Ladder Corrections to the Static Random-Phase-Approximation Dielectric Constant and to Positron Annihilation in Metals*

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A calculation of the static dielectric constant of a free-electron gas is made in which account is taken of repeated (ladder) electron-hole interactions. An analytic approximation is given for the results. The effective interaction thus obtained is used in a high-order (ladder) perturbation calculation of the two-photon annihilation rate of positrons in an electron gas. The results agree quite well with experimental data for $r_s \leq 5$. Comparison is made with similar calculations made using the static random-phase-approximation and Hubbard interactions.

I. INTRODUCTION

THE experimental results of Bell and Jørgensen[1] for the two-photon annihilation rate of positrons in the metals aluminum, lithium, sodium, potassium, and cesium are shown in Fig. 1. Calculations of theoretical annihilation rates by determining the electron density at the positron have been made by Ferrell,[2] using (a) a variational calculation and (b) a first-order perturbation calculation; by Butler,[3] using a first-order perturbation calculation; and by Kahana,[4] using a Bethe-Goldstone equation. These results are also shown in Fig. 1. All of these calculations have been more or less successful in predicting the decrease in the annihilation rate with the decrease in electron density, and Kahana's results agree well with experiment for $r_s < 4$. However, none of the calculations has been extended to low electron densities ($r_s > 4$) for one or both of two reasons: (1) at low electron densities, one needs to take account of repeated, i.e., ladder, electron-positron scattering before annihilation, and (2) at low electron densities, one needs to take account of repeated electron-hole scattering in the dielectric response function of the electron gas. Reason (1) is related to the increasing tendency of the electron and the positron to form a bound state as the density of the electron gas is decreased,[5] while reason (2) is related to the tendency of the electron-hole pair to form a bound state at low densities.[6] The Bethe-Goldstone approach of Kahana is equivalent to a ladder-series approach and therefore takes account of reason (1). Reason (2), however, must be taken account of by correcting the random-phase-approximation (RPA) dielectric constant.

Recently, Carbotte and Kahana[7] have carried out a ladder calculation in which they take account of electron and positron self-energies in a linear approximation. They indicate that their results are "of the same order of magnitude" as those quoted by Kahana[4] for two-photon momentum less than the Fermi momentum. (They also give results for the case where the two photons carry off a combined momentum greater than the Fermi momentum.) Their ladder calculations are for $r_s < 4$, and they include the dynamic RPA interaction in a linear approximation.

Our purpose in this paper is first of all to obtain a more realistic static dielectric constant than that described by the RPA. In this endeavor we will be guided

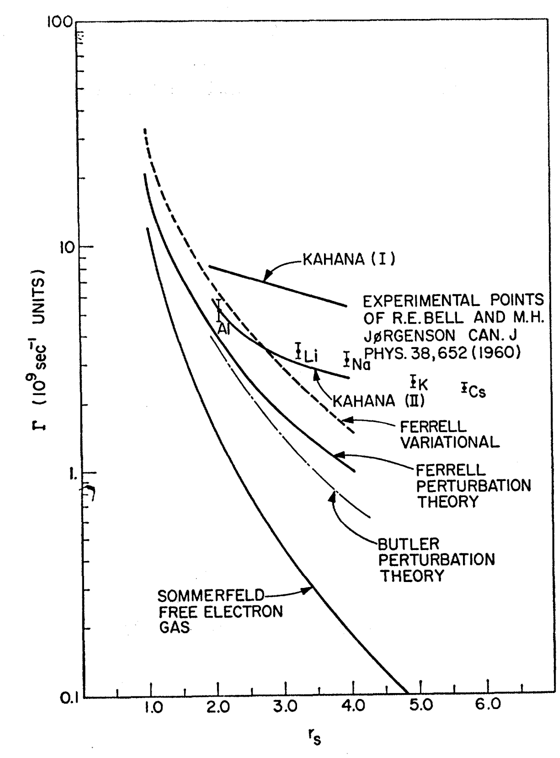

FIG. 1. Comparison of various theoretical annihilation rates with experimental results of Bell and Jørgensen.

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[1] R. E. Bell and M. H. Jørgensen, Can. J. Phys. 38, 652 (1960).

[2] R. A. Ferrell, Rev. Mod. Phys. 28, 308 (1956).

[3] D. Butler, Proc. Phys. Soc. (London) 80, 741 (1962).

[4] S. Kahana, Phys. Rev. 117, 123 (1960); Phys. Rev. 129, 1622 (1963).

[5] A. Held and S. Kahana, Can. J. Phys. 42, 1908 (1964).

[6] A. J. Glick, Phys. Rev. 129, 1399 (1963).

[7] J. P. Carbotte and S. Kahana, Phys. Rev. 139, A213 (1965).

by the work of Hubbard[8] and Glick.[6] To achieve our goal we use a Monte Carlo scheme to correct the RPA dielectric constant by inclusion of ladder interactions between electron-hole pairs which participate in the density fluctuations induced in an electron gas by an external perturbation. The effective interaction obtained from this ladder-corrected dielectric constant is used to make a high-order perturbation calculation of the annihilation rate of positrons in metals, again by a Monte Carlo approach.

In Sec. II we review briefly the theory of the dielectric constant from a field-theoretic point of view. In Sec. III we discuss the results of our calculation of the dielectric constant and summarize the results with an empirical expression for the irreducible part of the particle-hole-pair propagator. Comparison is made with the RPA and the Hubbard[8] dielectrics, and, incidentally, we obtain the screening constant of an electron gas. Section IV is a calculation of the annihilation rate of positrons in metals using the interaction of Sec. III.

II. REVIEW OF DIELECTRIC THEORY

In calculations[9] of the dielectric properties of an electron gas, the quantity of importance[10] is the irreducible polarization part of the electron-hole-pair propagator $F(x-x')$. This propagator, which is closely related to the linear response function of the electron gas, is the kernel expressing the linear relationship between $\rho_{\text{in}}(x)$, the electron density induced in the gas at x, and $V_{\text{ex}}(x')$, the potential produced in the system at x' by the external perturbing particle density $\rho_{\text{ex}}(x'')$ at x'', i.e.,

$$\rho_{\text{in}}(x)=\int d^4x' F(x-x')V_{\text{ex}}(x')\,, \tag{1}$$

where

$$V_{\text{ex}}(x')=\int d^4x'' V(x'-x'')\rho_{\text{ex}}(x'') \tag{2}$$

and $V(x'-x'')$ is the (static) Coulomb interaction. If we take the Fourier transform of (1), we obtain

$$\rho_{\text{in}}(\mathbf{k},\omega)=F(\mathbf{k},\omega)V(\mathbf{k})\rho_{\text{ex}}(\mathbf{k},\omega)\,, \tag{3}$$

and since

$$\rho_{\text{total}}(\mathbf{k},\omega)=\rho_{\text{ex}}(\mathbf{k},\omega)+\rho_{\text{in}}(\mathbf{k},\omega)=\rho_{\text{ex}}(\mathbf{k},\omega)/\epsilon(\mathbf{k},\omega)\,, \tag{4}$$

we have

$$\epsilon^{-1}(\mathbf{k},\omega)=1+V(\mathbf{k})F(\mathbf{k},\omega)\,, \tag{5}$$

where $\epsilon(\mathbf{k},\omega)$ is the wave vector- and frequency-dependent dielectric constant.

A first-order perturbation calculation reveals that

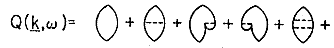

FIG. 2. The irreducible part of the particle-hole-pair propagator.

the particle-hole propagator $F(x-x')$ is given by

$$F(x-x')=(i\hbar)^{-1}\langle\phi_0|T\rho(x)\rho(x')|\phi_0\rangle\,, \tag{6}$$

where $|\phi_0\rangle$ is the Heisenberg exact, normalized ground state of the interacting system, $\rho(x)$ is the electron-number-density operator, and T is the time-ordering operator. If we write (6) in the interaction representation and perform a Fourier transform, we find that

$$F(\mathbf{k},\tau)=(i\hbar\Omega)^{-1}\sum_{\substack{\mathbf{p},\mathbf{p}'\\ \sigma,\sigma'}}\langle 0|T\{S(\infty,-\infty)a_{\mathbf{p}-\mathbf{k},\sigma}{}^{\dagger}(\tau)\times a_{\mathbf{p}\sigma}(\tau)a_{\mathbf{p}'+\mathbf{k},\sigma'}{}^{\dagger}(0)a_{\mathbf{p}',\sigma'}(0)\}|0\rangle_L\,, \tag{7}$$

where $S(\infty,-\infty)$ is the scattering matrix, $|0\rangle$ is the noninteracting ground state, and Ω is the quantization volume. The operators $a_{\mathbf{p}\sigma}{}^{\dagger}(\tau)$ and $a_{\mathbf{p}\sigma}(\tau)$ are electron creation and annihilation operators, respectively, and the subscript L indicates that we must consider only linked diagrams. On expanding (7) by means of many-body perturbation theory and Fourier transforming, it is found that

$$\begin{aligned}F(\mathbf{k},\omega)&=Q(\mathbf{k},\omega)+V(\mathbf{k})Q^2(\mathbf{k},\omega)+V^2(\mathbf{k})Q^3(\mathbf{k},\omega)+\cdots\\&=Q(\mathbf{k},\omega)/[1-V(\mathbf{k})Q(\mathbf{k},\omega)]\,,\end{aligned} \tag{8}$$

where $Q(\mathbf{k},\omega)$ is the irreducible polarization part of $F(\mathbf{k},\omega)$ and is the sum of contributions from diagrams which cannot be cut to produce other diagrams. Symbolically, $Q(\mathbf{k},\omega)$ is given by the sum in Fig. 2. Substitution of Eq. (8) into Eq. (5) yields

$$\epsilon(\mathbf{k},\omega)=1-V(\mathbf{k})Q(\mathbf{k},\omega)\,. \tag{9}$$

The simplest approximation to $Q(\mathbf{k},\omega)$ is the RPA[1]. in which one keeps only the first diagram in Fig. 2[1] This diagram is the basic polarization diagram in which an electron is excited out of the Fermi sphere and subsequently recombines with the hole which it left. To this approximation, the effective interaction $V(\mathbf{k})/\epsilon(\mathbf{k},\omega)$ is obtained by performing the sum indicated in Fig. 3. This interaction is exact for an infinitely dense electron

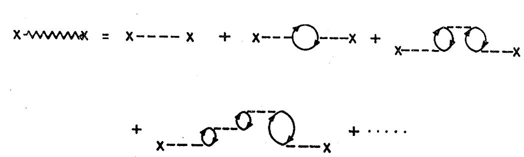

FIG. 3. The RPA interaction.

[8] J. Hubbard, Proc. Roy. Soc. (London) **A243**, 336 (1958).

[9] cf. T. D. Schultz, *Quantum Field Theory and the Many-Body Problem* (Gordon and Breach, Science Publishers, Inc., New York, 1964), pp. 79–96.

[10] D. F. DuBois, Ann. Phys. (N. Y.) **7**, 174 (1959).

[11] The RPA dielectric constant was first obtained by J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **28**, No. 8 (1954); see also P. Nozières and D. Pines, Nuovo Cimento **9**, 470 (1958); G. Rickayzen, Phys. Rev. **115**, 795 (1959); and H. Ehrenreich and M. Cohen, *ibid.* **115**, 786 (1959).

gas, but it is not exact at densities corresponding to those existing in the conduction band of real metals.

Hubbard[8] attempted to correct the RPA dielectric constant by replacing the basic-polarization bubble in Fig. 3 by the infinite sum shown in Fig. 4. If we express the RPA approximation to $Q(\mathbf{k},\omega)$ as $Q_{\mathrm{RPA}}(\mathbf{k},\omega) = F^{(0)}(\mathbf{k},\omega)$, then the analytic approximation obtained by Hubbard can be expressed

$$Q_H(\mathbf{k},\omega) = F^{(0)}(\mathbf{k},\omega)/\{1+\tfrac{1}{2}V'(\mathbf{k})F^{(0)}(\mathbf{k},\omega)\}, \quad (10)$$

where $V'(\mathbf{k}) = 4\pi e^2/(k^2+k_F^2)$ and k_F is the Fermi wave number. The static limit ($\omega \to 0$) of this propagator yields a negative dielectric constant for $r_s > 3$ when k is small.

III. CALCULATION OF AN INTERACTION TAKING ACCOUNT OF ELECTRON-HOLE SCATTERING

If we expand Eq. (7), keeping only the terms shown in Fig. 4, and replacing the Coulomb interaction in Fig. 4 by the RPA interaction,[6] then in the static limit we

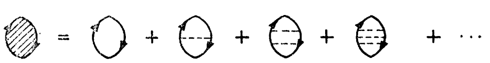

FIG. 4. Ladder correction to the RPA polarization.

have for our improved propagator

$$Q_{\mathrm{LAD}}(\mathbf{k},0) = -2\alpha \int d^3p \frac{f_0(\mathbf{p}) - f_0(\mathbf{p}-\mathbf{k})}{\mathbf{k}\cdot(\mathbf{k}-2\mathbf{p})}$$

$$\times \left\{ 1 + \alpha \int d^3p' \mathcal{V}_{\mathrm{RPA}}(|\mathbf{p}'-\mathbf{p}|,0) \frac{f_0(\mathbf{p}') - f_0(\mathbf{p}'-\mathbf{k})}{\mathbf{k}\cdot(\mathbf{k}-2\mathbf{p}')} \right.$$

$$+ \alpha^2 \int d^3p' \int d^3p'' \mathcal{V}_{\mathrm{RPA}}(|\mathbf{p}'-\mathbf{p}|,0) \mathcal{V}_{\mathrm{RPA}}(|\mathbf{p}''-\mathbf{p}'|,0)$$

$$\left. \times \frac{f_0(\mathbf{p}') - f_0(\mathbf{p}'-\mathbf{k})}{\mathbf{k}\cdot(\mathbf{k}-2\mathbf{p}')} \frac{f_0(\mathbf{p}'') - f_0(\mathbf{p}''-\mathbf{k})}{\mathbf{k}\cdot(\mathbf{k}-2\mathbf{p}'')} + \cdots \right\}, \quad (11)$$

where $\alpha = 2m/\hbar^2(2\pi)^3$ and $\mathcal{V}_{\mathrm{RPA}}(\mathbf{k},0) = V(\mathbf{k})/[1-V(\mathbf{k}) \times F^{(0)}(\mathbf{k},0)]$ is the static RPA interaction. $f_0(\mathbf{p})$ is the Fèrmi distribution function,

$$f_0(\mathbf{p}) = \langle 0 | a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} | 0 \rangle = 1, \quad p \leq k_F$$
$$= 0, \quad p > k_F. \quad (12)$$

Equation (12) can be evaluated by means of Monte Carlo procedures.[12] The results of such a calculation are shown in Figs. 5, 6, and 7. Figure 5 shows the effective interaction $\mathcal{V}_{\mathrm{LAD}}(zk_F,0) = \beta^2(\pi^3e^2a_0^2)/\{(z/2)^2 \times \epsilon_{\mathrm{LAD}}(zk_F,0)\}$, ($\beta = 0.166r_s$) in units of $(\pi^3e^2a_0^2)$ as a

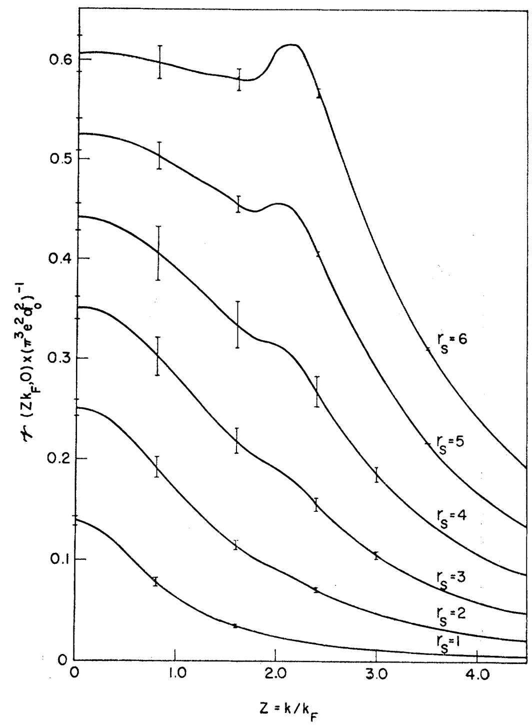

FIG. 5. The improved static interaction calculated by Monte Carlo techniques.

function of the momentum $z = k/k_F$ and for various values of r_s. Figure 6 shows a comparison of our calculated interaction with the RPA and Hubbard static interactions for $r_s = 3$. We note that for $2 \lesssim z$, the RPA is quite adequate; for $1 \lesssim z \lesssim 2$, the Hubbard interaction is valid; but for $z \lesssim 1$, both RPA and Hubbard interactions are inadequate. To a good approximation, our calculated results for the irreducible part of the static-particle-hole-pair propagator may be expressed by the

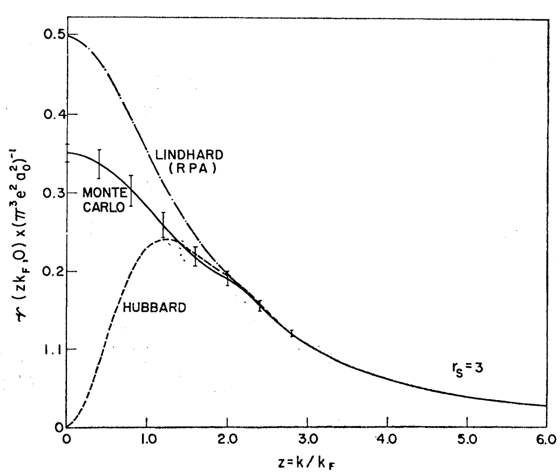

FIG. 6. Comparison of the RPA (Lindhard), Hubbard, and improved dielectric constants for $r_s = 3$.

[12] Details of the Monte Carlo calculation are discussed in Julian Crowell and R. H. Ritchie, Oak Ridge National Laboratory Report, ORNL-TM-1443 (unpublished).

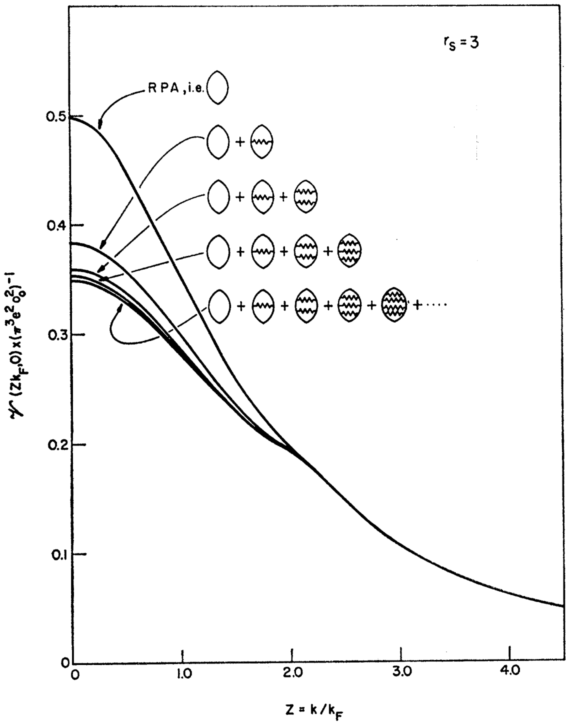

FIG. 7. Convergence of the ladder series used in calculating the improved dielectric constant.

equation

$$Q_{\text{LAD}}(\mathbf{k},0) = F^{(0)}(\mathbf{k},0)/\{1+[2\pi e^2/(k^2+\xi k_F^2)]F^{(0)}(\mathbf{k},0)\}, \quad (13)$$

where $\xi = 1.5 + 0.6 r_s$.

Figure 7 demonstrates the rate of convergence of the series in Eq. (11) for $r_s=3$. From the rather rapid convergence we may conclude that, although the first two or three electron-hole interactions represent important corrections to the static RPA, there seems to be little tendency for the formation of electron-hole bound states.

Before proceeding to a discussion of positron annihilation in metals, we note that the results of this section can be used to obtain the screening constant of an electron gas. The effective interaction can be written

$$\mathcal{V}_{\text{LAD}}(k,0) = 4\pi e^2/(k^2+k_s^2(k)), \quad (14)$$

where k_s is the screening wave vector. We have, taking $k_s = s k_F$,

$$s^2 = z^2[\epsilon_{\text{LAD}}(zk_F,0)-1]. \quad (15)$$

In the RPA approximation as $z\to 0$, we have $s=2\sqrt{\beta}\equiv s_{\text{FT}}$, where s_{FT} is the Fermi-Thomas wave vector.[13] We may define the screening constant C by the equation,

$$C \equiv \lim_{z\to 0} s^2(z). \quad (16)$$

[13] Cf. D. Pines, *Elementary Excitations in Solids* (W. A. Benjamin, Inc., New York, 1963), p. 96.

This quantity is plotted in Fig. 8 in units of the Fermi-Thomas screening constant $C_{\text{FT}}=4\beta$. For the sake of comparison, we also plot DuBois's equation[14] (4.14) which includes, to first order in r_s, the first four diagrams in Fig. 2. In our notation, DuBois's screening constant $C_{\text{D.B.}}$ may be written

$$C_{\text{D.B.}}/C_{\text{FT}} = 1+\beta = 1+0.166 r_s. \quad (17)$$

IV. POSITRON ANNIHILATION IN METALS

The two-photon annihilation rate for thermalized positrons in an electron gas may be obtained from the equation

$$\Gamma = \sum_{\mathbf{K}} \sum_{\omega(\mathbf{K})} 2 \sum_{p<k_F} \frac{1}{T} |\langle f|S|i\rangle|^2, \quad (18)$$

where

$$S = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{i\hbar}\right)^n \times\int_{-T/2}^{T/2} dt_1\cdots\int_{-T/2}^{T/2} dt_n P\{V(t_1)\cdots V(t_n)\} \quad (19)$$

and T is a time much greater than any periods of the system; the initial state $|i\rangle$ is the unperturbed Fermi sphere plus one positron, and the final state $|f\rangle$ is the unperturbed Fermi sphere with one hole. The summa-

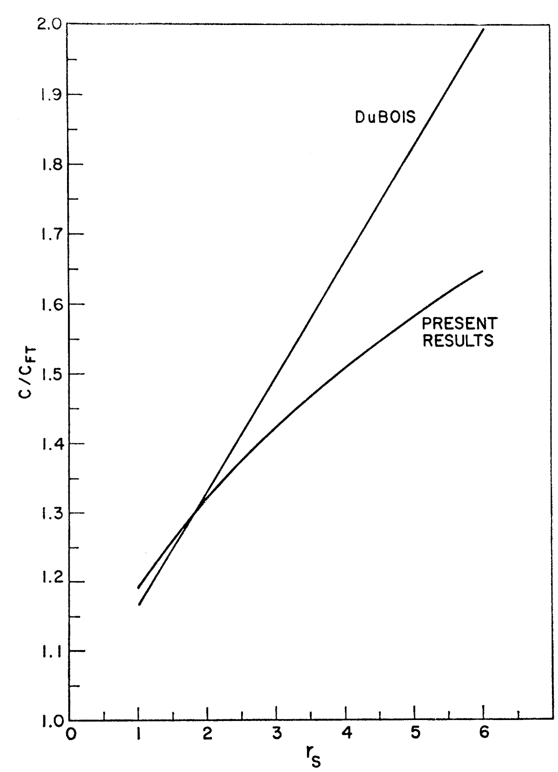

FIG. 8. The screening constant of an electron gas in units of the Fermi-Thomas screening constant.

[14] D. F. DuBois, Ann. Phys. (N. Y.) 8, 24 (1959).

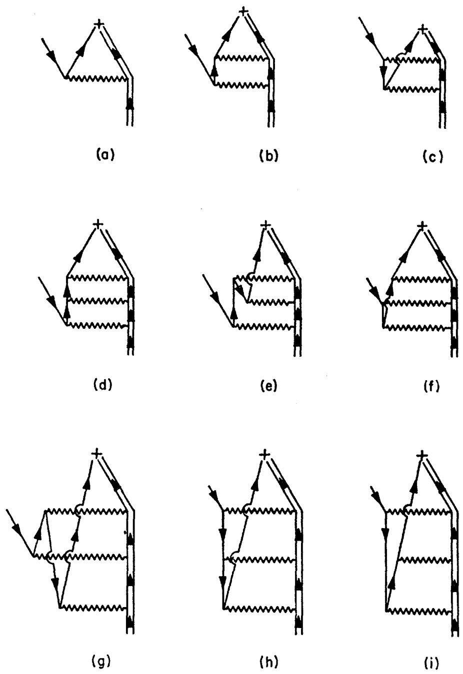

FIG. 9. First-, second-, and third-order Coulomb-scattering events between the electron (and/or hole) and the positron before annihilation.

tions in (18) are over final two-photon momenta **K** and energies $\omega(\mathbf{K})$, and over initial electron momenta **p**.

The interaction $V(t)$ which we shall use is given by

$$V(t)=\{V_{\mathrm{ep}}(t)+V_{\mathrm{RAD}}(t)\}e^{-\eta|t|}, \qquad (20)$$

where $V_{\mathrm{ep}}(t)$ is the electron-positron interaction, $V_{\mathrm{RAD}}(t)$ represents the interaction of the positron and the electron with the radiation field, and $\eta=0+$.

The electron-positron interaction is

$$V_{\mathrm{ep}}(t)=\int dt' \sum_{P_1P_2}\sum_{Q_1Q_2}\langle P_1Q_1|\mathcal{V}(x-x')|Q_2P_2\rangle \times a_{P_1}^{\dagger}(t)d_{Q_1}^{\dagger}(t')d_{Q_2}(t')a_{P_2}(t), \qquad (21)$$

where $a_P^{\dagger}(t)$ and $a_P(t)$ are electron-creation and destruction operators, $d_Q^{\dagger}(t)$ and $d_Q(t)$ are positron creation and destruction operators, and $P\equiv(\mathbf{p},\sigma)$. The interaction $\mathcal{V}(x-x')$ is the screened Coulomb interaction, i.e.,

$$\mathcal{V}(\mathbf{x}-\mathbf{x}',t-t')=\frac{1}{\Omega T}\sum_{\mathbf{k},\omega}\mathcal{V}(\mathbf{k},\omega) \times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega(t-t')], \qquad (22)$$

where

$$\mathcal{V}(\mathbf{k},\omega)=-4\pi e^2[k^2\epsilon(k,\omega)]^{-1}. \qquad (23)$$

$V_{\mathrm{RAD}}(t)$ may be represented here by

$$V_{\mathrm{RAD}}(t)=\hbar\left(\frac{\pi r_0^2c}{\Omega T}\right)^{1/2}\sum_P\sum_Q \delta^3{}_{\mathbf{K},\mathbf{p}+\mathbf{q}} \times e^{+i\omega(\mathbf{K})t}a_P(t)d_Q(t), \qquad (24)$$

where $\delta^3{}_{\mathbf{q},\mathbf{p}}$ is the three-dimensional Kroenecker-delta function. The operator V_{RAD} is assumed so small that we need retain only terms linear in it in the expansion of the S operator.

An expansion of the S matrix in (18) could include many different kinds of terms, some of which are shown in Fig. 9. We have not included any mass-renormalization terms, and they will not be studied in this paper. There is some indication[6] that m^*/m is nearly unity for the electron, but may be appreciably larger for the positron.[15] We have assumed in this paper that $m^*/m\simeq1$ for both the electron and the positron.[16] Further, we will restrict ourselves to diagrams of the type (a), (b), and (d), i.e., we will assume that the positron scatters the electron out of the Fermi sea and interacts repeatedly with it outside the sea before annihilating with it. This means that we will ignore diagrams such as (c), (e), (f), (g), (h), and (i). The omission of these exchange terms can be at least partially justified by studying the second-order annihilation rate. In Fig. 10 are shown the (summed) contributions to the annihilation rate of diagrams (a), (b), and (c) along with the zero-order Sommerfeld annihilation rate. This particular calculation assumes that the annihilating electrons have

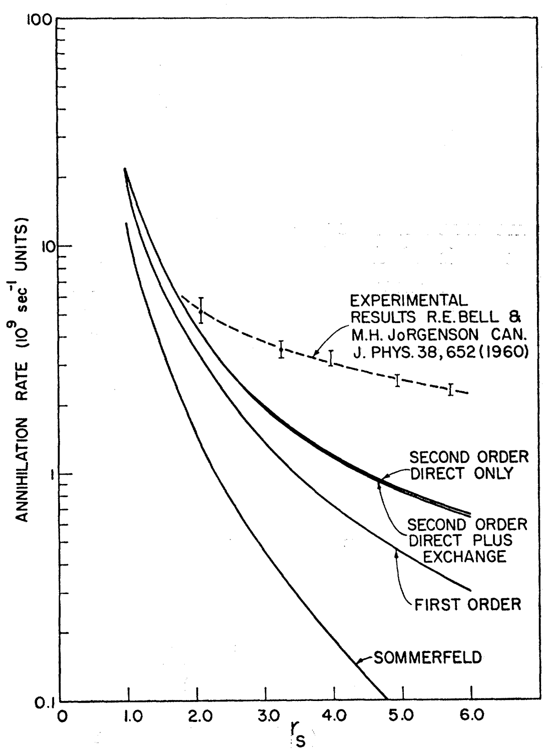

FIG. 10. Second-order perturbation-theory annihilation rate in a static RPA electron gas. Electrons are assumed to have zero momentum.

[15] A. T. Stewart and J. B. Shand, Bull. Am. Phys. Soc. **10**, 21 (1965).

[16] For discussion of the effect of self-energies on the annihilation rate, see Ref. 7.

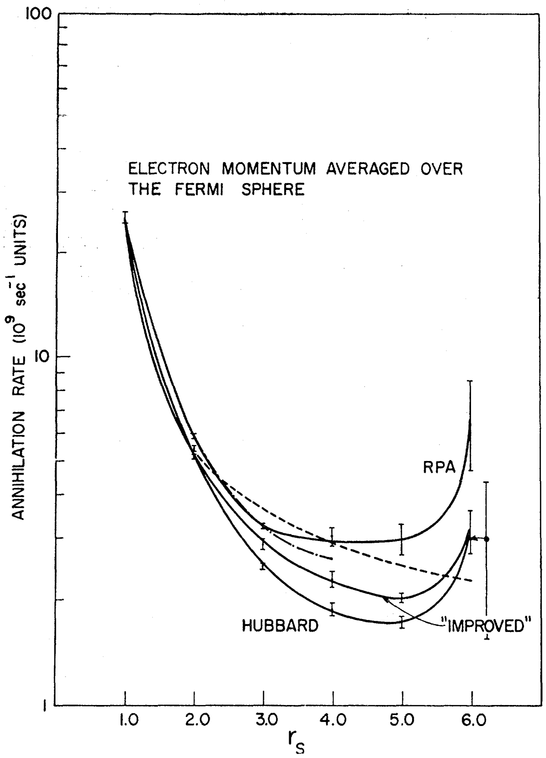

FIG. 11. High-order perturbation calculation of the annihilation rate using (1) the RPA interaction, (2) the Hubbard interaction, and (3) the improved "ladder" interaction. Dashed line—experimental results of R. E. Bell and M. H. Jørgensen (Ref. 1) Dot-dash line—theoretical results of S. Kahana [Phys. Rev. **129**, 1622 (1963)].

zero momentum. Carbotte and Kahana[7] also conclude, on the basis of the almost complete cancellation of positron-hole contributions with electron-hole contributions, that the exchange terms are unimportant.

We then return to our expansion of $\langle f|S|i\rangle$ and keep only the direct ladder terms, such as (a), (b), and (d) in Fig. 10. This leads to the following result:

$$\langle f|S|i\rangle=\langle f|S^{(0)}|i\rangle \times\left\{1+\frac{1}{(2\pi)^3\hbar}\int\frac{d^3k\,\mathcal{V}(\mathbf{k}-\mathbf{p},0)H(\mathbf{k})}{\omega(\mathbf{p})-\omega(\mathbf{k})-\omega(\mathbf{k}-\mathbf{p})}\right\}, \quad (25)$$

where $\omega(\mathbf{p})=\hbar p^2/2m$, and

$$H(\mathbf{k})=1+\frac{1}{(2\pi)^3\hbar}\int\frac{d^3k'\mathcal{V}(|\mathbf{k}'-\mathbf{k}|,0)H(\mathbf{k}')}{\omega(\mathbf{p})-\omega(\mathbf{k}')-\omega(\mathbf{k}'-\mathbf{p})}. \quad (26)$$

The matrix element $\langle f|S^{(0)}|i\rangle$ is the two-photon annihilation amplitude for a positron in a noninteracting free-electron gas; i.e., it is the Sommerfeld amplitude and is given by

$$\langle f|S^{(0)}|i\rangle=-i(\pi r_0^2cT/\Omega)^{1/2}\delta^3_{\mathbf{K},\mathbf{p}}\delta_{\omega(\mathbf{K}),\omega(\mathbf{p})}. \quad (27)$$

The integral equation (26) lends itself to a calculation by means of a Monte Carlo "transport game." The results of calculations of the annihilation rate by this method are shown in Fig. 11, where the three different curves were obtained using the RPA interaction, the Hubbard interaction, and our own interaction from Sec. IV above.

V. CONCLUSIONS

In the previous sections it has been shown that ladder interactions represent important corrections to both the static dielectric constant, where account must be taken of repeated hole-electron interactions in the density fluctuations, and to the positron-electron annihilation rate, where account must be taken of repeated interactions between the electron and the positron before annihilation.

Our inclusion of electron-hole interactions led us to a dielectric constant significantly different from the RPA and the Hubbard approximations for $k<2k_F$. This dielectric constant should be useful in the calculation of the effective mass of a charged particle and in other applications.

Our calculated annihilation results are in good agreement with experiment in the range $2\leq r_s\leq 5$, and indeed, should the conjecture[1] prove correct that 15% of the annihilations in metals take place with core electrons, then our results using our improved interaction agree quite closely with experiment.

The upturn in the theoretical annihilation curves in the neighborhood of $r_s\simeq 6$ may be real or may be due to statistical errors in the calculation. At low electron densities the probability of positronium formation increases, and more terms are required for convergence of the perturbation series. This in turn leads to a magnification of statistical fluctuations in the Monte Carlo process. Approximate calculations[12] indicate that a bound state does exist for r_s just greater than 6 in the RPA and Hubbard electron gases. So it may well be that the upturn is evidence for an incipient bound state.[5]

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