations of the ac field perfectly, the results for the intermediate state are independent of the treatment given to the normal-state skin effects. However, the skin effect is important when the rod is fully normal and also, since the boundaries do not in fact follow perfectly, in the real intermediate-state case. We will, therefore, write down the general equations including what we regard as a reasonable treatment of skin effects.

When the rod is in the normal state, we will use the solutions for the ac fields given by Landau and Lifshitz[5] for low frequencies. When the rod is in the intermediate state, we assume that the dimensions of the normal regions are large compared to the normal-state skin depth. Then we can treat the specimen as divided into regions in which the field penetration is exactly as in the fully normal case, plus regions in which the field does not penetrate, with moving boundaries between these regions. If this is the case, then the response of the rod can be discussed directly in terms of phase-boundary motion. Such a treatment is physically more interesting than Shoenberg's[3] description of the response of his spherical specimens in terms of an effective conductivity of the specimen as a whole. The effective-conductivity concept should be most appropriate to a case in which the specimen is divided into many very small normal and superconducting regions, as was undoubtedly the case for Shoenberg's spheres. In the enclosed superconductor, we find a system in which we can hope to give a simple interpretation of the DPE in terms of phase-boundary motion.

We need an expression for the ac flux contained in the rod, and this is easily obtained when the magnetic field is known throughout. According to Landau and

[3] D. Shoenberg, Proc. Cambridge Phil. Soc. **33**, 559 (1937).

[4] It has been shown that the strength of the trapped field will actually be somewhat higher than that of the applied field, since some of the flux in the walls of the shell will go into the hole. This effect is not large if the walls of the shell are thin compared to the radius, and in any case this effect is not important for our purpose. We shall speak of the trapped field H_T, recognizing that it may be larger than the applied field. See T. I. Smith and H. F. Rorschach, Jr., Rev. Mod. Phys. **36**, 277 (1964); R. I. Gayley and E. F. Young, Phys. Letters **20**, 104 (1966).

Lifshitz,[5] for a long, isotropic cylinder, subject to a low-frequency, axial, sinusoidal magnetic field, with the condition that the skin depth is much smaller than the radius of the cylinder but much larger than the electron mean free path,

$$H(\rho)=H(r)J_0(k\rho)/J_0(kr). \tag{3}$$

$H(r)$ is the instantaneous field strength at the surface ($\rho=r$), J_0 is the Bessel function of the first kind and order zero, $k=(1+i)/\delta$, and δ is the skin depth. Integrating the flux over the entire cylinder gives

$$\Phi=\frac{2\pi H(r)rJ_1(kr)}{kJ_0(kr)}=2\pi r\delta H(r)\frac{1}{1+i}. \tag{4}$$

The complex factor in this result shows that the flux is shifted in phase by $\pi/4$ relative to the applied field.

If we let the field at the surface consist of a dc component, H_{dc}, plus $H_S e^{i(\omega t+\phi)}$, where H_S and ϕ are real constants, then the flux in the rod is

$$\Phi=\pi r^2H_{\text{dc}}+\sqrt{2}\pi r\delta H_S e^{i(\omega t+\phi-\pi/4)}. \tag{5}$$

When the rod is in the intermediate state, with the fraction normal x, the fraction normal within the skin depth x_{sd}, and the trapped field H_T, the conservation-of-flux equation is

$$\pi R^2H_T=\pi H_2(R^2-p^2)+\pi(H_1+H_2)(p^2-r^2)+\pi r^2H_{\text{dc}}x+\sqrt{2}\pi r\delta x_{\text{sd}}H_S e^{i(\omega t+\phi-\pi/4)}. \tag{6}$$

For H_1, the field produced by the primary, and H_2 we write

$$H_1=H_Me^{i\omega t}, \tag{7}$$
$$H_2=H_{2M}e^{i(\omega t+\theta)}+H_{\text{dc}},$$

and then

$$H_1+H_2=H_{\text{dc}}+H_Se^{i(\omega t+\phi)}. \tag{8}$$

The conservation-of-flux equation is then

$$R^2H_T=H_{\text{dc}}[R^2-r^2(1-x)]+H_{2M}(R^2-p^2)e^{i(\omega t+\theta)}+(H_M+H_{2M}e^{i\theta})e^{i\omega t}[p^2-r^2+\sqrt{2}r\delta x_{\text{sd}}e^{-i\pi/4}]. \tag{9}$$

Let us first examine the case in which the rod is entirely normal, so that $x=x_{\text{sd}}=1$. The conservation of flux is then

$$R^2H_T=H_{\text{dc}}R^2+H_{2M}(R^2-p^2)\times e^{i(\omega t+\theta)}+(H_M+H_{2M}e^{i\theta})e^{i\omega t}\times[p^2-r^2+\sqrt{2}r\delta e^{-i\pi/4}]. \tag{10}$$

We see immediately that

$$H_{\text{dc}}=H_T. \tag{11}$$

Further, if we suppose that we know the primary current, and therefore H_M and ω, as well as δ and all dimensions, we can solve the above equation for H_{2M} and θ. The results are

$$H_{2M}=H_M\frac{\{[(R^2-r^2)(p^2-r^2+r\delta)+r\delta(p^2-r^2+2r\delta)]^2+r^2\delta^2(R^2-p^2)^2\}^{1/2}}{(R^2-r^2)^2+2r^2\delta^2+2r\delta(R^2-r^2)}, \tag{12}$$

and

$$\theta-\pi=\arctan\left[\frac{-r\delta(R^2-p^2)}{(R^2-r^2)(p^2-r^2+r\delta)+r\delta(p^2-r^2+2r\delta)}\right]. \tag{13}$$

The emf in the secondary can be computed most simply by noting that, if the net flux is constant, the rate of change of flux inside the secondary is just the negative of the rate of change of flux outside. Thus, we see that

$$(\text{emf})_{\text{Normal}}=\pi N_S\frac{\partial}{\partial t}[H_2(R^2-s^2)]=i\omega N_S(R^2-s^2)H_{2M}e^{i(\omega t+\theta)}, \tag{14}$$

where the quantity in square brackets is the flux outside of the secondary, N_S is the number of turns in the secondary, and $(\text{emf})_{\text{Normal}}$ is the emf in the secondary when the rod is normal but the shell is superconducting.

To find the emf when the rod is in the intermediate state, we can return to Eq. (6), solve for H_2, and substitute H_2 into Eq. (14). When the differentiation indicated in (14) is performed, it will be necessary to know the time dependence of x_{sd}, which means that it will be necessary to know the equation governing phase-boundary motion.[6] That the boundaries cannot follow the ac field perfectly is clear because we know that normal-state eddy currents will damp the motion of flux lines and thereby impede the motion of the boundaries.[7] Inertial effects due to the currents, restoring forces due to stretching of boundaries, and heat evolution or absorption due to the latent heat of transformation can also be expected to have some influence on the boundary motion. Further, in imperfect specimens, flux-pinning effects will arise.

Our purpose in this section is to develop the formulas for what we will call the ideal DPE, which is the DPE

[5] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960), p. 194.

[6] In addition, note that Eq. (6) contains the product of x_{sd} and $H(r)$, both of which in the general case are time-dependent. This means that H_2 and $H(r)$ will contain harmonics of ω, which in turn means that we cannot speak of a single skin depth δ. In the ideal case treated in this section, $H(r)$ is time-independent, and no harmonics will appear.

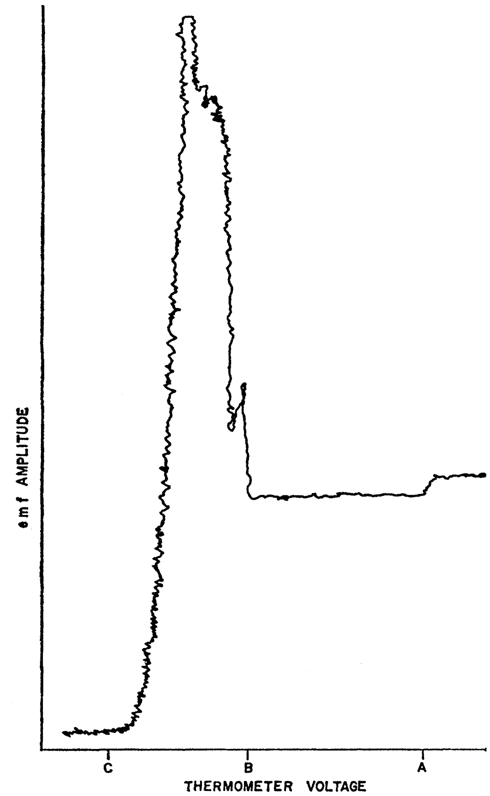

FIG. 3. The variation of the amplitude of the secondary voltage with temperature. The ac frequency was 10 cps, and the specimen was cooled in a field of 25 G. At point A, the shell entered the superconducting state; at point B, the rod entered the intermediate state; and below point C, the rod was fully superconducting.

that would be observed if the boundaries could follow perfectly. In other words, *we assume that the system is in thermodynamic equilibrium at each instant* and, in particular, that the temperature is constant and that the field at the surface of the rod is exactly H_C as long as the rod is in the intermediate state. This is what Shoenberg[3] referred to as "static case" behavior. The ideal DPE can be expected to correspond to reality only for perfect specimens and in the limit of low frequency. We have then, by assumption,

$$H_2 = H_C - H_1, \qquad (15)$$

and Eq. (14) becomes

$$(\text{emf})_{\text{inter}} = -\pi N_s \frac{d}{dt} H_1 (R^2 - s^2), \qquad (16)$$

where the subscript "inter" means that the rod is in the intermediate state.

It is interesting to notice that, since H_1 is the field strength produced by the primary, it appears as if the coil were turned inside out. It is as if the area outside of the coil (and inside the shell), $\pi(R^2-s^2)$, were the area of the coil. Notice that Eq. (14) is independent of choice of model for the details of what is going on inside the rod, requiring only the assumption contained in (15).

Equation (16) is the equation of the ideal DPE, and we see that we can make $(\text{emf})_{\text{inter}}$ as large as we please by making the shell larger and larger. In the limit as R becomes very large, the behavior of the system goes over to that of a bare rod, for which we expect an infinite spike of zero width in the emf of the secondary as the rod passes through the transition.

When the rod is completely superconducting we can set the x's in Eq. (6) equal to zero, and it is easy to show that

$$(\text{emf})_{\text{super}} = \frac{-\pi N_s (p^2 - r^2)(R^2 - s^2)}{(R^2 - r^2)} \frac{dH_1}{dt}, \qquad (17)$$

so that

$$(\text{emf})_{\text{inter}} / (\text{emf})_{\text{super}} = (R^2 - r^2)/(p^2 - r^2). \qquad (18)$$

Equation (16) expresses the DPE in a way that shows the large increase in emf that may occur when the rod goes from the normal to the intermediate state. Equation (18) has the feature that it is independent of the frequency and amplitude of the ac field.

III. THE DPE IN REAL SPECIMENS

We have made some preliminary observations of the DPE in a system like that shown in Fig. 2 with a tin shell and an indium rod. The dimensions were

$$R = 1.130 \text{ cm}, \quad r = 0.798 \text{ cm},$$
$$p \approx r + 0.0038 \text{ cm}, \quad s \approx r + 0.0076 \text{ cm},$$
$$\delta \approx 0.15 \text{ cm at 1 cps, and } 0.05 \text{ cm at 10 cps.}$$

With these parameters, $(\text{emf})_{\text{inter}}/(\text{emf})_{\text{super}}$ should be very nearly 100. The observed values were not very reproducible, but typical values were 20 at 1.5 cps and 10 at 10 cps. The largest value observed was 32, which happened to be obtained at 2 cps and also at 10 cps. At 1000 cps, $(\text{emf})_{\text{inter}}$ was never larger than $(\text{emf})_{\text{normal}}$, and this was sometimes true at 10 cps.

In Fig. 3 we show an example, traced from an X-Y recorder chart, of the amplitude of the emf in the secondary versus the voltage across a resistance thermometer. The enhanced signal found in the intermediate state is shown quite clearly, although, as in all of our observations, it is not as large as would be predicted from Eq. (16).

The failure of a real specimen to achieve the full DPE is perhaps more interesting than the DPE itself. In principle, the deviations of the DPE from ideal behavior give information about phase-boundary motion, although there are certain experimental difficulties that may make it difficult to obtain quantitative results.

We suppose, as Faber has found for a different situation,[7] that a boundary at a given depth from the free surface moves at a certain velocity $v_B(H')$ when the applied field deviates from its equilibrium value H_C by certain amount H'. If the applied field suddenly changes from H_C to H_C+H', the rate at which boundary motion can expel or admit flux depends on v_B and also on the number of boundaries. If there are many boundaries, the system will be able to follow fairly rapid changes in field. If there are few boundaries, there will be appreciable departures from equilibrium, even at relatively low frequencies.

In most familiar intermediate state situations we can expect the number of normal and superconducting regions to be very large, so that a small variation in the number present will not be of great importance. For example, in studying spheres, Shoenberg[3] apparently found the DPE to be fairly reproducible. In the case of the cylindrical enclosed superconductor, on the other hand, we showed in I that there is good reason to expect that the number of regions will be of order unity. If that is so, then a change of one or two in the total number of regions will result in quite a different over-all behavior. The system may behave quite differently on different trials, and extracting information will be correspondingly difficult.

However, it is perhaps not unreasonable to hope that, after the system has stabilized at some temperature, the number and distribution of superconducting regions will remain constant while the DPE is measured as a function of frequency. It may then be possible to determine in some detail the law governing the motion of the phase boundaries. With this in mind, we give an indication of the way that an approximate theory could be developed.

We will assume, as before, that the net flux within the shell is always constant and equal to the initially trapped amount, $\pi H_T R^2$. We will, however, drop the restriction that the field at the surface always be H_C while the rod is in the intermediate state. The reasoning behind this is that the constancy of the net flux is maintained by supercurrents in the outer shell, while the tendency of the system to keep the field at the rod at the value H_C is due to the fact that the phase boundaries will move when this is not the case. The latter process is the one that we suppose requires a significantly long relaxation time.

The field at the surface of the rod is H_1+H_2, and we define the quantity H' by the equation

$$H_1+H_2=H_C+H', \tag{19}$$

so that H' measures the amount by which the field departs from H_C.

[7] T. E. Faber, Proc. Roy. Soc. (London) **A219**, 75 (1953); T. E. Faber, *ibid.* **A223**, 174 (1954). See also T. E. Faber and A. B. Pippard, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (Interscience Publishers, Inc., New York, 1957), Vol. 1, p. 159; H. Cohen and F. Odeh, J. Math. Phys. **6**, 1411 (1965).

We assume that the normal regions are large compared to δ so that Eq. (9) can still be used. $x_{\rm sd}$ is a function of time, and if we assume that v_B is always directed tangentially,

$$\frac{dx_{\rm sd}}{dt}=\frac{N}{\pi r^2}\int_{\rho=0}^{r} v_B d\rho\,, \tag{20}$$

where N is the number of boundaries in the skin-depth region and ρ is the radial cylindrical coordinate. v_B is presumably given by a relation of the form

$$v_B=f(H',\rho)\,, \tag{21}$$

where f is the unknown function that is to be determined.

If Eq. (9) is differentiated once with respect to time, Eq. (20) is used to eliminate $dx_{\rm sd}/dt$, and Eq. (9) is then used to eliminate $x_{\rm sd}$, we can obtain an integro-differential equation in the fields. Ultimately, a given $f(H',\rho)$ will lead to a specific prediction of the frequency dependence of $(\text{emf})_{\rm inter}$ which can be compared with experiment.

Since Faber's work[7] has shown that different functions $f(H',\rho)$ will be appropriate under different conditions, and since it is not yet clear that the value of N can be stabilized well enough to allow meaningful measurements to be made, we will not carry this analysis any further at this time.

We should point out that of the others[3,7] who have made theoretical and experimental studies of boundary motion, only Shoenberg's work applies directly to the problem studied here, and his analysis is not in terms of phase-boundary motion. The other workers have not dealt with a steady-state oscillatory motion, with the system never very far from equilibrium. In addition, the emf measurement considered herein senses the rate at which flux moves in and out of the specimen, rather than boundary motion itself. The rapid propagation of a thin finger which displaces little flux, such as has been observed by Faber,[7] is relatively ineffective in producing an emf or in bringing the system closer to equilibrium.

There is an interesting variation to the approach described up to this point, involving a two frequency system. If a signal with frequency high enough so that the boundaries cannot follow it at all is superimposed on the low-frequency signal, then the high-frequency output will be modulated by the low-frequency boundary motion due to the corresponding change in inductance. We have tried this in a preliminary way, and the low-frequency inductance variation produces an easily observable modulation in the output of a detector that is tuned to the high frequency.

It should be noted that in this two-frequency method one sees the average boundary motion within the high-frequency skin depth. Since, as Faber has shown,[7] boundaries move more freely near a free surface, this may give a different picture than a method which averages over the low-frequency skin depth.

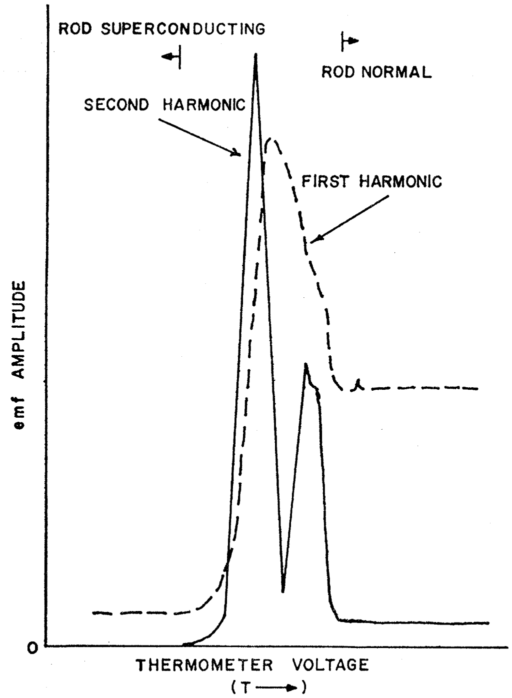

FIG. 4. The variation with temperature of the amplitudes of the first- and second-harmonic components of the secondary voltage. The specimen was cooled in a field of 25 G, with an ac frequency of 20 cps and an ac amplitude of 2 G. The vertical scales for each component are uncalibrated. The first harmonic shows the DPE, and the second harmonic has the structure expected when state-switching occurs.

The above discussion has presumed that the boundary motion is determined by a relation between v_B and H', due mainly to eddy-current damping. However, a most intriguing possibility is that the boundaries can be made to act as a taut membrane. If a specimen can be made which has a low enough conductivity to significantly reduce eddy currents, but in which the boundaries are not pinned by defects, the surface tension of the boundary may become important. If the ac field is applied only to the center of the rod, so that the boundaries stretch in the center rather than move as a unit, the analogy with the taut membrane should be very close. In a pure indium specimen, such as we have examined, the membrane is most likely heavily overdamped. If an underdamped condition can be achieved, it should be possible to determine the surface energy of the boundary from its resonant frequency.

Before concluding this discussion, we should point out an advantage of the ac approach over simply applying a dc current to the primary and watching the emf in the secondary as the system relaxes. If a dc field is applied, the phase boundaries will redistribute themselves throughout the body of the specimen, and the time constant exhibited will be in part that associated with the penetration of the field into the normal interior, in addition to that associated with boundary motion. In the ac case, after a steady-state has been achieved, an analysis such as has been suggested in this section enables one to more readily separate effects due to phase-boundary motion from those due to penetration of the field into the normal regions.

The studies discussed in this paper can be extended in an obvious way to the investigation of flux-tube motion in type-II superconductors or in films, and in fact measurements of this general type have recently been reported by Gittelman and Rosenblum.[8]

IV. HARMONIC GENERATION

There are two causes of harmonic generation in the system we have been studying. One of these is switching of the specimen from one state to another during the ac cycle, and the other is the inability of the boundaries to follow an ac field perfectly.

If the boundaries follow perfectly, the field strength inside the primary does not change. Only the area inside is changing, and this is the only source of change of flux in that region, i.e.,

$$\frac{d\Phi_p}{dt}=\frac{d}{dt}[\pi A_{\text{normal}}(H_1+H_2)]=\pi H_c\frac{d}{dt}(A_{\text{normal}})\,. \quad (22)$$

The normal area inside of the primary, A_{normal}, will be changing at the applied frequency, and no harmonics will appear.

When the boundaries do not move at all, A_{normal} is constant and it is (H_1+H_2) that changes. (H_1+H_2) will vary at the applied frequency, and again no harmonics will appear.

If, on the other hand, the boundaries follow, but only partially, then both A_{normal} and (H_1+H_2) will change with time. Since the product of these quantities appears in Eq. (22), it is clear that harmonics will be generated in the emf of the secondary. The harmonics that we have observed in our measurements, however, appear to be due primarily to state-switching.

Our analysis of harmonic generation due to state-switching assumes that the coils can be characterized by a mutual inductance M, which has the values M_N, M_S, and M_I when the rod is in the normal, superconducting, and intermediate states, respectively. The values of these quantities will be computed from the observed emfs.

We suppose that at some particular primary current, the value of which will depend on the trapped field, the temperature, and (when the boundaries do not follow perfectly) the frequency, the inductance switches abruptly from one value to another. If we examine the second harmonic as a function of temperature, there will be two temperatures at which the second harmonic is a maximum. One of these will be when the rod is normal during one half-cycle and in the intermediate state during the other. The other maximum will occur when

[8] Jonathan I. Gittelman and Bruce Rosenblum, Phys. Rev. Letters **16**, 734 (1966).

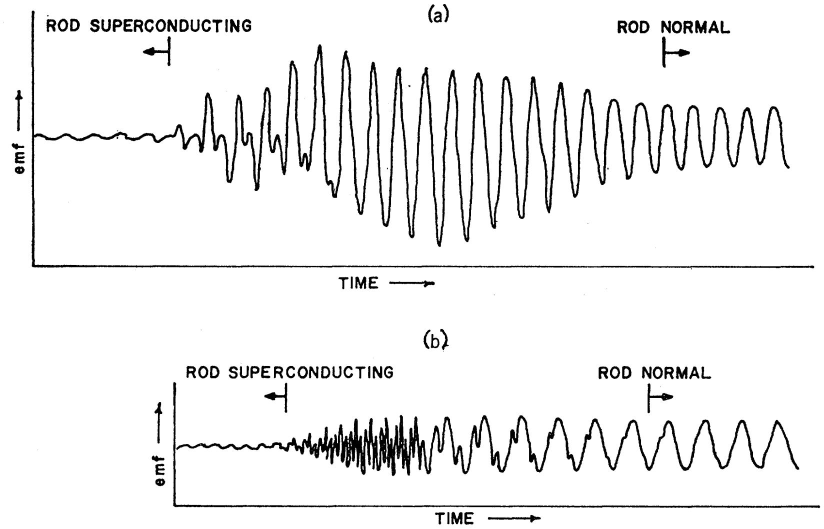

FIG. 5. Secondary emf versus time, traced from a recorder chart. (a) Warming in a 5 G field, with a 3-cps signal. (b) Warming in zero field, with a 2-cps signal. Note the absence of the DPE and of second harmonic.

the rod is in the intermediate state during one half-cycle and superconducting during the other. Elementary Fourier analysis shows that the ratio of the height of the former maximum to that of the latter is $(M_I-M_N)/(M_I-M_S)$, which is equal to

$$[(\text{emf})_{\text{inter}}-(\text{emf})_{\text{normal}}]/[(\text{emf})_{\text{inter}}-(\text{emf})_{\text{super}}].$$

In Fig. 4 we show the results of a measurement of the first and second harmonic as a function of temperature. The trapped field was 25 G, the ac frequency was 20 cycles, and the ac field amplitude was approximately 2 G. We see that

$$[(\text{emf})_{\text{inter}}-(\text{emf})_{\text{normal}}]/[(\text{emf})_{\text{inter}}-(\text{emf})_{\text{super}}]=0.53.$$

This is in close agreement with the ratio of the heights of the two second-harmonic peaks, which is 0.58. Thus the location and relative strengths of the peaks indicate that they are in fact due to state-switching, although the peaks are shifted slightly closer together than our simple analysis predicts.

This shift in the position of the peaks may be due to supercooling and superheating. For example, at the high-temperature end of the transition, supercooling will delay the transition from the normal to the intermediate state until the field is somewhat less than H_C. It is easy to show that this will shift the second-harmonic maximum to a lower temperature. Similarly, superheating at the low-temperature end of the transition will shift that peak to higher temperatures. The ratios of the heights of the peaks will be unaffected, however, in agreement with Fig. 4.

The harmonic content is also displayed in Fig. 5(a), where we have put the secondary voltage directly onto an X-Y recorder, with a time sweep on the horizontal axis. The input signal had a frequency of 3 cps. Such a trace cannot be used for quantitative measurement of the harmonic content unless the frequency response of the recorder is taken into account, but it does show the general behavior quite clearly. We see that at the low-temperature end of the transition the second-harmonic amplitude is comparable to that of the first.

In Fig. 5(b) we show the behavior when there is no dc field trapped in the specimen. In this case the frequency was 2 cps. We see that there is no DPE, and that there is third harmonic but no second harmonic. There can be no second harmonic, since the rod cannot distinguish between the positive and negative halves of the ac cycle in the absence of a dc bias field. In this respect, the behavior of the system is much like that of a flux-gate magnetometer.

V. CONCLUSIONS

The similarity shown in I between the enclosed superconductor and an ellipsoid extends to ac properties, since in both of these types of specimen the differential paramagnetic effect is observed. The qualitative features of the ac behavior of the enclosed superconductor can be understood in simple terms by analyzing the ideal case.

The observed DPE is less than ideal, presumably because phase-boundary motion is retarded by eddy-current damping. A model is proposed, based on the assumption that the intermediate state consists of a few large, normal regions, which, in principle, allows one to relate the DPE directly to the velocity of the boundaries. This simple model is expected to apply only to the enclosed arrangement.

Harmonic generation can be expected due to departures from equilibrium and due to state-switching. The observed harmonics are due primarily to switching and give a behavior much like that of a flux-gate magnetometer.